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Abstract — *A unified approach is presented for the synthesis of continuous-time fractional orthogonal bases including Laguerre-like, Kautz-like and the Generalized-Orthogonal-Basis-like (GOB-like) bases. They extend the definitions of their rational counterpart to fractional differentiation orders. Modes can either be chosen to be real or by pairs complex conjugate. Completeness of fractional Laguerre-like basis is demonstrated.*
Keywords — *Fractional calculus, fractional derivative, dynamical system, orthogonal functions, Laguerre basis, Kautz basis, generalized orthogonal basis (GOB), identification.*

1 Introduction and mathematical background

1.1 Context and motivation

Over the last fifteen years, identification and control of linear stable dynamic systems using orthogonal functions have widely been used; see for instance [1, 2, 3, 4] and all references therein. The most popular orthogonal functions used in control engineering are: Laguerre functions, having a single real pole; Kautz functions, having two complex conjugate poles; and the Generalized Orthogonal Basis (GOB) functions which extend the two former definitions to any number of real or complex conjugate poles.

The interest of fractional differentiation (real, complex, integer or not) is motivated by studies on real systems such as thermal [5] and electrochemical [6] which reveal inherent fractional differentiation behavior. The use of classical models, based on integer order differentiation, is thus inappropriate in modeling these fractional systems. Thus, models using fractional differentiation have been developed [7, 8, 9] .

El-Sayed [10] has proposed a direct extension of the definition of Laguerre functions by simply allowing their differentiation orders to be real. However, Abbott [11] has proven that Laguerre functions are divergent as soon as their differentiation order is non-integer commenting on El-Sayed's work.

Recently, we have proposed an interpolation of Laguerre functions to fractional differentiation orders keeping them convergent when differentiation orders are non-integers [12]. To our knowledge, this fractional Laguerre-like basis is the first fractional orthogonal basis ever synthesized for control engineering purposes. It is however limited to the use of a unique mode.

The aim of this paper is to present a unified construction for all fractional orthogonal bases including Laguerre-like, Kautz-like and GOB-like bases, allowing the user to choose any number of real or complex conjugate modes. For that purpose, Gram-Schmidt orthogonalization procedure is applied on adequate functional series leading to extend the definition of Laguerre, Kautz and GOB bases to fractional differentiation orders.

1.2 Representation of fractional systems

Fractional mathematical models are based on fractional differential equations:

$$\begin{array}{l} y\left(t\right)+b_1\mathbf{D}^{\beta_1}y\left(t\right)+\cdots+b_{m_B}\mathbf{D}^{\beta_{m_B}}y\left(t\right)= \\ a_0\mathbf{D}^{\alpha_0}u\left(t\right)+a_1\mathbf{D}^{\alpha_1}u\left(t\right)+\cdots+a_{m_A}\mathbf{D}^{\alpha_{m_A}}u\left(t\right) \end{array}, \tag{1}$$

where differentiation orders $\beta_1,\ \ldots,\ \beta_{m_B},\ \alpha_0,\ \ldots,\ \alpha_{m_A}$ are allowed to be non-integer positive numbers. The concept of differentiation to an arbitrary order (non-integer),

$$\mathbf{D}^{\gamma}\triangleq\left(\frac{d}{dt}\right)^{\gamma}\qquad\forall\gamma\in\mathbb{R}_+^*,$$

was defined in the 19^{th} century. The main contribution to the establishment of the definition is due to Riemann and Liouville. They extend differentiation by using not only integer but also non-integer (real or complex) orders. The γ fractional order derivative of $x(t)$ is defined as being an integer derivative of order $m=\lfloor\gamma\rfloor+1$ ($\lfloor.\rfloor$ stands for the floor operator) of a non-integer integral of order $1-(m-\gamma)$ [13]:

$$\mathbf{D}^{\gamma}x\left(t\right)\triangleq\frac{1}{\Gamma\left(m-\gamma\right)}\left(\frac{d}{dt}\right)^{m}\int_0^t\frac{x\left(\tau\right)d\tau}{\left(t-\tau\right)^{1-(m-\gamma)}}, \tag{2}$$

where $t>0$, $\gamma>0$.

A more concise algebraic tool can be used to represent fractional systems: the Laplace transform. The Laplace transform of a γ order derivative $(\gamma\in\mathbb{R}_+)$ of a signal $x(t)$ relaxed at $t=0$ (i.e. all derivatives of $x(t)$ equal 0 when $t<0$) is given by [14]:

$$\mathscr{L}\left\{\mathbf{D}^{\gamma}x\left(t\right)\right\}=s^{\gamma}\mathscr{L}\left\{x\left(t\right)\right\}.$$

This property allows to write the fractional differential equation (1), provided all signals $u(t)$ and $y(t)$ are relaxed at $t=0$, in a transfer function form:

$$F\left(s\right)\triangleq\frac{\sum\limits_{i=0}^{m_A}a_is^{\alpha_i}}{1+\sum\limits_{j=1}^{m_B}b_js^{\beta_j}}, \tag{3}$$

where $(a_i,b_j)\in\mathbb{C}^2,\quad(\alpha_i,\beta_j)\in\mathbb{R}_+^2,\forall i=0,1,\ldots,m_A,\forall j=1,2,\ldots,m_B.$

Definition: A transfer function $F(s)$ is commensurate of order γ iff it can be written as $F(s) = S(s^\gamma)$, where $S \stackrel{\Delta}{=} \frac{T}{R}$ is a rational function with T and R two coprime polynomials. $\square$

