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Abstract

An acyclic coloring of a graph $G$ is a coloring of its vertices such that: (i) no two adjacent vertices in $G$ receive the same color and (ii) no bicolored cycles exist in $G$. A list assignment of $G$ is a function $L$ that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. Let $G$ be a graph and $L$ be a list assignment of $G$. The graph $G$ is acyclically $L$-list colorable if there exists an acyclic coloring $\phi$ of $G$ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If $G$ is acyclically $L$-list colorable for any list assignment $L$ with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is acyclically $k$-choosable. In this paper, we prove that every planar graph without cycles of lengths 4 to 12 is acyclically 3-choosable.

1 Introduction

A proper coloring of a graph is an assignment of colors to the vertices of the graph such that two adjacent vertices do not use the same color. A $k$-coloring of $G$ is a proper coloring of $G$ using $k$ colors; a graph admitting a $k$-coloring is said to be $k$-colorable. An acyclic coloring of a graph $G$ is a proper coloring of $G$ such that $G$ contains no bicolored cycles; in other words, the graph induced by every two color classes is a forest. A list assignment of $G$ is a function $L$ that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. Let $G$ be a graph and $L$ be a list assignment of $G$. The graph $G$ is acyclically $L$-list colorable if there is an acyclic coloring $\phi$ of $G$ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. If $G$ is acyclically $L$-list colorable for any list assignment $L$ with $|L(v)| \geq k$ for all $v \in V(G)$, then $G$ is acyclically $k$-choosable. The acyclic choice number of $G$, $\chi^a_c(G)$, is the smallest integer $k$ such that $G$ is acyclically $k$-choosable. Borodin et al. [1] first investigated the acyclic choosability of planar graphs proving that:

**Theorem 1** [1] Every planar graph is acyclically 7-choosable.

and put forward to the following challenging conjecture:

**Conjecture 1** [1] Every planar graph is acyclically 5-choosable.


In 1976, Steinberg conjectured that every planar graph without cycles of lengths 4 and 5 is 3-colorable (see Problem 2.9 [7]). This problem remains open. In 1990, Erdős suggested the following relaxation of Steinberg’s Conjecture: what is the smallest integer $i$ such that every planar graph without cycles of lengths 4 to $i$ is 3-colorable? The best known result is $i = 7$ [2]. This question is also studied in the choosability case: what is the smallest integer $i$ such that every planar graph without cycles of lengths 4 to $i$ is 3-choosable? In [12], Voigt proved that Steinberg’s Conjecture can not be extended to list coloring; hence, $i \geq 6$. Nevertheless, in 1996, Borodin [4] proved that every planar graph without cycles of lengths 4 to 9 is 3-colorable; in fact, 3-choosable. So, $i \leq 9$.

In this paper, we study the question of Erdős in the acyclic choosability case:
Problem 1 What is the smallest integer $i$ such that every planar graph without cycles of lengths 4 to $i$ is acyclically 3-choosable?

Note that it is proved that every planar graph without cycles of lengths 4 to 6 is acyclically 4-choosable [10]. Also, the relationship between the maximum average degree of $G$ (or the girth of $G$) and its acyclic choice number was studied (see for example [9, 8, 5]).

Our main result is the following:

Theorem 2 Every planar graph without cycles of lengths 4 to 12 is acyclically 3-choosable.

Hence, in Problem 1, $6 \leq i \leq 12$.

