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Viable harvest of monotone bioeconomic models

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Abstract Some monospecies age class models, as well as specific multi-species models (with so-called technical interactions), exhibit useful monotonicity properties. This paper deals with discrete time monotone bioeconomic dynamics in the presence of state and control constraints. In practice, these latter "acceptable configurations" represent production and preservation requirements to be satisfied for all time, and they also possess monotonicity properties. A state x is said to belong to the viability kernel if there exists a trajectory, of states and controls, starting from x and satisfying the constraints. Under monotonicity assumptions, we present upper and lower estimates of the viability kernel. This helps delineating domains where a viable management is possible. Numerical examples, in the context of fisheries management, for the Chilean sea bass (*Dissostichus eleginoides*) and Alfonsino (*Beryx splendens*) are given.

Keywords control · viability · monotonicity · reference points · multi-criteria · sustainable management · fisheries

1 Introduction

This paper deals with the control of discrete–time dynamical systems of the form $x(t+1) = G\big(x(t), u(t)\big)$, $t \in \mathbb{N}$, with state $x(t) \in \mathbb{X}$ and control $u(t) \in \mathbb{U}$, in the presence of state and control constraints $\big(x(t), u(t)\big) \in \mathbb{D}$. The subset $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ describes "acceptable configurations of the system". Such problems of dynamic control

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under constraints refers to viability Aubin (1991) or invariance Clarke et al (1995) framework. From the mathematical viewpoint, most of viability and weak invariance results are addressed in the continuous time case. However, some mathematical works deal with the discrete-time case. This includes the study of numerical schemes for the approximation of the viability problems of the continuous dynamics as in Aubin (1991); Saint-Pierre (1994); Quincampoix and Saint-Pierre (1995). In the control theory literature, problems of constrained control have also been addressed in the discrete time case (see the survey paper Blanchini (1999)); reachability of target sets or tubes for nonlinear discrete time dynamics is examined in Bertsekas and Rhodes (1971).

We consider sustainable management issues which can be formulated within such a framework as in Béné and Doyen (2000, 2003); Béné et al (2001); Eisenack et al (2006); Martinet and Doyen (2007); Mullon et al (2004); Rapaport et al (2006); De Lara et al (2007); De Lara and Doyen (2008).

The time index t is an integer and the time period $[t, t+1[$ may be a year, a month, etc. The dynamics is generally a population dynamics, with state vector $x(t)$ being either the biomass of a single species, or a couple of biomasses for a predator–prey system, or a vector of abundances at ages for one or for several species, or abundances at different spatial patches, etc. The control $u(t)$ may represent catches, fishing mortality or fishing effort. The "acceptable set" $\mathbb{D}$ such that $\big(x(t), u(t)\big) \in \mathbb{D}$ may include biological, ecological and economic objectives as in Béné et al (2001). For instance, if the state x is a vector of abundances at ages and the control u is a fishing effort, $\mathbb{D} = \{(x,u) \mid B(x) \geq b^\flat \ , \ E(x,u) \geq e^\flat\}$ represents acceptable configurations where conservation is ensured by an biological indicator $B(x) \geq b^\flat$ (spawning stock biomass above a reference point, for instance) and economics is taken into account via minimal catches $E(x,u) \geq e^\flat$ (catches $E(x,u)$ above a threshold).

The viability kernel $\mathbb{V}(G, \mathbb{D})$ associated with the dynamics G and the acceptable set[1] $\mathbb{D}$ is known to play a basic role for the analysis of such problems and the design of viable control feedbacks. Unfortunately, its computation is not an easy task in general. However, following an approach initiated in De Lara et al (2006), the viability kernel may be estimated from below or form above under specific monotonicity assumptions which are pertinent for a class of bioeconomic models.

In Sect. 2, we recall the viability issues in discrete time, and we introduce monotone bioeconomic models. Sect. 3 provides our main theoretical results on estimates of the viability kernel. An application to fishery management is provided in Sect. 4 with numerical estimates for the Chilean sea bass (*Dissostichus eleginoides*), harvested in the south of Chile, and Alfonsino (*Beryx splendens*), harvested in the Juan Fernández archipelago.

[1] In Aubin (1991), the viability kernel $\mathbb{V}_K(G)$ is defined with respect to the dynamics G and to a subset $K \subset \mathbb{X}$ of the state space $\mathbb{X}$, and the constraints on the controls are contained in the definition in G. We prefer to put together the set of state constraints with the set of admissible controls, although these sets play very different roles. Indeed, in practice, constraints are expressed *via* indicators which are functions of both variables (state and control), especially for production constraints which depend on the catches. Thus, the set $\mathbb{D}$ makes the conflicting requirements, between preservation and production, more visible than with the Aubin's formalism.

2 Viability issues and monotone bioeconomic models

In this introductory section, we recall the viability issues in discrete time, then introduce monotone bioeconomic models.

2.1 Viability in discrete time

Let us consider a nonlinear control system described in discrete time by the difference equation

$$\begin{cases} x(t+1) = G(x(t), u(t)), & t = t_0, t_0+1, \dots \\ x(t_0) \quad \text{given}, \end{cases}$$

where the *state variable* $x(t)$ belongs to the finite dimensional state space $\mathbb{X} \subset \mathbb{R}^{n_{\mathbb{X}}}$, the *control variable* $u(t)$ is an element of the *control set* $\mathbb{U} \subset \mathbb{R}^{n_{\mathbb{U}}}$ while the *dynamics* G maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$. In our context, $x(t)$ will typically represent the vector of abundances per age class of a population, while $u(t)$ will be a harvest (induced mortality, harvesting effort, etc.).

A decision maker describes *acceptable configurations of the system* through a set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the *acceptable set*

$$(x(t), u(t)) \in \mathbb{D}\,, \quad \forall t = t_0, t_0+1, \dots$$

where $\mathbb{D}$ includes both system states and controls constraints. Typical instances of such an acceptable set are given by inequalities requirements

$$\mathbb{D} = \{(x,u) \in \mathbb{X} \times \mathbb{U} \mid \forall i = 1, \dots, p\,, \quad \mathcal{L}_i(x,u) \geq l_i\}, \tag{1}$$

where the functions $\mathcal{L}_1, \dots, \mathcal{L}_p$ may be interpreted as *indicators*, and the real numbers $l_1, \dots, l_p$ as the corresponding *thresholds*. For management issues, the set $\mathbb{D}$ will be the mathematical expression of preservation and/or production objectives.

