# On sl3 KZ equations and W3 null-vector equations

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# *On $s\ell_3$ KZ equations and $\mathcal{W}_3$ null-vector equations*

**Sylvain Ribault**

**Introduction.** Many interesting 2d CFTs are based on affine Lie algebras and their cosets. For example, the affine $\widehat{s\ell_2}$ algebra is the symmetry algebra of the $SL(2,\mathbb{R})$ WZNW model which is related to string theory in $AdS_3$, its Euclidean version the $H_3^+$ model, and strings in the 2d BH. The simplest nonrational theory in this $s\ell_2$ family is Liouville theory.

*Families of non-rational 2d CFTs*
$s\ell_2$ and $s\ell_3$ families
The theories, their sym. alg., target space dim.

*Families of non-rational 2d CFTs*

Similarly we can define an $s\ell_3$ family consisting of the $SL(3)$ WNZW model, its cosets, and other related models. The simplest theory in the family is the $s\ell_3$ conformal Toda theory, a theory of 2 interacting bosons, where $e_1, e_2$ are simple roots of the Lie algebra $s\ell_3$. Several CFTs in the $s\ell_2$ family have been solved, and their correlation functions are written in terms of the same special functions. But CFTs in the $s\ell_3$ family are richer and more difficult to solve, due to phenomena like

- Correlation functions involving degenerate fields but obeying no differential equations
- Infinite fusion multiplicities

Now the $s\ell_2$ family not only consists of theories with similar symmetry algebras, but there are also relations between correlation functions of these theories. In particular a formula for arbitrary correlation functions of the $H_3^+$ model in terms of certain correlation functions in Liouville theory was found. Intuitively, the reason is: affine $s\ell_2$ representations are labelled by just one parameter (the spin), so even if a theory like the $SL(2,\mathbb{R})$ WZNW model has a 3d target space, its dynamics are effectively 1d, due to the large symmetry of the theory.

Such an $H_3^+$-Liouville relation is interesting because Liouville theory was solved in the sense of the conformal bootstrap. This can be used to

- Solve the $H_3^+$ model with a boundary
- Hopefully, solve the 2d black hole and $AdS_3$ WZNW models
- Construct new CFTs by generalizing the $H_3^+$-Liouville relation, as I will mention later

Similarly, the $SL(3,\mathbb{R})$ WZNW model has a 8d target space, but we expect effectively 2d dynamics, since the Cartan subgroup is 2d. The question is whether its correlation functions can be expressed in terms of correlation functions of $s\ell_3$ Toda theory. This would be a large simplification, which would however not solve the $SL(3)$ WZNW model, as the $s\ell_3$ conformal Toda theory has not been solved, in spite of recent progress due to Fateev and Litvinov.

Plan of the talk:

1. Conjectured relation between correlation functions in the $SL(N)$ WZNW model and $s\ell(N)$ conformal Toda theory
2. Case $N=2$ and an application
3. Case $N=3$

*Plan of the talk*

*Plan of the talk*
General $N$
$N=2$
$N=3$

**Relation between correlation functions.** Consider an $m$-point function of affine primary fields in a theory with $\widehat{s\ell_N}$ symmetry. The theory is parametrized by the level $k>N$. The fields are parametrized by their position $z$ on the Riemann sphere, the spin $j$ with $N-1$ (real) components and isospins $x$ with $\frac{N(N-1)}{2}$ (complex) components, see $N-1+N(N-1)=N^2-1$.

We want to relate this to a correlation function in $s\ell_N$ conformal Toda theory (with vertex operators $V_\alpha \sim e^{(\alpha,\phi)}$). The price to pay for working in a simpler theory is that we work with a more complicated correlation function. This allows the isospin variables $x$ to be represented as the positions $y_a$ of certain extra fields, in addition to the $m$ fields with momenta $\alpha(j_i)$ which correspond to the fields $\Phi^j$.

*Correlation functions*

*Correlation functions*
$\Omega_m$
$s\ell_2$ isospins
$\tilde{\Omega}_m$
relations

The problem is that the relation between $x_i$ and $y_a$ is complicated: $y_a$ are not just a function of $x_i$, but they are obtained by a complicated integral transformation, namely Sklyanin's SOV. The conjectured relation also involves a simples twist function $\Theta_m$, with parameters $\lambda,\mu,\nu$ to be determined as functions of the level $k$.