Section 2 is dedicated to the proof of Theorem 2. Follow some notations we will use:

Notations Let $G$ be a planar graph. We use $V(G)$, $E(G)$ and $F(G)$ to denote the set of vertices, edges and faces of $G$ respectively. Let $d(v)$ denote the degree of a vertex $v$ in $G$ and $r(f)$ the length of a face $f$ in $G$. A vertex of degree $k$ (resp. at least $k$, at most $k$) is called a $k$-vertex (resp. $\geq k$-vertex, $\leq k$-vertex). We use the same notations for faces: a $k$-face (resp. $\geq k$-face, $\leq k$-face) is a face of length $k$ (resp. at least $k$, at most $k$). A $k$-face having the boundary vertices $x_1, x_2, ..., x_k$ in the cyclic order is denoted by $[x_1x_2...x_k]$. For a vertex $v \in V(G)$, let $n_i(v)$ denote the number of $i$-vertices adjacent to $v$ for $i \geq 1$, and $m_3(v)$ the number of 3-faces incident to $v$. A 3-vertex is called a $3^*$-vertex if it is incident to a 3-face and adjacent to a 2-vertex (for example in Figure 1, the vertex $t$ is a $3^*$-vertex). A 3-face $[rst]$ with $d(r) = d(s) = d(t) = 3$ and with a $3^*$-vertex on its boundary is called a 3-face. Two 3-faces $[rst]$ and $[uvw]$ are called linked if there exists an edge $tv$ which connects these two 3-faces such that $d(t) = d(v) = 3$ (see Figure 2). A vertex $v$ is linked to a 3-face $[rst]$ if there exists an edge between $v$ and one vertex of the boundary of $[rst]$, say $t$, such that $d(t) = 3$ (for example in Figure 1, the vertex $v$ is linked to the 3-face $[rst]$). Let $n^*(v)$ be the number of 3*-face linked to $v$.

![Figure 1](image1.png)

Figure 1: The vertex $t$ is a $3^*$-vertex and the vertex $v$ is linked to the 3-face $[rst]$

![Figure 2](image2.png)

Figure 2: The two 3-faces $[rst]$ and $[uvw]$ are linked

2 Proof of Theorem 2

To prove Theorem 2 we proceed by contradiction. Suppose that $H$ is a counterexample with the minimum order to Theorem 2 which is embedded in the plane. Let $L$ be a list assignment with $|L(v)| = 3$ for all $v \in V(H)$ such that there does not exist an acyclic coloring $c$ of $H$ with for all $v \in V(H)$, $c(v) \in L(v)$.

Without loss of generality we can suppose that $H$ is connected. We will first investigate the structural properties of $H$ (Section 2.1), then using Euler’s formula and the discharging technique we will derive a contradiction (Section 2.2).
2.1 Structural properties of \( H \)

**Lemma 1** The minimal counterexample \( H \) to Theorem 2 has the following properties:

(C1) \( H \) contains no 1-vertices.

(C2) A 3-face has no 2-vertex on its boundary.

(C3) A 2-vertex is not adjacent to a 2-vertex.

(C4) A 3-face has at most one 3*-vertex on its boundary.

(C5) A 3-face \([rst]\) with \( d(r) = d(s) = d(t) = 3 \) is linked to at most one 3*-face.

(C6) Two 3*-faces cannot be linked.

**Proof**

(C1) Suppose \( H \) contains a 1-vertex \( u \) adjacent to a vertex \( v \). By minimality of \( H \), the graph \( H' = H\setminus\{u\} \) is acyclically 3-choosable. Consequently, there exists an acyclic \( L \)-coloring \( c \) of \( H' \). To extend this coloring to \( H \) we just color \( u \) with \( c(u) \in L(u)\setminus\{c(v)\} \). The obtained coloring is acyclic, a contradiction.
Lemma 2 Let $H$ be a connected plane graph with $n$ vertices, $m$ edges and $r$ faces. Then, we have

(C2) Suppose $H$ contains a 2-vertex $u$ incident to a 3-face $[uvw]$. By minimality of $H$, the graph $H' = H \setminus \{u\}$ is acyclically 3-choosable. Consequently, there exists an acyclic $L$-coloring $c$ of $H'$. We show that we can extend this coloring to $H$ by coloring $u$ with $c(u) \in L(u) \setminus \{c(v), c(w)\}$.