Viability is defined as the ability to choose, at each time step $t = t_0, t_0+1, \dots$, a control $u(t) \in \mathbb{U}$ such that the system configuration remains acceptable. More precisely, the system is viable if the following feasible set is not empty:

$$\mathbb{V}(G, \mathbb{D}) := \left\{ x \in \mathbb{X} \;\middle|\; \begin{array}{l} \exists\ (u(t_0), u(t_0+1), \dots) \ \text{ and } \ (x(t_0), x(t_0+1), \dots) \\ \text{ satisfying } \ x(t_0) = x\,, \quad x(t+1) = G(x(t), u(t)) \\ \text{and } (x(t), u(t)) \in \mathbb{D}\,, \quad \forall t = t_0, t_0+1, \dots \end{array} \right\}.$$

For a decision maker, knowing the viability kernel has practical interest since it describes the set of states from which controls can be found that maintain the system in an acceptable configuration forever. However, computing this kernel is not an easy task in general.

We shall focus on estimates of viability kernels when the dynamics G and the acceptable sets have specificic monotony properties. For this purpose, we shall introduce a generic form for dynamics and acceptable sets corresponding to what we shall call *monotone population models harvesting issues*.

2.2 Monotone bioeconomic models

In what follows, the state space $\mathbb{X}$ and the control space $\mathbb{U}$ are subsets $\mathbb{X} \subset \mathbb{R}^{n_\mathbb{X}}$ and $\mathbb{U} \subset \mathbb{R}^{n_\mathbb{U}}$ supplied with the componentwise order: $x' \geq x$ if and only if each component of $x' = (x'_1, \ldots, x'_{n_\mathbb{X}})$ is greater than or equal to the corresponding component of $x = (x_1, \ldots, x_{n_\mathbb{X}})$: $x' \geq x \iff x'_i \geq x_i,\ i = 1, \ldots, n_\mathbb{X}$. A mapping $f : \mathbb{R}^a \to \mathbb{R}^b$ is said to be increasing if $x \geq x' \Rightarrow f(x) \geq f(x')$. Similar definition holds for decreasing.

Dynamics

Monospecies dynamic population models generally have the following qualitative properties: i) the higher the state abundance vector, the higher at next period; ii) the higher the harvest, the lower the state abundance vector at next period. Some specific multispecies models, without ecological but with so called technical interactions, share such properties. This motivates the following definitions.

We say that the dynamics $G : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is *increasing with respect to the state* if it satisfies $\forall\ (x, x', u) \in \mathbb{X} \times \mathbb{X} \times \mathbb{U}$, $x' \geq x \Rightarrow G(x', u) \geq G(x, u)$, and is *decreasing with respect to the control* if $\forall (x, u, u') \in \mathbb{X} \times \mathbb{U} \times \mathbb{U}$, $u' \geq u \Rightarrow G(x, u') \leq G(x, u)$.

We shall coin $G : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ a *monotone bioeconomic dynamics* if G is increasing with respect to the state and decreasing with respect to the control.

Bounded control set

Assuming they exist, we denote by $u^\flat$, $u^\sharp \in \mathbb{U}$ the lower and upper bounds of the set $\mathbb{U}$, *i. e.* $u^\flat \leq u \leq u^\sharp$ for all $u \in \mathbb{U}$.

Upper and lower dynamics without control

Let the dynamics G be a monotone bioeconomic dynamics. Define the *upper*[2] *dynamics without control* $G^\flat : \mathbb{X} \to \mathbb{X}$ by $G^\flat(x) = G(x, u^\flat)$. Its t iterate ($t = t_0, t_0 + 1, \ldots$) will be denoted by $(G^\flat)^{(t)}$. In the same way, the *lower dynamics without control* is defined by $G^\sharp(x) = G(x, u^\sharp)$. With these notations, we have that

$$G^\sharp(x) \leq G(x, u) \leq G^\flat(x)\,, \quad \forall\ (x, u) \in \mathbb{X} \times \mathbb{U}\ .$$

Acceptable set

We say that a set $S \subset \mathbb{X}$ is an *upper set* (or is an *increasing set*) if it satisfies the following property: $\forall x \in S$, $\forall x' \in \mathbb{X}$, $x' \geq x \Rightarrow x' \in S$. In the same way, a set $K \subset \mathbb{X} \times \mathbb{U}$ is said to be an *upper set* if $\forall (x, u) \in K$, $\forall x' \in \mathbb{X}$, $x' \geq x \Rightarrow (x', u) \in K$.

An acceptable set $\mathbb{D}$ is said to be *a production acceptable set* if $\mathbb{D}$ is *increasing with respect both to the state and to the control*, that is $\forall\ (x, x', u, u') \in \mathbb{X} \times \mathbb{X} \times \mathbb{U} \times \mathbb{U}$, $x' \geq x,\ u' \geq u, \quad (x, u) \in \mathbb{D} \Rightarrow (x', u') \in \mathbb{D}$. Particular instances are given by acceptable sets of the form (1) where the indicators $\mathcal{L}_1, \ldots, \mathcal{L}_p$ are increasing with respect to both variables (state and control). For instance, requiring a minimum yield may be captured by the acceptable set $\mathbb{D}_{\mathsf{yield}} = \{(x, u) \mid Y(x, u) \geq y^\flat\}$ where $y^\flat \in \mathbb{R}$ is a minimum yield

[2] Because $G \leq G^\flat$.

threshold and where the yield function $Y : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is increasing with respect to both variables (state and control).

An acceptable set $\mathbb{D}$ is said to be *a preservation acceptable set* if $\mathbb{D}$ is *increasing with respect to the state and decreasing with respect to the control*, that is $\forall\ (x, x', u, u') \in \mathbb{X}\times\mathbb{X}\times\mathbb{U}\times\mathbb{U}$, $x' \geq x,\ u' \leq u,\ \ (x,u) \in \mathbb{D} \Rightarrow (x', u') \in \mathbb{D}$. Particular instances are given by acceptable sets of the form (1) where the indicators $\mathcal{L}_1, \ldots, \mathcal{L}_p$ are increasing with respect to the state but decreasing with respect to the control. For instance, the ICES[3] precautionary approach may be stated in the viability framework with the following preservation acceptable set $\mathbb{D}_{\mathsf{protect}} = \{(x,u) \in \mathbb{X} \times \mathbb{U} \mid SSB(x) \geq B_{lim},\ F(u) \leq F_{\mathsf{lim}}\}$ as in De Lara et al (2007). Here, $SSB(x)$ is the spawning stock biomass, increasing with respect to the state, while the fishing mortality $F(u)$ is increasing[4] with respect to the control.