All differentiation orders are multiples of the commensurate order, allowing to obtain a rational transfer function. In this paper, the commensurate order is left free to vary in $\mathbb{R}_+^*$. Taking as an example $F(s)$ defined in (3), assuming that $F(s)$ is commensurate of order γ, and using $F(s) = S(s^\gamma)$, one can write:

$$S(s) = \frac{T(s)}{R(s)} = \frac{\sum_{m=0}^{m_A} a_m s^{\frac{\alpha_m}{\gamma}}}{1 + \sum_{m=1}^{m_B} b_m s^{\frac{\beta_m}{\gamma}}}. \tag{4}$$

All powers of s in (4) are integers. A sufficient condition for $F(s)$ to be commensurate is that all differentiation orders belong to the set of rational numbers $\mathbb{Q}$. It covers a wide range of fractional transfer functions.

The transfer function (3) is said to be of explicit form because all fractional derivatives apply to s (as compared to implicit forms where fractional derivatives apply to $(s+\xi)^\gamma$, $\forall \xi \in \mathbb{C}^*$, $\forall \gamma \in \mathbb{R}_+^* \backslash \mathbb{N}$).

1.3 Stability condition

Matignon [15, theorem 2.21 p.150] has established the stability condition of any commensurate explicit fractional model. Here is a revisited version of his theorem.

Theorem: A commensurate (of order γ) transfer function $F(s) = S(s^\gamma) = \frac{T(s^\gamma)}{R(s^\gamma)}$ is BIBO stable iff

$$0 < \gamma < 2 \tag{5}$$

and for every $s \in \mathbb{C}$ such that $R(s) = 0$

$$|\arg(s)| > \gamma \frac{\pi}{2}. \tag{6}$$

1.4 Fractional transfer functions belonging to $H_2(\mathbb{C}^+)$

Contrary to the rational case, stability condition does not guarantee for a fractional transfer function to belong to $H_2(\mathbb{C}^+)$, Hardy space of functions $F(s)$ analytic on the open right half-plane $\mathbb{C}^+$ and such that $\|F\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\overline{F(j\omega)}d\omega < \infty$. As demonstrated in [16], a fractional transfer function (say (3)) belongs to $H_2(\mathbb{C}^+)$ iff stability conditions (5) and (6) are satisfied and the difference between denominator and numerator degrees is greater than one half:

$$\beta_{m_B} - \alpha_{m_A} > \frac{1}{2}. \tag{7}$$

Condition (7) is essential when choosing fractional generating functions for the orthogonal basis to be synthesized.

1.5 Function expansion on orthogonal bases

The classical Laguerre, Kautz, and GOB functions form complete orthonormal bases in $L_2[0,\infty[$, Lebesgue space of squared integrable functions $f^2 = \int\limits_0^\infty f\,(t)\,dt < \infty$, according to the usual definition of the scalar product [17]:

$$\langle l_n, l_m \rangle = \int\limits_0^\infty l_n\,(t)\, l_m\,(t)\,dt = \delta_{nm}, \tag{8}$$

which can also be computed in the frequency domain by using Plancherel's theorem:

$$\langle l_n, l_m \rangle = \langle L_n, L_m \rangle = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} L_n\,(\mathrm{j}\omega)\,\overline{L_m\,(\mathrm{j}\omega)}d\omega = \delta_{nm}. \tag{9}$$

Thus, any transfer function $F(s)$ belonging to $H_2(\mathbb{C}^+)$ can be written as a linear combination of orthogonal functions spanning $H_2(\mathbb{C}^+)$:

$$F(s) = \sum_{n=0}^{\infty} a_n L_n\,(s). \tag{10}$$

The Fourier coefficients a_n being convergent as n tends to infinity, (10) is usually truncated to an order N. Hence, $F(s)$ is approximated by the finite sum:

$$F(s) \approx F_N(s) \triangleq \sum_{n=0}^{N} a_n L_n\,(s). \tag{11}$$

The Fourier coefficients are computed by minimizing the least squares criterion:

$$J = \int\limits_0^\infty (f(t) - f_N(t))^2 dt, \tag{12}$$

which corresponds to the L_2 norm of the approximation error, according to definition (8) of the scalar product :

$$J = \|f - f_N\|_2^2\,. \tag{13}$$

Minimizing J, when orthogonal functions are involved, leads to the computation of Fourier coefficients by evaluating the scalar product either in the time or the frequency domain:

$$a_n = \langle f, l_n \rangle = \langle F, L_n \rangle \tag{14}$$

2 Orthonormal bases construction

2.1 Gram-Schmidt procedure

Given an arbitrary functional series $\{F_m\}_{m=1}^{M}$, where $F_m \in H^2(\mathbb{C}^+)\ \forall m$, orthonormal functions $\{G_m\}_{m=1}^{M}$ are obtained, according to the Gram-Schmidt orthogonalization procedure, as a linear combination of the generating functions F_m:

$$\mathbf{G} = \Delta \times \mathbf{F}, \tag{15}$$

where Δ is a real-valued MxM matrix,

$$\mathbf{G} = \left[\begin{array}{ccccc} G_1 & G_2 & \cdots & G_{M-1} & G_M \end{array} \right]^{\mathbf{T}},$$

and

$$\mathbf{F} = \left[\begin{array}{ccccc} F_1 & F_2 & \cdots & F_{M-1} & F_M \end{array} \right]^{T}.$$