(C3) Suppose $H$ contains a 2-vertex $u$ adjacent to a 2-vertex $v$. Let $t$ and $w$ be the other neighbors of $u$ and $v$ respectively. By minimality of $H$, the graph $H' = H \setminus \{u\}$ is acyclically 3-choosable. Consequently, there exists an acyclic $L$-coloring $c$ of $H'$. We show that we can extend this coloring to $H$. Assume first that $c(t) \neq c(v)$. Then we just color $u$ with $c(u) \in L(u) \setminus \{c(t), c(v)\}$. Now, if $c(t) = c(v)$, we color $u$ with $c(u) \in L(u) \setminus \{c(v), c(w)\}$. In the two cases, the obtained coloring is acyclic, a contradiction.

(C4) Suppose $H$ contains a 3-face $[rst]$ with two 3*-vertices $s$ and $t$. Suppose that $t$ (resp. $s$) is adjacent to a 2-vertex $v$ (resp. $x$) with $v \neq r, s$ by (C2) (resp. $x \neq r, t$). Let $u$ (resp. $y$) be the other neighbor of $v$ (resp. $x$) with $u \neq r, s$ (resp. $y \neq r, t$). By the minimality of $H$, $H' = H \setminus \{v\}$ is acyclically 3-choosable. Consequently, there exists an acyclic $L$-coloring $c$ of $H'$. We show now that we can extend $c$ to $H$. If $c(u) \neq c(t)$, we color then $v$ with a color different from $c(u)$ and $c(t)$ and the coloring obtained is acyclic. Otherwise, $c(u) = c(t)$. If we cannot color $v$, this implies without loss of generality $L(v) = \{1, 2, 3\}$, $c(u) = c(t) = c(x) = 1$, $c(r) = 2$ and $c(s) = c(y) = 3$. Observe that necessarily $L(t) = \{1, 2, 3\}$ (otherwise we can recolor $t$ with $\alpha \in L(t) \setminus \{1, 2, 3\}$ and color $v$ properly, i.e. $v$ receives a color distinct of those of these neighbors). For a same reason $L(s) = \{1, 2, 3\}$ and $L(x) = \{1, 2, 3\}$. Now, we recolor $t$ with the color $3$, $s$ with the color $1$ and $x$ with the color $2$, then we can color $v$ with the color $2$. It is easy to see that the coloring obtained is acyclic.

(C5) Suppose $H$ contains a 3-face $[rst]$ incident to three vertices such that two of them are linked to two 3*-faces $[ijk]$ and $[l mn]$. Suppose $[ijk]$ and $[l mn]$ are linked to $[rst]$ respectively by the edges $sj$ and $tl$. Call $y$ the third neighbor of $i$, $x$ the third neighbor of $r$, and $p$ the third neighbor of $m$. Suppose that the 2-vertex $u$ (resp. $v$) is adjacent to $k$ and $z$ (resp. $n$ and $w$). For example, $H$ contains the graph depicted by Figure 3. By the minimality of $H$, $H' = H \setminus \{v\}$ is acyclically 3-choosable. Consequently, there exists an acyclic $L$-coloring $c$ of $H'$. We show now that we can extend $c$ to $H$. If $c(w) \neq c(n)$, we color then $v$ with a color different from $c(w)$ and $c(n)$ and the coloring obtained is acyclic. Otherwise, $c(w) = c(n)$. If we cannot color $v$, this implies without loss of generality $L(v) = \{1, 2, 3\} = L(l) = L(m)$, $c(w) = c(n) = c(t) = c(p) = 1$, and by permuting the colors of $l$ and $m$, we are sure that $L(r) = \{1, 2, 3\} = L(s)$ and $c(x) = c(j) = 1$, then by permuting the colors of $r$ and $s$, we are sure that $L(i) = \{1, 2, 3\} = L(k)$, $L(y) = c(u) = 1$, and $L(z) \in \{2, 3\}$. Let $\alpha = \{2, 3\} \setminus \{c(z)\}$. We recolor $k, s, l, v$ with $\alpha$ and $m, r, i$ with $c(z)$. The coloring obtained is acyclic.