Notice that both production and preservation acceptable sets are upper sets.

For any acceptable set $\mathbb{D}$, introduce the *state constraints set*

$$\mathbb{V}_0 := \mathrm{Proj}_{\mathbb{X}}(\mathbb{D}) = \{x \in \mathbb{X} \mid \exists u \in \mathbb{U}\,,\ (x,u) \in \mathbb{D}\}\ , \tag{2}$$

obtained by projecting the acceptable set $\mathbb{D}$ onto the state space $\mathbb{X}$. Introduce also

$$\begin{cases} \mathbb{V}_0^\sharp := \{x \in \mathbb{X} \mid (x, u^\sharp) \in \mathbb{D}\} \subset \mathbb{V}_0\ , \quad \mathbb{D}^\sharp := \mathbb{V}_0^\sharp \times \{u^\sharp\} \\ \mathbb{V}_0^\flat := \{x \in \mathbb{X} \mid (x, u^\flat) \in \mathbb{D}\} \subset \mathbb{V}_0\ , \quad \mathbb{D}^\flat := \mathbb{V}_0^\flat \times \{u^\flat\}\ . \end{cases} \tag{3}$$

Notice that if $\mathbb{D}$ is a production acceptable set, we have $\mathbb{V}_0 = \mathbb{V}_0^\sharp$, and if $\mathbb{D}$ is a preservation acceptable set, we have $\mathbb{V}_0 = \mathbb{V}_0^\flat$.

3 Viability kernel estimates for monotone bioeconomic models

In this part, we shall provide lower and upper estimates of the viability kernel $\mathbb{V}(G, \mathbb{D})$ thanks to the following sets $\mathbb{V}(G^\flat, \mathbb{D}^\flat)$, $\mathbb{V}(G^\flat, \mathbb{D}^\sharp)$, $\mathbb{V}(G^\sharp, \mathbb{D}^\flat)$ and $\mathbb{V}(G^\sharp, \mathbb{D}^\sharp)$. These latter sets are easier to compute than the viability kernel $\mathbb{V}(G, \mathbb{D})$ because the dynamics $G^\flat$ and $G^\sharp$ have no control. Indeed, by (2), one obtains that, for any acceptable set $\mathbb{D}$,

$$\begin{aligned} \mathbb{V}(G^\flat, \mathbb{D}) &= \bigcap_{t=0}^{+\infty} \{x \in \mathbb{X} \mid (G^\flat)^{(t)}(x) \in \mathbb{V}_0\} \\ \mathbb{V}(G^\sharp, \mathbb{D}) &= \bigcap_{t=0}^{+\infty} \{x \in \mathbb{X} \mid (G^\sharp)^{(t)}(x) \in \mathbb{V}_0\}\ . \end{aligned} \tag{4}$$

Proposition 1 *Suppose that G is a monotone bioeconomic dynamics and that the control set $\mathbb{U}$ has lower and upper bounds $u^\flat$, $u^\sharp \in \mathbb{U}$.*

1. *If $\mathbb{D}$ is a production acceptable set, then*

$$\mathbb{V}(G^\sharp, \mathbb{D}^\flat) \subseteq \mathbb{V}(G^\flat, \mathbb{D}^\flat) \cup \mathbb{V}(G^\sharp, \mathbb{D}^\sharp) \subseteq \mathbb{V}(G, \mathbb{D}) \subseteq \mathbb{V}(G^\flat, \mathbb{D}^\sharp)\ . \tag{5}$$

[3] International Council for the Exploration of the Sea.

[4] Hence $-F(u)$ is decreasing with respect to the control. To be consistent with the notation in (1), it suffice to rewrite $\mathbb{D}_{\mathsf{protect}} = \{(x,u) \in \mathbb{X} \times \mathbb{U} \mid SSB(x) \geq B_{lim},\ -F(u) \geq -F_{\mathsf{lim}}\}$.

2. If $\mathbb{D}$ is a preservation acceptable set, then

$$\mathbb{V}(G^\sharp, \mathbb{D}^\sharp) \subseteq \mathbb{V}(G, \mathbb{D}) = \mathbb{V}(G^\flat, \mathbb{D}^\flat) \; . \tag{6}$$

Proof First, let us notice that, whatever the acceptable set $\mathbb{D}$ and the dynamics G, we have the inclusion

$$\mathbb{V}(G^\flat, \mathbb{D}^\flat) \cup \mathbb{V}(G^\sharp, \mathbb{D}^\sharp) \subseteq \mathbb{V}(G, \mathbb{D}) \; . \tag{7}$$

Indeed, $\mathbb{V}(G^\flat, \mathbb{D}^\flat) = \bigcap_{t=0}^{+\infty} \{x \in \mathbb{X} \mid \left((G^\flat)^{(t)}(x), u^\flat\right) \in \mathbb{D}\} \subseteq \mathbb{V}(G, \mathbb{D})$, since $x \in \mathbb{V}(G^\flat, \mathbb{D}^\flat)$ means that the stationary control $u(t) = u^\flat$ makes that the trajectory $(x(t), u(t)) = ((G^\flat)^{(t)}(x), u^\flat)$ belongs to $\mathbb{D}$. The same may be done with the control $u^\sharp$.

Second, when G is a monotone bioeconomic dynamics and $\mathbb{D}$ is either a production or a preservation acceptable set, we have the inclusions

$$\mathbb{V}(G^\sharp, \mathbb{D}) \subset \mathbb{V}(G, \mathbb{D}) \subset \mathbb{V}(G^\flat, \mathbb{D}) \; . \tag{8}$$

This is a straightforward application of Proposition 11 in De Lara et al (2006), because both $G^\flat$ and G are increasing with respect to the state, and because $\mathbb{D}$ is an upper set (production and preservation acceptable sets are upper sets).

Now, we come to the proof.