The vector **G**, gathering the functional series $\{G_m\}_{m=1}^{M}$, satisfies thus:

$$\left\langle \mathbf{G}, \mathbf{G}^T \right\rangle = \mathbf{I}, \tag{16}$$

where $\left\langle \mathbf{G}, \mathbf{G}^T \right\rangle$ is a matrix of scalar products which element (i,j) is $\langle G_i, G_j \rangle$ and **I** denotes an M by M identity matrix. Using (15):

$$\left\langle \mathbf{G}, \mathbf{G}^T \right\rangle = \Delta \left\langle \mathbf{F}, \mathbf{F}^T \right\rangle \Delta^T = \mathbf{I}, \tag{17}$$

and the following quadratic form is obtained:

$$\Delta^T \Delta = \left\langle \mathbf{F}, \mathbf{F}^T \right\rangle^{-1},$$

Δ, a lower triangular matrix, is computed using the Cholesky decomposition:

$$\Delta = Cholesky \left(\left\langle \mathbf{F}, \mathbf{F}^T \right\rangle^{-1} \right). \tag{18}$$

Orthonormal functions are then deduced from (15):

$$\mathbf{G} = Cholesky \left(\left\langle \mathbf{F}, \mathbf{F}^T \right\rangle^{-1} \right) \times \mathbf{F} \tag{19}$$

The remaining difficulty is to compute the matrix of scalar products $\left\langle \mathbf{F}, \mathbf{F}^T \right\rangle$ for fractional transfer functions known to be multivalued complex functions as soon as non-integer differentiation is involved.

2.2 Scalar product of any pair of fractional explicit and commensurate transfer functions

Assume any pair $(G(s), H(s)) \in H_2^2(\mathbb{C}^+)$ of fractional explicit and commensurate transfer functions. The scalar product of G and H is expressed in the frequency domain as:

$$\langle G, H \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} G\left(\mathrm{j}\omega\right) \overline{H\left(\mathrm{j}\omega\right)} d\omega. \tag{20}$$

Define the following change of variable, where γ is the commensurate order:

$$x = \omega^{\gamma} \Rightarrow d\omega = \frac{1}{n} x^{\frac{1}{\gamma}-1} dx. \tag{21}$$

Define q and ρ respectively as integer and non-integer parts of $\frac{1}{\gamma}$. Then,

$$\langle G, H \rangle = \frac{1}{\pi n} \int_{0}^{\infty} x^{q+\rho-1} \frac{A\left(x\right)}{B\left(x\right)} dx. \tag{22}$$

Two cases are distinguished.

2.2.1 $\deg(B) \leq \deg(A) + \frac{1}{n}$

In this case,

$$\langle G, H \rangle = \infty, \tag{23}$$

because the difference between denominator and numerator degrees in (22) is greater than one, which yields after integration a positive power of x in the numerator and the integral diverges.

2.2.2 $\deg(B) > \deg(A) + \frac{1}{n}$

Depending on the nullity of ρ, the solution to integral (20) is different. Again, two cases are distinguished.

a – $0 < \rho < 1$. This condition means that $\frac{1}{\gamma}$ is non-integer. A partial fraction expansion is carried out on $x^q \frac{A(x)}{B(x)}$ and gives:

$$x^{\rho-1}\left[x^q \frac{A(x)}{B(x)}\right] = \sum_{k=1}^{r} \sum_{l=1}^{v_k} \frac{a_{k,l} x^{\rho-1}}{(x+s_k)^l}. \tag{24}$$

Replacing back in (22) gives:

$$\langle G, H \rangle = \frac{1}{n\pi} \sum_{k=1}^{r} \sum_{l=1}^{v_k} a_{k,l} s_k^{-l} \int_0^\infty \frac{x^{\rho-1} dx}{\left(1 + s_k^{-1} x\right)^l}. \tag{25}$$

Similar function was integrated in [18] and is also reported in [19, as formula 3.194,4 p. 285]. Plugging the result gives:

$$\langle G, H \rangle = \frac{\sum\limits_{k=1}^{r} \sum\limits_{l=1}^{v_k} (-1)^{l-1} a_{k,l} s_k^{\rho-l} \begin{pmatrix} \rho - 1 \\ l - 1 \end{pmatrix}}{n \sin(\rho\pi)}. \tag{26}$$

b – $\rho = 0$. This condition means that $\frac{1}{\gamma}$ is integer. The following expansion is carried out:

$$x^{q-1} \frac{A(x)}{B(x)} = \sum_{k=2}^{r} \frac{c_k}{(x+s_1)(x+s_k)} + \sum_{k=1}^{r} \sum_{l=2}^{v_k} \frac{b_{k,l}}{(x+s_k)^l}, \tag{27}$$

where s_1 is an arbitrary chosen pole. Plugging (27) in (22) and computing the integral yields the following result:

$$\langle G, H \rangle = \sum_{k=2}^{r} \frac{c_k \left(\ln(s_k) - \ln(s_1)\right)}{n\pi (s_k - s_1)} + \sum_{k=1}^{r} \sum_{l=2}^{v_k} \frac{b_{k,l} s_k^{1-l}}{n\pi (l-1)}, \tag{28}$$

completing the computations of the scalar product of any fractional explicit and commensurate transfer functions.