(C6) Suppose $H$ contains a 3-face $[rst]$ incident to three 3-vertices such that one vertex is linked to a 3*-face $[ijk]$ and one vertex is a 3*-vertex, say $t$. Call $y$ the third neighbor of $i$, $x$ the third neighbor of $r$. Suppose that the 2-vertex $u$ (resp. $v$) is adjacent to $k$ and $z$ (resp. $t$ and $w$). For example, $H$ contains the graph depicted by Figure 4. By the minimality of $H$, $H' = H \setminus \{v\}$ is acyclically 3-choosable. Consequently, there exists an acyclic $L$-coloring $c$ of $H'$. We show now that we can extend $c$ to $H$. If $c(w) \neq c(t)$, we color then $v$ with a color different from $c(w)$ and $c(t)$ and the coloring obtained is acyclic. Otherwise, $c(w) = c(t)$. If we cannot color $v$, this implies without loss of generality $L(v) = \{1, 2, 3\} = L(r) = L(s)$, $c(w) = c(t) = c(x) = c(j) = 1$, and by permuting the colors of $r$ and $s$, we are sure that $L(i) = \{1, 2, 3\} = L(k)$, $c(y) = c(u) = 1$, and $c(z) \in \{2, 3\}$. Let $\alpha = \{2, 3\} \setminus \{c(z)\}$. We recolor $k, s, v$ with $\alpha$ and $r, i$ with $c(z)$. The coloring obtained is acyclic.

□
\[ \sum_{v \in V(H)} (11d(v) - 26) + \sum_{f \in F(H)} (2r(f) - 26) = -52 \]  

**Proof**

Euler’s formula \( n - m + f = 2 \) can be rewritten as \( (22m - 26n) + (4m - 26f) \) = -52. The relation \[ \sum_{v \in V(H)} d(v) = \sum_{f \in F(H)} r(f) = 2m \] completes the proof. \( \square \)

### 2.2 Discharging procedure

Let \( H \) be a counterexample to Theorem 2 with the minimum order. Then, \( H \) satisfies Lemma 1.

We define the weight function \( \omega : V(H) \cup F(H) \rightarrow \mathbb{R} \) by \( \omega(x) = 11d(x) - 26 \) if \( x \in V(H) \) and \( \omega(x) = 2r(x) - 26 \) if \( x \in F(H) \). It follows from Equation (1) that the total sum of weights is equal to -52. In what follows, we will define discharging rules (R1) and (R2) and redistribute weights accordingly. Once the discharging is finished, a new weight function \( \omega^* \) is produced. However, the total sum of weights is kept fixed when the discharging is achieved. Nevertheless, we will show that \( \omega^*(x) \geq 0 \) for all \( x \in V(H) \cup F(H) \). This leads to the following obvious contradiction:

\[ 0 \leq \sum_{x \in V(H) \cup F(H)} \omega^*(x) = \sum_{x \in V(H) \cup F(H)} \omega(x) = -52 < 0 \]

and hence demonstrates that no such counterexample can exist.

The discharging rules are defined as follows:

(R1.1) Every \( \geq 3 \)-vertex \( v \) gives 2 to each adjacent 2-vertex.

(R1.2) Every \( \geq 4 \)-vertex \( v \) gives 9 to each incident 3-face and 1 to each linked 3*-face.

(R2.1) Every 3*-vertex \( v \) gives 5 to its incident 3-face.

(R2.2) Every 3-vertex \( v \), different from a 3*-vertex, which is not linked to a 3*-face, gives 7 to its incident 3-face (if any).

(R2.3) Every 3-vertex \( v \), different from a 3*-vertex, linked to a 3*-face gives 1 to each linked 3*-face and gives 6 to its incident 3-face (if any).

In order to complete the proof, it suffices to prove that the new weight \( \omega^*(x) \) is non-negative for all \( x \in V(H) \cup F(H) \).

Let \( v \in V(H) \) be a \( k \)-vertex. Then, \( k \geq 2 \) by (C1).