1. On the one hand, we have that $\mathbb{V}(G^\sharp, \mathbb{D}^\flat) \subseteq \mathbb{V}(G^\flat, \mathbb{D}^\flat)$ by (8) with $\mathbb{D}$ replaced by $\mathbb{D}^\flat$. By (7), this gives the two lower estimates of the viability kernel $\mathbb{V}(G, \mathbb{D})$ in (5). On the other hand, since $\mathbb{D}$ is a production acceptable set, we have $\mathbb{V}_0 = \mathbb{V}_0^\sharp$, and thus, by (3) and (4),

$$\mathbb{V}(G^\flat, \mathbb{D}) = \bigcap_{t=0}^{+\infty} \{x \in \mathbb{X} \mid (G^\flat)^{(t)}(x) \in \mathbb{V}_0\} = \bigcap_{t=0}^{+\infty} \{x \in \mathbb{X} \mid (G^\flat)^{(t)}(x) \in \mathbb{V}_0^\sharp\}$$
$$= \mathbb{V}(G^\flat, \mathbb{D}^\sharp) \; .$$

   As we have seen by (8) that $\mathbb{V}(G, \mathbb{D}) \subseteq \mathbb{V}(G^\flat, \mathbb{D})$, this gives $\mathbb{V}(G, \mathbb{D}) \subseteq \mathbb{V}(G^\flat, \mathbb{D}^\sharp)$, hence the upper estimate of the viability kernel $\mathbb{V}(G, \mathbb{D})$ in (5).
2. The lower estimate of the viability kernel $\mathbb{V}(G, \mathbb{D})$ in (6) comes from (7).
   Now, let us prove the equality $\mathbb{V}(G^\flat, \mathbb{D}^\flat) = \mathbb{V}(G, \mathbb{D})$ in (6). On the one hand, by (7) we know that $\mathbb{V}(G^\flat, \mathbb{D}^\flat) \subseteq \mathbb{V}(G, \mathbb{D})$. On the other hand, since $\mathbb{D}$ is a preservation acceptable set, we have $\mathbb{V}_0 = \mathbb{V}_0^\flat$, and thus

$$\mathbb{V}(G^\flat, \mathbb{D}) = \bigcap_{t=0}^{+\infty} \{x \in \mathbb{X} \mid (G^\flat)^{(t)}(x) \in \mathbb{V}_0\} = \bigcap_{t=0}^{+\infty} \{x \in \mathbb{X} \mid (G^\flat)^{(t)}(x) \in \mathbb{V}_0^\flat\}$$
$$= \mathbb{V}(G^\flat, \mathbb{D}^\flat) \; .$$

   By (8), this gives $\mathbb{V}(G^\flat, \mathbb{D}^\flat) \subseteq \mathbb{V}(G, \mathbb{D}) \subseteq \mathbb{V}(G^\flat, \mathbb{D}) = \mathbb{V}(G^\flat, \mathbb{D}^\flat)$.

When the acceptable set is given by means of indicators functions and thresholds as in (1), and the upper dynamics $G^\flat$ has a steady state satisfying some requirements, we obtain the following practical conditions for nonemptyness of the viability kernel.

Corollary 1 *Suppose that G is a monotone bioeconomic dynamics, that the control set $\mathbb{U}$ has lower and upper bounds $u^\flat$, $u^\sharp \in \mathbb{U}$ and that the acceptable set $\mathbb{D}$ is given by* (1) *with indicators $\mathcal{L}_i$'s being upper semi-continuous functions in the first (state) variable. Assume also that the upper dynamics $G^\flat$ has a steady state $\overline{x}(u^\flat)$ and there exists $L < 1$ such that*

$$\|G^\flat(x) - \overline{x}(u^\flat)\| \leq L\|x - \overline{x}(u^\flat)\| \qquad \forall x \in \mathbb{V}_0 \tag{9}$$

for some norm $\|\cdot\|$ in $\mathbb{X}$.

1. *If $\mathbb{D}$ is a production acceptable set, one has*

$$\exists i = 1, \ldots, p\ , \quad \mathcal{L}_i(\overline{x}(u^\flat), u^\sharp) < l_i \Rightarrow \mathbb{V}(G, \mathbb{D}) = \emptyset\ . \tag{10}$$

2. *If $\mathbb{D}$ is a preservation acceptable set, one has*

$$\mathbb{V}(G, \mathbb{D}) \neq \emptyset \Leftrightarrow \mathcal{L}_i(\overline{x}(u^\flat), u^\flat) \geq l_i \qquad \forall i = 1, \ldots, p\ . \tag{11}$$

Proof We proceed to prove Statement 1 by contra-reciprocal argument. Let us suppose that $\mathbb{V}(G, \mathbb{D}) \neq \emptyset$ and take x in this set which is included in $\mathbb{V}_0$. From Proposition 1, x belongs to $\mathbb{V}(G^\flat, \mathbb{D}^\sharp)$ or, equivalently

$$(G^\flat)^{(t)}(x) \in \mathbb{V}_0^\sharp = \mathbb{V}_0\ , \quad \forall t \geq t_0 \Leftrightarrow \mathcal{L}_i\big((G^\flat)^{(t)}(x), u^\sharp\big) \geq l_i \quad \forall i = 1, \ldots, p \quad \forall t \geq t_0\ .$$

Since $(G^\flat)^{(t)}(x) \in \mathbb{V}_0$ for all $t \geq t_0$, condition (9) implies $(G^\flat)^{(t)}(x) \to \overline{x}(u^\flat)$. Then, from the upper semi-continuity property of functions $\mathcal{L}_i(\cdot, u^\sharp)$, we obtain the desired inequalities

$$\mathcal{L}_i(\overline{x}(u^\flat), u^\sharp) \geq l_i\ , \quad \forall i = 1, \ldots, p\ .$$

The proof of necessary condition ($\Rightarrow$) in Statement 2 is analogous. For the sufficient condition ($\Leftarrow$) directly one can prove that $\overline{x}(u^\flat) \in \mathbb{V}(G, \mathbb{D})$ taking the stationary control $u(t) = u^\flat$.

The previous corollary provides necessary conditions (in the case of a production acceptable set) and necessary and sufficient conditions (in the case of a preservation acceptable set) to assure the non-emptiness of the viability kernel. The quantities $\mathcal{L}_i(\overline{x}(u^\flat), u^\sharp)$ (for production acceptable sets) and $\mathcal{L}_i(\overline{x}(u^\flat), u^\flat)$ (for preservation acceptable sets) can be interpreted as *maximal thresholds* for the acceptable configurations. That is, no trajectory $x(\cdot) = (x(t_0), x(t_0 + 1), \ldots)$ can generate values $\mathcal{L}_i(x(t), u(t))$ above these values for all periods of time t, whatever the initial state x_0 and the control trajectory $u(\cdot) = (u(t_0), u(t_0 + 1), \ldots)$ be.