2.3 Choosing fractional generating functions

The method described above allows to orthogonalize any series of generating functions provided they are not collinear according to the scalar product (9). However, to extend the definition of classical bases (Laguerre, Kautz, and BOG) to fractional differentiation orders, it is necessary to choose adequate generating functions. Hence, each generating function will introduce either a real or a complex mode. If a generating function introduces a complex mode, then the next generating function must introduce its conjugate so as to obtain a real-valued impulse response. Special care must be taken for the first generating function which must satisfy stability condition and belong to $H^2(\mathbb{C}^+)$. Once these functions chosen, (19) is applied in order to obtain the orthonormal basis functions.

If the first mode is chosen to be real, the first generating function is set to:

$$F_{m_0}(s) = \frac{1}{(s^\gamma + \lambda_{m_0})^{m_0}}. \tag{29}$$

However, if a complex mode λ_{m_0} is chosen, then, its conjugate $\overline{\lambda_{m_0}}$ must be chosen afterwards. The first two generating functions are then set to:

$$\begin{aligned} F'_{m_0}(s) &= \frac{1}{(s^\gamma + \lambda_{m_0})^{m_0}} \\ F''_{m_0}(s) &= \frac{1}{\left(s^\gamma + \overline{\lambda_{m_0}}\right)^{m_0}} \end{aligned} . \tag{30}$$

For $F_{m_0}(s)$, $F'_{m_0}(s)$, and $F''_{m_0}(s)$ to be stable, according to §1.3, the following conditions must be satisfied:

$$|\arg(-\lambda_{m_0})| > \gamma\frac{\pi}{2} \qquad \text{and} \qquad \gamma \in]0, 2[. \tag{31}$$

Moreover, for $F_{m_0}(s)$, $F'_{m_0}(s)$, and $F''_{m_0}(s)$ to belong to $H^2(\mathbb{C}^+)$, according to §1.4, the difference between denominator and numerator degrees must be greater than $\frac{1}{2}$. Consequently, m_0 is the smallest integer satisfying the aforementioned condition:

$$m_0 = \left\lfloor \frac{1}{2\gamma} \right\rfloor + 1. \tag{32}$$

Generating functions $F'_{m_0}(s)$ and $F''_{m_0}(s)$, having complex impulse responses, are not adapted to represent real-valued impulse response systems. Therefore, they are replaced by two new generating functions $\tilde{F}'_{m_0}$ and $\tilde{F}''_{m_0}$ which are linear combinations of F'_{m_0} and F''_{m_0}:

$$\begin{bmatrix} \tilde{F}'_{m_0}(s) \\ \tilde{F}''_{m_0}(s) \end{bmatrix} = \begin{bmatrix} c_0 & c_1 \\ c'_0 & c'_1 \end{bmatrix} \begin{bmatrix} F'_{m_0}(s) \\ F''_{m_0}(s) \end{bmatrix}, \tag{33}$$

where c_0, c_1, c'_0 and c'_1 are non null complex numbers chosen so that $\tilde{F}'_{m_0}(s)$ and $\tilde{F}''_{m_0}(s)$

have real-valued impulse responses, which yields: $c_0 = \overline{c_1}$ and $c'_0 = \overline{c'_1}$. Consequently,

$$\begin{array}{c} \tilde{F}'_{m_0}(s) = \dfrac{\sum\limits_{k=0}^{m_0} \beta_k s^{\gamma k}}{\left(\left(s^\gamma + \overline{\lambda_{m_0}}\right)\left(s^\gamma + \lambda_{m_0}\right)\right)^{m_0}} \\ \tilde{F}''_{m_0}(s) = \dfrac{\sum\limits_{k=0}^{m_0} \beta'_k s^{\gamma k}}{\left(\left(s^\gamma + \overline{\lambda_{m_0}}\right)\left(s^\gamma + \lambda_{m_0}\right)\right)^{m_0}} \end{array}, \tag{34}$$

where

$$\begin{array}{l} \beta_k = 2\binom{m_0}{k}\left(\mathscr{Re}\left(c_0\right)\mathscr{Re}\left(\lambda_{m_0}^{m_0-k}\right) + \mathscr{Im}\left(c_0\right)\mathscr{Im}\left(\lambda_{m_0}^{m_0-k}\right)\right) \\ \beta'_k = 2\binom{m_0}{k}\left(\mathscr{Re}\left(c'_0\right)\mathscr{Re}\left(\lambda_{m_0}^{m_0-k}\right) + \mathscr{Im}\left(c'_0\right)\mathscr{Im}\left(\lambda_{m_0}^{m_0-k}\right)\right) \end{array}. \tag{35}$$

Then, the generating functions of index $m > m_0$ are defined recursively based on the definition of the preceding function, noted $\Phi_{m-1}(s)$, where:

$$\Phi_{m-1} = \begin{cases} F_{m-1}, & \text{if } \lambda_{m-1} \text{ is real} \\ \tilde{F}'_{m-1} \text{ or } \tilde{F}''_{m-1}, & \text{if } \lambda_{m-1} \text{ is complex} \end{cases}. \tag{36}$$