Remember that $\Omega_m$ should obey differential equations, the KZ equations. This provides a way to test the conjecture.

Status of our conjecture: compatible with KZ in $s\ell_2$, and $s\ell_3$ in the limit $k\to 3$. Proved in a specific model in $H_3^+$-Liouville case.

*The conjecture*

*The conjecture*
$\Theta_m$
Conjecture
Status of conjecture

**$s\ell_2$ case.** Let me explain what is Sklyanin's separation of variables and why we use it. Our WZNW correlation functions $\Omega_m$ obey KZ equations, which are written in terms of the Hamiltonians of the Gaudin model, an integrable model. These Hamiltonians $H_i$ are differential operators wrt the isospin variables $x_i$. They are built from a set of differential operators $\overrightarrow{D}$ which obey an $s\ell_N$ algebra, see $s\ell_2$ example. (But we do not really need to specify $\overrightarrow{D}$ explicitly in the calculations that follow.) Sklyanin's SOV was introduced as a way to simplify the simultaneous diagonalization of these Hamiltonians.

| *KZ equations and Gaudin model* |
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| *KZ equations and Gaudin model* |
| KZ and Gaudin |
| $s\ell_2$ isospins |
| Lax matrix |

To construct Sklyanin's variables, the essential object is the operator-valued Lax matrix $L(u)$ where $u$ is the spectral parameter. This is an $N \times N$ matrix which we write explicitly in the case of $s\ell_2$. From $L(u)$ we construct the function $B(u)$ whose zeroes are $y_a$, and $A(u)$ such that $p_a = A(y_a)$ are the associated momenta. In $s\ell_2$ these are simply matrix elements of the Lax matrix. Having thus defined Sklyanin variables, we can deduce the kernel $S$ of the transformation between $x_i$ and $y_a$ variables.

| *SOV in the Gaudin model* |
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| *SOV in the Gaudin model* |
| $s\ell_2$ Lax matrix |
| $B(u)$ |
| $A(u)$ |
| Kernel $S$ |

In addition, Sklyanin's variables obey a kinematical identity called the characteristic equation, which can be written in terms of Lax matrix elements and then in terms of Gaudin Hamiltonians. When applied to a function of $y_a$ like $S^{-1}\Omega_m$, the momentum $p_a$ becomes $\frac{\partial}{\partial y_a}$, and we obtain a differential identity. But then it is possible to inject the KZ equations (obeyed by $\Omega_m$) in this identity. This allows us to rewrite the KZ equations in terms of Sklyanin's variables.

Then compute $S^{-1}\frac{\delta}{\delta z_\ell}S$, doable in $s\ell_2$. Resulting equations are equivalent to second-order BPZ, with correct choice of degenerate field.

| *$s\ell_2$ KZ in Sklyanin variables* |
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| *$s\ell_2$ KZ in Sklyanin variables* |
| $A(u)$, $B(u)$, characteristic equation |
| Apply to $S^{-1}\Omega_m$, inject KZ |
| Compute $S^{-1}\frac{\delta}{\delta z_\ell}S$ |

This agreement is a strong test, and actually part of the proof, of the $H_3^+$-Liouville relation. Beyond helping solve the $H_3^+$ model, this relation has led to the construction of a new family of solvable CFTs. The idea is to modify the Liouville correlator $\tilde{\Omega}_m$ in our ansatz $\Omega_m = S \cdot \Theta_m \tilde{\Omega}_m$, by replacing $V_{-\frac{1}{2b}}$ with $V_{-\frac{r}{2b}}$. We do not get an $m$-point function in $H_3^+$, is it an $m$-point function in some new CFT?

I propose a Lagrangian for the new CFT in terms of the same bosonic fields $\phi, \beta, \bar{\beta}, \gamma, \bar{\gamma}$ which appear in the $H_3^+$ model.

| *Family of non-rational CFTs* |
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| *Family of non-rational CFTs* |
| The ansatz |
| The lagrangian |
| Questions for the symmetry algebra |
| Central charge |

And indeed one can argue that the correlation functions associated to this Lagrangian agree with our Ansatz, and are those of a CFT. So we obtain a family of CFTs with two parameters $r, b$ and central charge $c(r, b)$. This can be interpreted as an extension of the very definition of the $s\ell_2$ family, as composed not from theories built from the affine $s\ell_2$ algebra or some coset thereof, but from theories whose correlation functions are related to Liouville theory correlation functions.