Remark 1 An alternative (non equivalent) condition to (9) in the above Corollary 1 would be supposing that the steady state $\overline{x}(u^\flat)$ is globally asymptotic stable on $\mathbb{V}_0$ for the dynamics $G^\flat$. However, this is a strong assumption. A weaker one would restrict global asymptotic stability on the subset $\mathbb{V}(G, \mathbb{D}) \subset \mathbb{V}_0$ (see the proof of Corollary 1). Nevertheless, it is not elegant neither practical to make any assumption on the viability kernel $\mathbb{V}(G, \mathbb{D})$, which is an object of study and which might be empty.

4 Application to fishery management

Now, we apply and specify the previous results in the case of an age-structured abundance population model, especially with a Beverton-Holt stock-recruitment relationship. With this, we provide numerical estimates for two Chilean fisheries.

4.1 An age class dynamical model

We consider an age-structured abundance population model with a possibly non linear stock-recruitment relationship, derived from fish stock management (see Quinn and Deriso (1999), and also De Lara et al (2007) for more details).

Time is measured in years, and the time index $t \in \mathbb{N}$ represents the beginning of year t and of yearly period $[t, t+1[$. Let $A \in \mathbb{N}^*$ denote a maximum age, and $a \in \{1, \ldots, A\}$ an age class index, all expressed in years. The state is the vector $N = (N_a)_{a=1,\ldots,A} \in \mathbb{R}_+^A$, the *abundances* at age: for $a = 1, \ldots, A-1$, $N_a(t)$ is the number of individuals of age between $a-1$ and a at the beginning of yearly period $[t, t+1[$; $N_A(t)$ is the number of individuals of age greater than $A-1$. The control $\lambda(t)$ is the *fishing effort multiplier*, supposed to be applied in the middle of period $[t, t+1[$. The control dynamical model is

$$N(t+1) = G\big(N(t), \lambda(t)\big), \quad t = t_0, t_0+1, \ldots, \qquad N(t_0) \text{ given},$$

where the vector function $G = (G_a)_{a=1,\ldots,A}$ is defined for any $N \in \mathbb{R}_+^A$ and $\lambda \in \mathbb{R}_+$ by

$$\begin{cases} G_1(N, \lambda) &= \varphi\big(SSB(N)\big)\,, \\ G_a(N, \lambda) &= e^{-(M_{a-1} + \lambda F_{a-1})} N_{a-1}, \quad a = 2, \ldots, A-1\,, \\ G_A(N, \lambda) &= e^{-(M_{A-1} + \lambda F_{A-1})} N_{A-1} + \pi \times e^{-(M_A + \lambda F_A)} N_A\,. \end{cases} \tag{12}$$

In the above formulas, M_a is the natural *mortality rate* of individuals of age a, F_a is the mortality rate of individuals of age a due to harvesting between t and $t+1$, supposed to remain constant during period $[t, t+1[$ (the vector $(F_a)_{a=1,\ldots,A}$ is termed the *exploitation pattern*), and the parameter $\pi \in \{0, 1\}$ is related to the existence of a so-called *plus-group* (if we neglect the survivors older than age A then $\pi = 0$, else $\pi = 1$ and the last age class is a plus group). The function φ describes a *stock-recruitment relationship*. The *spawning stock biomass SSB* is defined by

$$SSB(N) := \sum_{a=1}^{A} \gamma_a w_a N_a\,, \tag{13}$$

that is summing the contributions of individuals to reproduction, where $(\gamma_a)_{a=1,\ldots,A}$ are the *proportions of mature individuals* (some may be zero) at age and $(w_a)_{a=1,\ldots,A}$ are the *weights at age* (all positive).

4.2 An acceptable set reflecting conflicting preservation and production objectives

We shall consider an acceptable set $\mathbb{D}$ which reflects conflicting objectives of *preservation* – measured by the spawning stock biomass being high enough – and of *production*, measured by the following yield indicator.

The exploitation is described by catch-at-age C_a and yield Y, defined for a given vector of abundance N and a given control λ by the so called *Baranov catch equations* (Quinn and Deriso, 1999, p. 255-256). The catches are the number of individuals captured over the period $[t-1, t[$:

$$C_a\big(N, \lambda\big) = \frac{\lambda F_a}{\lambda F_a + M_a} \left(1 - e^{-(M_a + \lambda F_a)}\right) N_a\,.$$

The production in term of biomass at the beginning of period $[t, t+1[$ is then

$$Y\big(N,\lambda\big) = \sum_{a=1}^{A} w_a\, C_a(N,\lambda)\;. \tag{14}$$

We focus our analysis on the acceptable set

$$\mathbb{D}_{\mathsf{yield}}(y_{\mathsf{min}}, B_{\mathsf{lim}}) := \{(N,\lambda) \mid Y(N,\lambda) \geq y_{\mathsf{min}},\ SSB(N) \geq B_{\mathsf{lim}}\}\,, \tag{15}$$

where the yield function Y is given by (14) and *SSB* by (13). Contrarily to the ICES precautionary approach as analyzed in De Lara et al (2007), we not only focus on *preservation issues* ($SSB(N) \geq B_{\mathsf{lim}}$) but also on *production issues* by asking for a minimal yield ($Y(N,\lambda) \geq y_{\mathsf{min}}$).

4.3 Monotonicity properties

The set $\mathbb{D}_{\mathsf{yield}}(y_{\mathsf{min}}, B_{\mathsf{min}})$ is a production acceptable set. Indeed, on the one hand, the yield Y is increasing with respect both to the state and to the control. On the other hand, the spawning stock biomass *SSB* is increasing with respect to the state (and does not depend on the control).

The dynamics (12) is a monotone bioeconomic one whenever the recruitment function φ in (12) is non decreasing.