Hence, if a real mode λ_m is introduced , the generating function is set to:

$$F_m(s) = \frac{1}{s^\gamma + \lambda_m}\Phi_{m-1}(s). \tag{37}$$

Otherwise, if two complex conjugate modes λ_m and $\overline{\lambda_m}$ are chosen, the two generating functions having real-valued impulse responses are:

$$\begin{bmatrix} \tilde{F}'_m(s) \\ \tilde{F}''_m(s) \end{bmatrix} = \begin{bmatrix} c_0 & \overline{c_0} \\ c'_0 & \overline{c'_0} \end{bmatrix} \begin{bmatrix} \frac{1}{s^\gamma + \lambda_m} \\ \frac{1}{s^\gamma + \overline{\lambda_m}} \end{bmatrix} \Phi_{m-1}(s), \tag{38}$$

yielding:

$$\begin{array}{c} \tilde{F}'_m(s) = \dfrac{\beta_1 s^\gamma + \beta_0}{\left(s^\gamma + \lambda_m\right)\left(s^\gamma + \overline{\lambda_m}\right)}\Phi_{m-1}(s) \\ \tilde{F}''_m(s) = \dfrac{\beta'_1 s^\gamma + \beta'_0}{\left(s^\gamma + \lambda_m\right)\left(s^\gamma + \overline{\lambda_m}\right)}\Phi_{m-1}(s) \end{array}, \tag{39}$$

where

$$\begin{array}{lcl} \beta_1 = 2\mathscr{Re}\left(c_0\right) & \text{and} & \beta_0 = \mathscr{Re}\left(\mathrm{c}_0\right)\mathscr{Re}\left(\lambda_\mathrm{m}\right) + \mathscr{I}\mathrm{m}\left(\mathrm{c}_0\right)\mathscr{I}\mathrm{m}\left(\lambda_\mathrm{m}\right) \\ \beta'_1 = 2\mathscr{Re}\left(c'_0\right) & \text{and} & \beta'_0 = \mathscr{Re}\left(\mathrm{c}'_0\right)\mathscr{Re}\left(\lambda_\mathrm{m}\right) + \mathscr{I}\mathrm{m}\left(\mathrm{c}'_0\right)\mathscr{I}\mathrm{m}\left(\lambda_\mathrm{m}\right) \end{array}, \tag{40}$$

and the complex parameters c_0 and c'_0 are chosen so that $c_0 \neq d c'_0, \forall d \in \mathbb{R}_+^*$.

Taking into account (31) and (32), $F_m(s)$, $F'_m(s)$ and $F''_m(s)$ belong to $H^2(\mathbb{C}^+)$ *iff*:

$$|\arg(-\lambda_m)| > \gamma\frac{\pi}{2} \tag{41}$$

It is interesting to point out that, in the special case where all the modes λ_m are chosen to be alike, Laguerre-like generating functions are obtained as illustrated in [12]:

$$F_m(s) = \frac{1}{(s^\gamma + \lambda)^m}, \qquad m \geq m_0 = \left\lfloor \frac{1}{2\gamma} \right\rfloor + 1. \tag{42}$$

In the special case where all pairs of complex conjugate modes are chosen to be alike, the fractional Kautz-like generating functions are obtained as illustrated in [20].

3 Completeness

To date, completeness of fractional Laguerre-like basis is proven. Completeness of fractional Kautz-like basis is announced in a conjecture and completeness of fractional GOB-like basis is yet to be proven.

3.1 Completeness of Fractional Laguerre-like basis

Theorem: Define $F_m(s)$ as in (42) with $\lambda \in \mathbb{R}^+$. Then, the linear space spanned by the series $\{F_m\}_{m=m_0,m_{0+1},\dots,\infty}$, where $m_0 = \lfloor \frac{1}{2\gamma} \rfloor + 1$ and $m \in \mathbb{N}^+$, is dense in $H^2(\mathbb{C}^+)$. $\square$

Proof: Let

$$w(s) = \frac{1}{s^\gamma + \lambda}$$

w is a bijective conformal mapping from $\mathbb{C}^+$ to a finite modulus open domain Ω bounded by a Jordan curve Γ. Γ is a finite union of circle arcs.

Let $G(s) = (s^\gamma + \lambda)^{m_0} F(s)$ where $m_0 = \lfloor \frac{1}{2\gamma} \rfloor + 1$. Let $E(\Omega)$ be the space of all functions analytical on Ω and continuous on $\overline{\Omega} = \Omega \cup \Gamma$. Then, $G \circ w^{-1} \in E(\Omega)$. Moreover, applying $Runge - Walsh$ theorem [17, p.7 theorem 1.3.4], $G \circ w^{-1}$ can always be approximated by a polynomial $P(z)$:
$\forall z \in \overline{\Omega}, \forall \varepsilon > 0, \exists\, P(z)$, such that $\|G \circ \omega^{-1} - P(z)\| \leq \varepsilon$.
Setting $P(z) = \sum\limits_{n=0}^{N} a_n (z)^n$, $N \in \mathbb{N}$, and substituting z by $w(s)$ yields:

$$\forall s \in \overline{\mathbb{C}^+} : \quad |G(s) - \sum_{n=0}^{N} a_n (w(s))^n| \leq \epsilon$$

$$|(s^\gamma + \lambda)^{m_0} F(s) - \sum_{n=0}^{N} a_n (w(s))^n| \leq \epsilon$$

Since $\lambda > 0$ and $\Re(s) \geq 0$ (convergence domain of Laplace transform), then $(s^\gamma + \lambda)^{m_0} \neq 0$ and:

$$|F(s) - (s^\gamma + \lambda)^{-m_0} \sum_{n=0}^{N} a_n (w(s))^n| \leq \epsilon |(s^\gamma + \lambda)^{-m_0}|$$

$$|F(s) - \sum_{n=0}^{N} a_n (w(s))^{n+m_0}| \leq \epsilon |w(s)^{m_0}|$$

Moreover, $w^m \in H^2(\mathbb{C}^+)$ $(m \in \mathbb{N})$ iff $m \geq m_0$. Thus, $\forall n \in \mathbb{N}, w^{n+m_0} \in H^2(\mathbb{C}^+)$. Furthermore, since $F \in H^2(C^+)$,

$$\int_{-j\infty}^{j\infty} |F(s) - \sum_{n=0}^{N} a_n (w(s))^{n+m_0}|^2 ds \leq$$
$$\epsilon^2 \int_{-j\infty}^{j\infty} |w(s)|^{2m_0} ds$$