**The $s\ell_3$ case.** We can follow similar steps as in the $s\ell_2$ case for defining and using Sklyanin variables. The function $B(u)$, whose zeroes are the Sklyanin variables, is rather complicated, and the kernel $S$ of the integral transformation between the isospin variables $x_i$ and the Sklyanin variables $y_a$ is not known. However, we do not really need this kernel, but rather the characteristic equation. This one can be obtained explicitly.

| *SOV in the $s\ell_3$ Gaudin model* |
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| Lax matrix |
| $A(u)$ and $B(u)$ |
| Characteristic equation |

*SOV in the $s\ell_3$ Gaudin model*

Then we use the characteristic equation to rewrite the KZ equations in terms of Sklyanin variables. The resulting equations involve the spins of the fields $\Phi^j$ which appear in the $m$-point function $\Omega_m$. The spins appear through the $s\ell_3$ invariants $\Delta_j, q_j$ which correspond to the quadratic and cubic Casimirs of $s\ell_3$.

The corresonding null-vector equations in $s\ell_3$ Toda theory appear if the fields $V_{\alpha_d}$ in $\tilde{\Omega}_m$ is chosen to have null vectors at level $1, 2, 3$, namely $V_{-b\omega_1}$ with $\omega_1$ the fundamental weight. They are rather similar to the KZ equations, but involve extra complicated differential operators $D_1, D_2$, which however depend neither on the field parameters $j, q_j, \Delta_j$, nor on $b^2 = \frac{1}{k-3}$.

| *$s\ell_3$ KZ in Sklyanin variables* |
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| KZ in Slyanin variables |
| NVE if $V_{\alpha_d} = V_{-b\omega_1}$ |
| Agreement of 3 terms |

*$s\ell_3$ KZ in Sklyanin variables*

While the second-order operator $D_2$ might well agree with the $\frac{\delta}{\delta z_i}$ terms in KZ which are technically hard to calculate, the first-order operator $D_1$ has no analog in KZ. The only way to get rid of $D_1$ seems to take the critical level limit $k \to 3$, in which case the rest of the equations have finite limits and agree with each other. This is rather unnatural from the CFT point of view, but maybe natural from the mathematical point of view of the Langlands correspondence, which is supposed to relate differential operators – like the KZ equations – to representations of the affine algebra, in our case the fundamental representation where the Lax matrix lives. (Or the antifundamental representation, if $V_{-b\omega_1}$ is replaced with $V_{-b\omega_2}$.)

In the general case $k \neq 3$ we might speculate that our ansatz $S{\cdot}\Theta_m\tilde{\Omega}_m$, while not being an $m$-point function in the $SL(3)$ WZNW model, is an $m$-point function in some new CFT.

**Questions from the Hamburg audience.**

- *Is the relation also valid in the case of torus partition functions?* In the $s\ell_2$ case, yes. This has been shown by Hikida and Schomerus, who generalized the $H_3^+$-Liouville relation to higher genus Riemann surfaces. The higher the genus, the more degenerate fields $V_{-\frac{1}{2b}}$ appear in the relation. So, while the relation does not apply to the sphere partition function as it would involve $-2$ such fields, it applies to the torus partition function. The torus partition functions of $H_3^+$ and Liouville essentially agree, with no degenerate fields involved. This however requires a careful regularization of the divergences due to the non-rational nature of the theories.

- *Is the relation related to the quantum Drinfeld-Sokolov reduction?* The Drinfeld-Sokolov reduction indeed relates the same theories, for example the $H_3^+$ model and Liouville theory, with the same values of the parameters (central charges). But this reduction consists in eliminating the isospin variables $x_i$, by say giving them fixed values. Here we are not eliminating them, but transforming them using Sklyanin's separation of variables. They reappear in Liouville theory as the positions of the degenerate fields. So we are really able to study arbitrary $H_3^+$ correlation functions.

  (Comment by J. Teschner) However, the relation might well be understandable as a generalization of Drinfeld-Sokolov reduction, where the nilpotent generator $J^+$ is constrained in a more flexible way.

- *What is the generalization to $s\ell_N$?* The obvious conjecture is: for all $N > 2$ the relation works only in the critical level limit $k = N$.