We now focus on the existence of equilibrium points. As is classical, consider the following proportions of equilibrium recruits which survive up to age a:

$$\begin{cases} s_1(\lambda) := 1 \\ s_a(\lambda) := \exp\Big(-\big(M_1 + \cdots + M_{a-1} + \lambda(F_1 + \cdots + F_{a-1})\big)\Big)\,, \quad a = 2, \ldots, A-1 \\ s_A(\lambda) := \dfrac{1}{1 - \pi e^{-(M_A + \lambda F_A)}} \exp\Big(-\big(M_1 + \cdots + M_{A-1} + \lambda(F_1 + \cdots + F_{A-1})\big)\Big) \end{cases}$$

Let also $\mathrm{spr}(\lambda) := \sum_{a=1}^{A} \gamma_a w_a s_a(\lambda)$ be the *spawning per recruit* at equilibrium. When the recruitment function is Beverton-Holt $\varphi(B) = \frac{B}{\alpha+\beta B}$ (which includes the constant case, taking $\alpha = 0$, and the linear case, taking $\beta = 0$), there exists an equilibrium point for any control $\lambda \geq 0$. It is given by $\overline{N}(\lambda) = (\overline{N}_a(\lambda))_{a=1,\ldots,A}$, where $\overline{N}_a(\lambda) = Z(\lambda) s_a(\lambda)$ and $Z(\lambda) = \max\left\{0, \frac{\mathrm{spr}(\lambda) - \alpha}{\beta \mathrm{spr}(\lambda)}\right\}$ if $\beta > 0$, $Z(\lambda) = 0$ if $\beta = 0$.

4.4 Minimal viable production issues

The following statement establishes *maximum sustainable thresholds* for the indicators *SSB* and Y. It is an application of Corollary 1.

Proposition 2 *Assume that the stock-recruitment relationship φ is Beverton-Holt $\varphi(B) = \frac{B}{\alpha+\beta B}$ (allowing the cases $\alpha = 0$ or $\beta = 0$), that the fishing effort λ is bounded from below and above by $0 \leq \lambda^\flat \leq \lambda \leq \lambda^\sharp$. If*

$$\phi_G(\lambda^\flat) := \varphi'\left(SSB(\overline{N}(\lambda^\flat))\right) \max_{a=1,\ldots,A} \gamma_a w_a + \max_{a=1,\ldots,A} e^{-(M_a + \lambda^\flat F_a)} < 1 \tag{16}$$

then, ensuring a minimal viable production and spawning stock biomass requires that the production and preservation thresholds y_{min} *and* B_{lim} *be not too high:*

$$\left.\begin{array}{r} y_{\mathsf{min}} > Y(\overline{N}(\lambda^\flat), \lambda^\sharp) \\ or \quad B_{\mathsf{lim}} > SSB(\overline{N}(\lambda^\flat)) \end{array}\right\} \Rightarrow \mathbb{V}\big(G, \mathbb{D}_{\mathsf{yield}}(y_{\mathsf{min}}, B_{\mathsf{lim}})\big) = \emptyset \ .$$

Proof In order to apply Corollary 1, let us prove that, for $B_{\mathsf{lim}} > SSB(\overline{N}(\lambda^\flat))$, one has the following property

$$\|G(N, \lambda^\flat) - \overline{N}(\lambda^\flat)\|_1 \leq \phi_G(\lambda^\flat)\|N - \overline{N}(\lambda^\flat)\|_1 \ , \tag{17}$$

for all N in $\mathbb{V}_0$ (projection on $\mathbb{R}_+^A$ of the acceptable set $\mathbb{D}_{\mathsf{yield}}(y_{\mathsf{min}}, B_{\mathsf{lim}})$) where $\|N\|_1$ is the norm $\sum_{a=1}^A |N_a|$ in $\mathbb{R}^A$. For any N one has

$$\|G(N, \lambda^\flat) - \overline{N}(\lambda^\flat)\|_1 \leq \Big|\varphi(SSB(N)) - \varphi(SSB(\overline{N}(\lambda^\flat))\Big| + \sum_{a=1}^A e^{-(M_a + \lambda^\flat F_a)} |N_a - \overline{N}(\lambda^\flat)_a| \ . \tag{18}$$

If $N \in \mathbb{V}_0$ then $SSB(N) \geq B_{\mathsf{lim}} > SSB(\overline{N}(\lambda^\flat))$ and therefore one obtains

$$\Big|\varphi(SSB(N)) - \varphi(SSB(\overline{N}(\lambda^\flat))\Big| \leq \max_{B \in [SSB(\overline{N}(\lambda^\flat)), SSB(N)]} |\varphi'(B)| \sum_{a=1}^A \gamma_a w_a |N_a - \overline{N}(\lambda^\flat)_a| \ .$$

By the concavity of φ (which implies that φ' is decreasing), this gives

$$\begin{aligned} \Big|\varphi(SSB(N)) - \varphi(SSB(\overline{N}(\lambda^\flat))\Big| &\leq \Big|\varphi'(SSB(\overline{N}(\lambda^\flat)))\Big| \sum_{a=1}^A \gamma_a w_a |N_a - \overline{N}(\lambda^\flat)_a| \\ &\leq \left(\varphi'\left(SSB(\overline{N}(\lambda^\flat))\right) \max_{a=1,\ldots,A} \gamma_a w_a\right) \|N - \overline{N}(\lambda^\flat)\|_1 \ . \end{aligned}$$

The above inequality together with (18) and the definition of (16) make it possible to obtain (17) and, then, the condition (9) of Corollary 1.

The above result can be interpreted as follows.

- There is no vector of abundance which allows to obtain, starting from it, catches greater than the *maximal production threshold* $Y(\overline{N}(\lambda^\flat), \lambda^\sharp)$, during all the periods.
- Starting from any vector of abundance, whatever the harvest, the minimum level of spawning stock biomass (*SSB*) observed during all the periods will be lower than (or equal to) the *maximal preservation threshold* $SSB(\overline{N}(\lambda^\flat))$.

4.5 Numerical applications to Chilean fisheries

We provide numerical estimates obtained for the species Chilean sea bass (*Dissostichus eleginoides*), harvested in the south of Chile, and Alfonsino (*Beryx splendens*), harvested in the Juan Fernández archipelago. The dynamics of the Chilean sea bass can be described by the model (12) with a Beverton-Holt stock-recruitment relationship φ. For the Alfonsino, females and males are distinguished, each following a dynamics (12) with a Beverton-Holt stock-recruitment relationship φ. Thus, for this species, the state

is the abundances at age for females and males and the resulting dynamics is a monotone bioeconomic one. For both species, the mortality is supposed to be the same at all ages, and will be denoted by M. In Tables 2, 3, and 4 in the Appendix, we present numerical data for both species, provided by the *Centro de Estudios Pesqueros - Chile (CEPES)*.