Consequently, $\forall F(s) \in H^2(\mathbb{C}^+), \forall \varepsilon > 0, \exists\, P(w(s)) = \sum_{n=0}^{N} a_n (w(s))^{n+m_0}$ such that

$$||F(s) - \sum_{n=0}^{N} a_n (w(s))^{n+m_0}||_2 \leq \varepsilon$$

Thus the series $\{w^m\}_{m \geq m_0}$ is dense in $H^2(\mathbb{C}^+)$ which completes the proof. $\square$

Consequently, the orthogonal functions $\{G_m\}_{m=m_0,m_{0+1},\ldots,\infty}$, linear combinations of $\{F_m\}_{m=m_0,m_{0+1},\ldots,\infty}$, are dense in $H^2(\mathbb{C}^+)$ too. Therefore,

$$\forall H(s) \in H^2(\mathbb{C}^+): \quad H(s) = \sum_{m=0}^{\infty} a_m G_m(s). \tag{43}$$

Moreover, all results announced in §1.5 are valid for the Laguerre-like fractional orthogonal basis which can hence be used to model any finite energy system.

3.2 Completeness of Fractional Kautz-like basis

Conjecture: Define $\tilde{F}'_{m_0}(s)$ and $\tilde{F}''_{m_0}(s)$, for $m_0 = \lfloor \frac{1}{2\gamma} \rfloor + 1$, $\forall \gamma \in]0, 2[$, as in (34) and $\tilde{F}'_m(s)$ and $\tilde{F}''_m(s)$ for $m > m_0$ as in and (39) with $\lambda_{m_0} = \lambda_{m_0+1} = \lambda_{m_0+2} = \ldots = \lambda$ and $|\arg(-\lambda)| > \gamma \frac{\pi}{2}$. Then, the linear space spanned by the series $\{\tilde{F}'_m, \tilde{F}''_m\}_{m=m_0,m_{0+1}\ldots,\infty}$ is dense in $H^2(\mathbb{C}^+)$. $\square$

The proof of this conjecture would permit to model any finite energy system using fractional Kautz-like basis.

3.3 Completeness of Fractional GOB-like basis

To date, no result can be announced concerning the completeness of fractional GOB-like basis. However, it is interesting to point out that the classical GOB, which can also be obtained by orthogonalizing F_m,$\tilde{F}'_m$, and $\tilde{F}''_m$ when $\gamma = 1$ and $m_0 = 1$, is dense in $H^2(\mathbb{C}^+)$ iff [21]:

$$\sum_{m=1}^{\infty} \frac{\mathscr{Re}\,(\lambda_m)}{1 + |\lambda_m|^2} = \infty \quad \text{for} \quad \gamma = 1. \tag{44}$$

An extended condition needs undoubtedly to be found for the completeness of the fractional GOB-like basis for any $\gamma \in]0, 2[$.

4 System identification using the synthesized orthogonal bases

The fractional orthogonal bases are now used in a system identification context based on fixed denominator models. The procedure is described below.

- First, differentiation order γ and all the modes are fixed either using an a priori knowledge on system's behavior or a rough estimation of a fractional ARX model [22]. The knowledge of these parameters allows then to fix all the generating functions F_m based on (29), (34), (37) and (39).
- Next, orthogonalization procedure described in §2.1 is applied on the aforementioned generating functions.
- Finally, Fourier coefficients of the orthogonal basis are computed, using a least squares method.

The identified model $H(s)$ is expressed as the sum of Fourier coefficients multiplied by the orthonormal functions:

$$H(s) = \sum_{m=m_0}^{M} g_m G_m(s) = \mathbf{g}^T \mathbf{G}(s), \tag{45}$$

where

$$\mathbf{g} = [g_{m_0}, g_{m_0+1}, \ldots, g_M]^T,$$

and

$$\mathbf{G}(s) = [G_{m_0}(s), G_{m_0+1}(s), \ldots, G_M(s)]^T.$$

The truncation order M is fixed so as to obtain a satisfactory approximation and can be increased if the identified model is not satisfactory.

Assume $u(t), y(t), t \in [0, T]$ input and output data issued from linear finite-energy system. Then the identification procedure consists of computing optimal coefficient vector **g** by minimizing the least squares error:

$$J = \frac{1}{T} \int_0^T (\varepsilon(t))^2 \, dt, \tag{46}$$

where

$$\varepsilon(t) = \sum_{m=m_0}^{M} g_m u_{G_m}(t) - y(t) \tag{47}$$

The filtered outputs $u_{G_{m(t)}}$ and $\mathbf{u_G}(t)$ are defined respectively as:

$$u_{G_m}(t) = G_m(t) \otimes u(t) \quad (\otimes \text{ being the convolution product})$$

$$\mathbf{u_G}(t) = \left[u_{G_{m_0}}(t), u_{G_{m_0+1}}(t), \cdots, u_{G_M}(t)\right].$$