- If \( k = 2 \), then \( \omega(v) = -4 \) and \( v \) is adjacent to two \( \geq 3 \)-vertices by (C3). By (R1.1), \( \omega^*(v) = -4 + 2 \cdot 2 = 0 \).

- If \( k = 3 \), then \( \omega(v) = 7 \). Since \( H \) contains no 4-cycles, \( v \) is incident to at most one 3-face. Assume first that \( v \) is not incident to a 3-face. Then by (R1.1) and (R2.3), \( v \) gives at most 3 times 2. Hence, \( \omega^*(v) \geq 7 - 3 \cdot 2 \geq 1 \). Assume now that \( v \) is incident to a 3-face. If \( v \) is a 3*-vertex, then \( \omega^*(v) = 7 - 5 - 2 = 0 \) by (R1.1) and (R2.1). If \( v \) is linked to a 3*-face then \( \omega^*(v) \geq 7 - 6 - 1 = 0 \) by (R2.3). If \( v \) is not adjacent to a 2-vertex and not linked to a 3*-face then \( \omega^*(v) = 7 - 7 = 0 \) by (R2.2).

- If \( k \geq 4 \), then \( \omega(v) = 11k - 26 \). Observe by (C1), (C2) and definitions of \( n^*(v) \) and of linked vertices that:

\[ m_3(v) \leq \left\lfloor \frac{k}{2} \right\rfloor \quad \text{and} \quad k - 2m_3(v) \geq n_2(v) + n^*(v) \]
\[ \begin{align*}
k \geq 2m_3(v) + n_2(v) + n^*(v) \quad (2)
\end{align*}\]

It follows by (R1.1), (R1.2) and Equation (2) that:

\[ \begin{align*}
\omega^*(v) &= 11k - 26 - 9m_3(v) - n^*(v) - 2n_2(v) \\
&\geq 11k - 26 - 9m_3(v) - \frac{9}{2}n^*(v) - \frac{9}{2}n_2(v) \\
&\geq 11k - 26 - \frac{9}{2}k \\
&\geq \frac{13}{2}k - 26 \\
&\geq 0
\end{align*}\]

Suppose that \( f \) is a \( k \)-face. Then, \( k = 3 \) or \( k \geq 13 \) by hypothesis.

- If \( k \geq 13 \), then \( \omega^*(f) = \omega(f) = 2k - 26 \geq 0 \).
- If \( k = 3 \), then \( \omega(f) = -20 \). Suppose \( f = [rst] \). By (C2), \( f \) is not incident to a 2-vertex; hence, \( d(r) \geq 3, d(s) \geq 3, d(t) \geq 3 \). By (C4) \( f \) is incident to at most one \( 3^* \)-vertex. Now, observe that if one of the vertices \( r, s, t \) is a \( \geq 4 \)-vertex, then by (R1.2) (R2.1) (R2.2) (R2.3) \( \omega^*(f) \geq -20 + 9 + 5 + 6 = 0 \). So assume \( d(r) = d(s) = d(t) = 3 \) and let \( r_0, s_0, t_0 \) be the other neighbors of \( r, s, t \), respectively. Suppose that \( f \) is a \( 3^* \)-face and let \( r \) be its unique \( 3^* \)-vertex. By (C6) none of \( s \) and \( t \) is linked to a \( 3^* \)-face. Moreover \( s_0 \) and \( t_0 \) give \( 1 \) to \( f \) by (R1.2) and (R2.3). Hence \( \omega^*(f) = -20 + 5 + 7 + 2 = 1 \). Finally assume that \( f \) is not a \( 3^* \)-face. By (C5) at most one of \( r, s, t \) is linked to a \( 3^* \)-face. Hence \( \omega^*(f) \geq -20 + 6 + 2 \cdot 7 = 0 \), by (R1.2), (R2.2) and (R2.3).

We proved that, for all \( x \in V(H) \cup F(H) \), \( \omega^*(x) \geq 0 \). This completes the proof of Theorem 2.

References