Table 1 sums up the maximal production and preservation thresholds obtained from Proposition 2 for both species and the values of $\phi_G(\lambda^\flat)$ defined by (16).

Definition	Notation	*Chilean sea bass*	*Alfonsino*
Maximal threshold for a sustainable catch (tons)	$Y(\overline{N}(\lambda^\flat), \lambda^\sharp)$	15 166	16 158
Maximal threshold for a sustainable *SSB* (tons)	$SSB(\overline{N}(\lambda^\flat))$	56 521	52 373
Constant defined by (16)	$\phi_G(\lambda^\flat)$	0.852	0.818

Table 1 Maximal sustainable thresholds for *Chilean sea bass* and for *Alfonsino*.

4.6 Chilean sea bass

Figure 1 displays the Chilean sea bass landings, between 1988 and 2006. The horizontal line represents the maximal threshold $Y(\overline{N}(\lambda^\flat), \lambda^\sharp)$. Hence, it may be seen that the catches obtained in 1992 were not sustainable: even if the species were abundant, such landings could not be maintained forever.

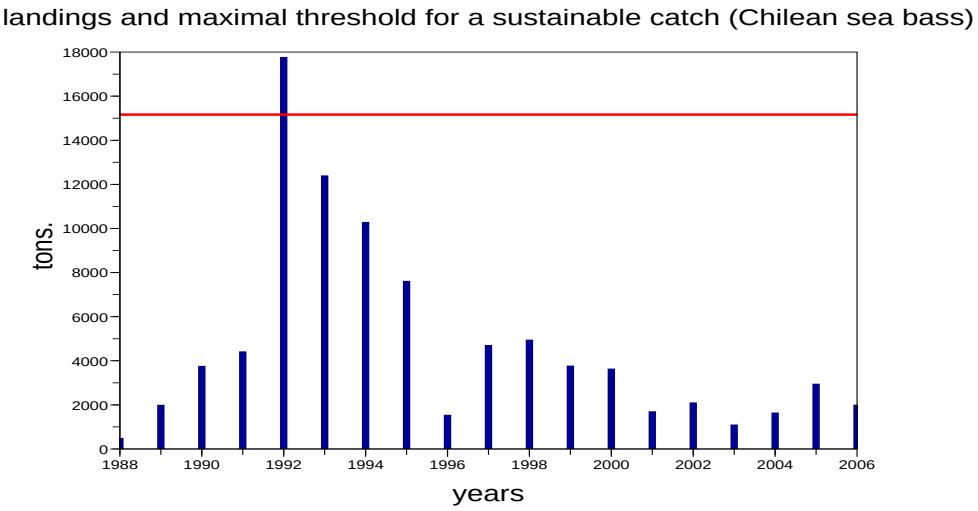


Fig. 1 Chilean sea bass: landings (1988-2006) [tons] and $Y(\overline{N}(\lambda^\flat), \lambda^\sharp)$.

Figure 2 displays the Chilean sea bass spawning stock biomass (*SSB*), between 1988 and 2006. The horizontal line represents the maximal threshold $SSB(\overline{N}(\lambda^\flat))$. The *SSB* observed during the first six years could not have been sustained forever.

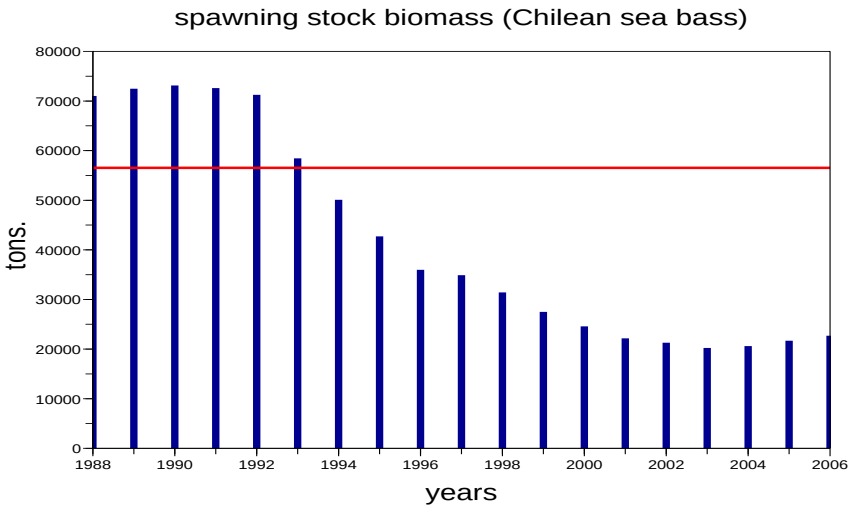


Fig. 2 Chilean sea bass: SSB (1988-2006) [tons] and $SSB(\overline{N}(\lambda^{\flat}))$.

4.7 Alfonsino

For the Alfonsino, Figures 3 and 4, both spawning stock biomasses and landings are below the maximal threshold. Thus, we cannot conclude that these levels indicate a non viable fishery management.

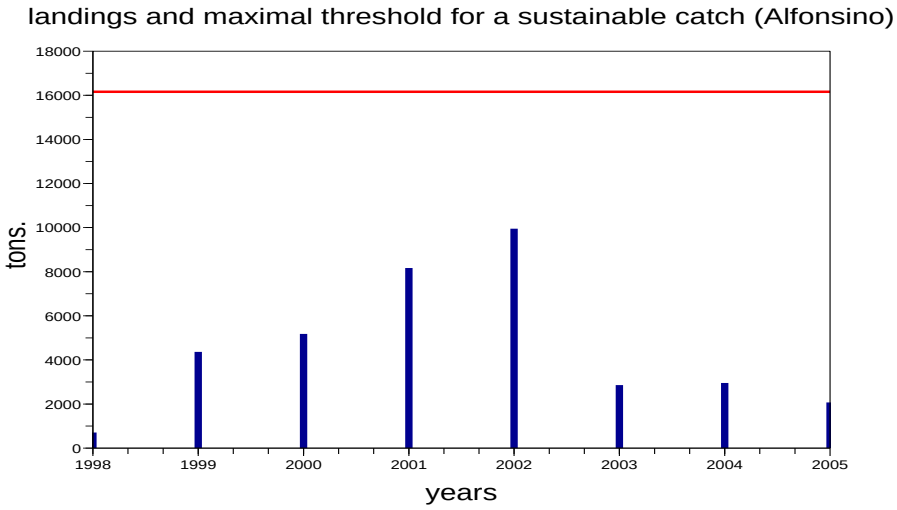


Fig. 3 Alfonsino: landings (1998-2005) [tons] and $Y(\overline{N}(\lambda^{\flat}), \lambda^{\sharp})$.