The optimum values of Fourier coefficients are given by the least squares formula:

$$\hat{\mathbf{g}} = \left[\int_0^T \left(u_G(t)^T u_G(t)\right) dt\right]^{-1} \int_0^T u_G(t)^T y\,(t)\, dt, \tag{48}$$

or after a numerical discretization, by defining $\mathbf{Y}$ as a column vector of system's outputs and $\mathbf{X}$ as a regression matrix which columns are filtered outputs, (48) can be approximated by:

$$\hat{\mathbf{g}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y}. \tag{49}$$

5 Examples

5.1 Example 1 – Orthogonalization of a set of functions

The objective is to synthesize an orthogonal basis with the following parameters: the commensurate order γ is set to 1.5 and the eigenvalues to 1, 2, and $2 \pm \mathrm{i}$. The first two generating functions are obtained using respectively (29) and (37):

$$F_1(s) = \frac{1}{s^{1.5}+1} \tag{50}$$

$$F_2(s) = \frac{1}{s^{1.5}+2} \times \frac{1}{s^{1.5}+1}. \tag{51}$$

The next two generating functions are obtained using (39) with $c_0 = 1$ and $c_0' = \mathrm{i}$:

$$F_3(s) = \frac{2s^{1.5}+2}{s^3+4s^{1.5}+5} \times \frac{1}{s^{1.5}+2} \times \frac{1}{s^{1.5}+1} \tag{52}$$

$$F_4(s) = \frac{-1}{s^3+4s^{1.5}+5} \times \frac{1}{s^{1.5}+2} \times \frac{1}{s^{1.5}+1} \tag{53}$$

The functions of the orthonormal basis are computed by applying formula (19):

$$G_1(s) = \frac{1.14}{s^{1.5}+1} \tag{54}$$

$$G_2(s) = \frac{1.53s^{1.5}-0.11}{s^3+3.00s^{1.5}+2.00} \tag{55}$$

$$G_3(s) = \frac{0.38s^{4.5}-0.60s^3+1.58s^{1.5}-2.45}{s^6+7.00s^{4.5}+19.00s^3+23.00s^{1.5}+10.00} \tag{56}$$

$$G_4(s) = \frac{1.52s^{4.5}+3.00s^3+3.13s^{1.5}+2.15}{s^6+7.00s^{4.5}+19.00s^3+23.00s^{1.5}+10.00}. \tag{57}$$

Their impulse responses are plotted in figure (1).

5.2 Example 2 – Application in system identification context

To illustrate the use of Fractional GOB-like functions in system identification, the proposed procedure will be applied to identify a real industrial system, the nature of which cannot be divulged due to confidentiality reasons. System's input and output signals are plotted on figure (2). In addition to these signals, frequency domain analysis showed that differentiation orders can be set to multiples of 0.6, and that 3 modes can be fixed at 0.6 and $4.1e^{\pm j0.112\pi}$. Consequently, one Laguerre-like and two Kautz-like generating functions are fixed. Applying orthogonalization procedure yields the following orthonormal functions:

$$G_1(s) = \frac{0.78}{s^{0.6}+0.60} \tag{58}$$

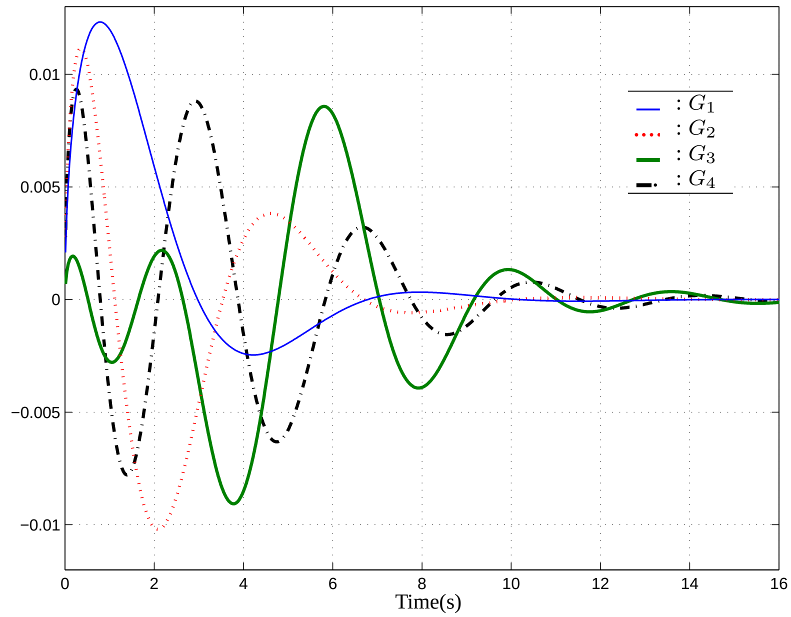

Figure 1: Impulse responses of the orthogonal functions (54), (55), (56), and (57)

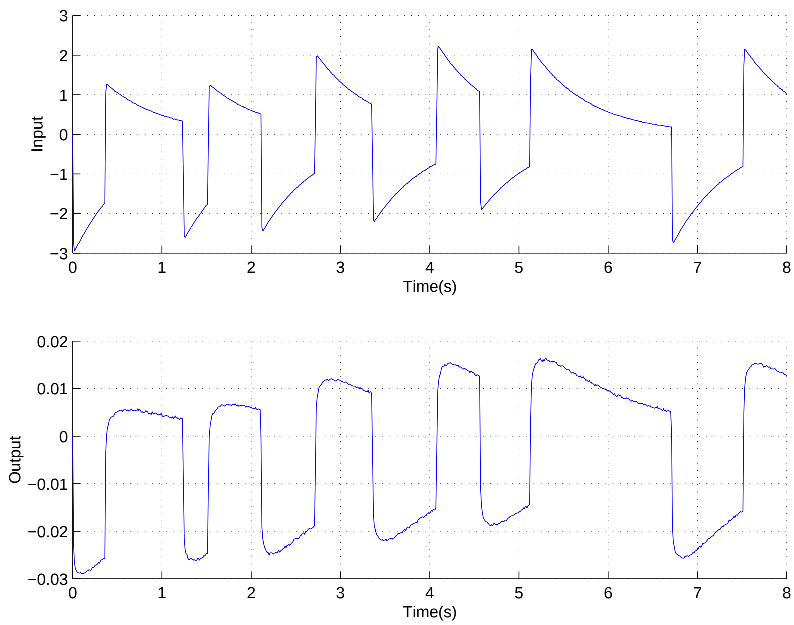

Figure 2: Input and output identification data

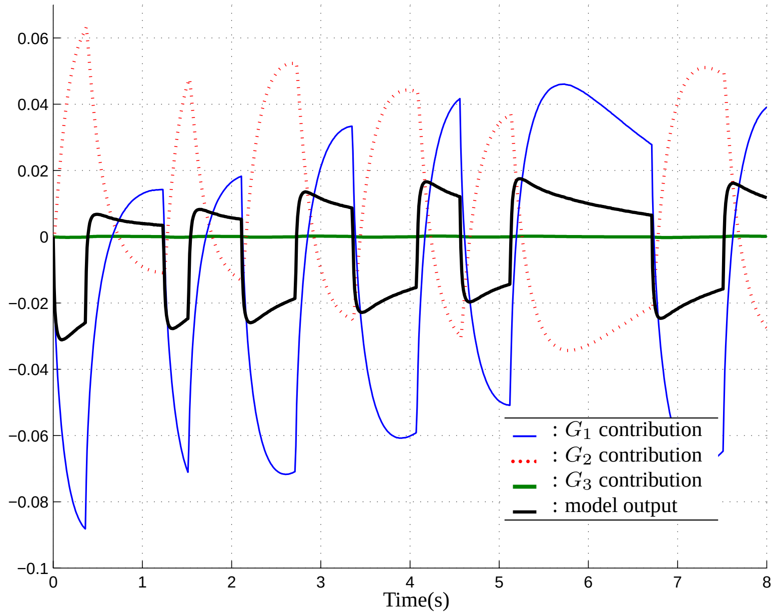

Figure 3: Output of the optimal model and contribution of each function of the basis.

$$G_2(s) = \frac{0.70s^{1.2} - 8.83s^{0.6} - 16.09}{s^{1.8} + 8.40s^{1.2} + 21.49s^{0.6} + 10.09} \tag{59}$$

$$G_3(s) = \frac{0.22s^{1.2} - 24.82s^{0.6} + 24.49}{s^{1.8} + 8.40s^{1.2} + 21.49s^{0.6} + 10.09} \tag{60}$$

Then, Fourier coefficients are computed by minimizing least squares criterion (46), which gives the following identified model:

$$\hat{H}(s) = 0.1138G_1(s) + 0.0647G_2(s) - 0.0002G_3(s) \tag{61}$$

Contribution of each vector of the basis in model's output is plotted in figure (3). Due to the weak contribution of G_3, it is omitted and the number of the orthogonal functions is reduced to 2 in the final model. As shown on validation data of figure (4), the identified model gives satisfactory results.

6 Conclusion

A unified procedure is presented in this paper for the synthesis of fractional orthogonal bases. The rational Laguerre, Kautz and GOB functions are interpolated to fractional differentiation orders. Fractional Laguerre basis is proven to be dense in $H_2(C^+)$. The obtained Fractional GOB-like functions was successfully used in the context of system identification with fixed denominator models.

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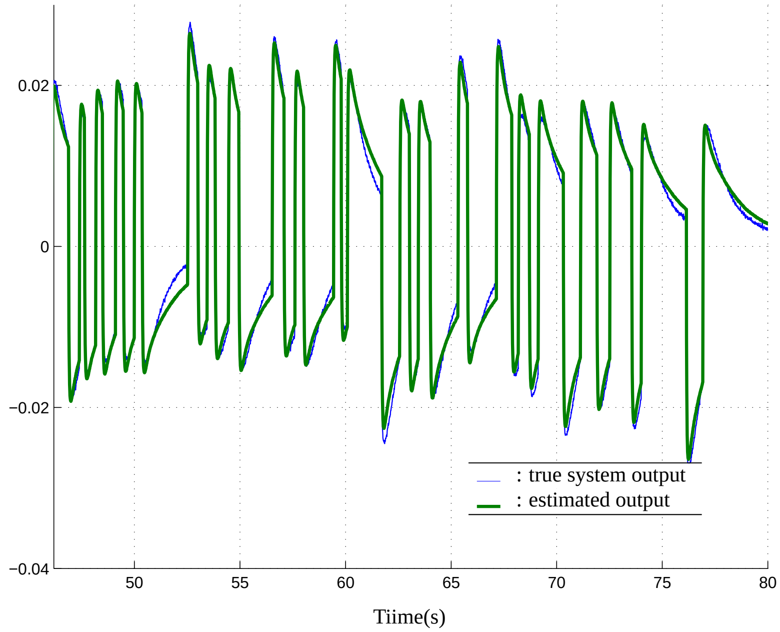

Figure 4: System's and models's outputs on validation data

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