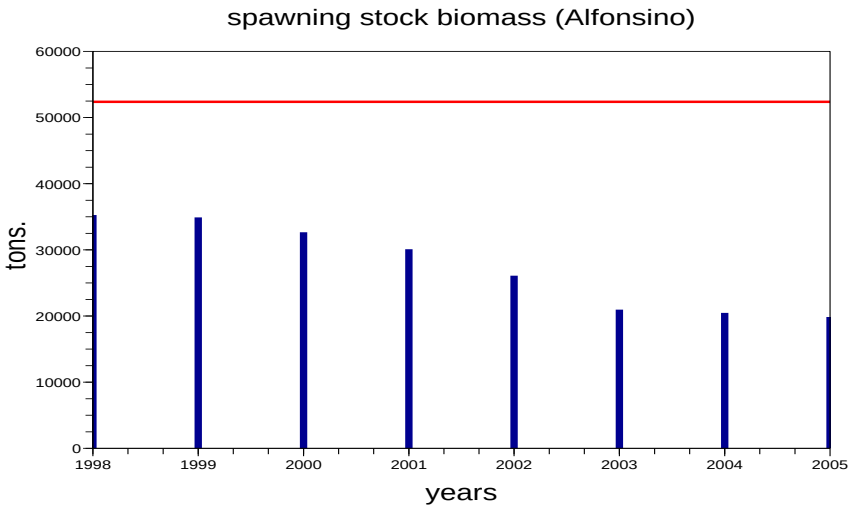


Fig. 4 Alfonsino: SSB (1998-2005) [tons] and $SSB(\overline{N}(\lambda^\flat))$.

5 Conclusion

Some monospecies age class models, as well as specific multi-species models (with so-called technical interactions), exhibit useful monotonicity properties. We have shown how these latter may help providing estimates of the viability kernel for so-called production and preservation acceptable sets. When the acceptable set is defined by inequalities requirements given by indicator functions, we provide conditions on the corresponding thresholds to test whether the viability kernel is empty or not.

This theoretical framework is applied to fishery management analysis. We obtain upper bounds for production which are interesting for managers in that they only depend on the model parameters, and not on the current stocks. Our formulas for so-called maximal sustainable thresholds give sensible values: Chilean sea bass data violate these bounds, while Alfonsino data are within.

We have thus provided a general method to analyze up to what points can conflicting production and preservation objectives be sustainably achieved for a class of models including monospecies age class and multi-species with technical interactions.

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A Appendix: model data

Definition	Notation	*Chilean sea bass*	*Alfonsino*
Maximum age	A	36	40 (20 male & 20 female)
Beverton-Holt parameter (adimensional)	α	$1.4 \cdot 10^{-3}$	$9.16 \cdot 10^{-5}$
Beverton-Holt parameter (gr)	β	$4.65 \cdot 10^{-7}$	$7.46 \cdot 10^{-8}$
Natural mortality	M	0.16	0.2
Presence of plus group	π	1	1
Lower limit for fishing effort multiplier	$\lambda^{\flat}$	0	0
Upper limit for fishing effort multiplier	$\lambda^{\sharp}$	0.39	0.885

Table 2 Parameter definitions and values for two case studies.

Age a	Mean weight-at-age (gr) $(w_a)_a$	Maturity ogive $(\gamma_a)_a$	Fishing mortality-at-age $(F_a)_a$
1	3	0	0.0005
2	77	0	0.0013
3	326	0	0.0051
4	809	0	0.0183
5	1547	0	0.0494
6	2536	0	0.1080
7	3753	0	0.2067
8	5169	0.1	0.3467
9	6748	0.2	0.5277
10	8454	0.3	0.7127
11	10253	0.5	0.8675
12	12113	0.7	0.9611
13	14006	0.8	1.0000
14	15905	0.9	0.9831
15	17791	1	0.9302
16	19646	1	0.8661
17	21456	1	0.7933
18	23209	1	0.7254
19	24897	1	0.6614
20	26514	1	0.6040
21	28056	1	0.5537
22	29520	1	0.5091
23	30906	1	0.4701
24	32213	1	0.4434
25	33441	1	0.4115
26	34594	1	0.3867
27	35673	1	0.3651
28	36681	1	0.3464
29	37621	1	0.3341
30	38496	1	0.3206
31	39309	1	0.3089
32	40064	1	0.2986
33	40763	1	0.2896
34	41411	1	0.2817
35	42011	1	0.2747
36	45409	1	0.2408

Table 3 Parameters at age for the Chilean sea bass.

Age a (male & female)	Mean weight-at-age (gr) $(w_a)_a$	Maturity ogive $(\gamma_a)_a$	Fishing mortality-at-age $(F_a)_a$
1	138 /140	0	0,006 /0,006
2	252/ 256	0	0,026/ 0,024
3	397/ 406	0	0,083/ 0,075
4	565/ 583	0	0,217/ 0,192
5	752/ 783	0	0,441 /0,390
6	951/ 999	1	0,691/ 0,619
7	1156/ 1227	1	0,883/ 0,809
8	1364/ 1462	1	0,997 /0,932
9	1571/ 1699	1	1,000/ 1,000
10	1774/ 1935	1	0,564/ 0,884
11	1970/ 2168	1	0,229/ 0,486
12	2158/ 2395	1	0,098/ 0,233
13	2337/ 2614	1	0,045/ 0,118
14	2506/ 2825	1	0,022/ 0,064
15	2664/ 3026	1	0,012/ 0,037
16	2812/ 3216	1	0,006/ 0,022
17	2950/ 3396	1	0,004/ 0,014
18	3078/ 3565	1	0,002/ 0,010
19	3196/ 3724	1	0,001/ 0,007
20	3304/ 3872	1	0,001/ 0,005

Table 4 Parameters at age for the Alfonsino (male & female). Maturity ogive parameters are the same for males and females.