An explicit formula for the free exponential modality of linear logic

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Abstract. The exponential modality of linear logic associates a commutative comonoid $!A$ to every formula A in order to duplicate it. Here, we explain how to compute the free commutative comonoid $!A$ as a sequential limit of equalizers in any symmetric monoidal category where this sequential limit exists and commutes with the tensor product. We then apply this general recipe to two familiar models of linear logic, based on coherence spaces and on Conway games. This algebraic approach enables to unify for the first time apparently different constructions of the exponential modality in spaces and games. It also sheds light on the subtle duplication policy of linear logic. On the other hand, we explain at the end of the article why the formula does not work in the case of the finiteness space model.

1 Introduction

Linear logic is based on the principle that every hypothesis A_i should appear exactly once in a proof of the sequent

$$A_1, \ldots, A_n \quad \vdash \quad B. \tag{1}$$

This logical restriction enables to represent the logic in monoidal categories, along the idea that every formula denotes an object of the category, and every proof of the sequent (1) denotes a morphism

$$A_1 \otimes \cdots \otimes A_n \quad \longrightarrow \quad B$$

where the tensor product is thus seen as a linear kind of conjunction. Note that, for clarity's sake, we use the same notation for a formula A and for its interpretation (or denotation) in the monoidal category.

This linearity policy on proofs is far too restrictive in order to reflect traditional forms of reasoning, where it is accepted to repeat or to discard an hypothesis in the course of a logical argument. This difficulty is nicely resolved by providing linear logic with an exponential modality, whose task is to strengthen every formula A into a formula $!A$ which may be repeated or discarded. From a semantic point of view, the formula $!A$ is most naturally interpreted as a *comonoid* of the monoidal category. Recall that a comonoid (C, d, u) in a monoidal category $\mathscr{C}$ is defined as an object C equipped with two morphisms

$$d \quad : \quad C \quad \longrightarrow \quad C \otimes C \qquad\qquad u \quad : \quad C \quad \longrightarrow \quad \mathbf{1}$$

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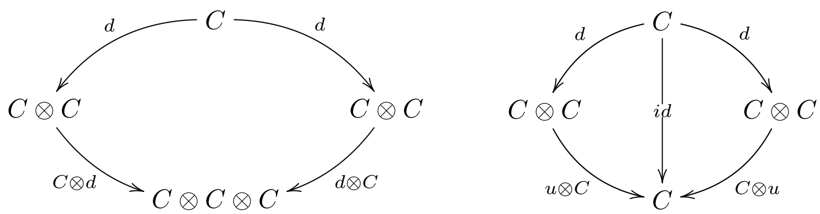
where $\mathbf{1}$ denotes the monoidal unit of the category. The morphism d and u are respectively called the *multiplication* and the *unit* of the comonoid. The two morphisms d and u are supposed to satisfy *associativity* and *unitality* properties, neatly formulated by requiring that the two diagrams



commute. Note that we draw our diagrams as if the category were *strictly* monoidal, although the usual models of linear logic are only *weakly* monoidal.

The comonoidal structure of the formula $!A$ enables to interpret the *contraction rule* and the *weakening rule* of linear logic

$$\dfrac{\dfrac{\pi}{\Gamma, !A, !A, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B}\ \text{Contraction} \qquad\qquad \dfrac{\dfrac{\pi}{\Gamma, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B}\ \text{Weakening}$$

by pre-composing the interpretation of the proof π with the multiplication d in the case of contraction

$$\Gamma \otimes !A \otimes \Delta \xrightarrow{d} \Gamma \otimes !A \otimes !A \otimes \Delta \xrightarrow{\pi} B$$

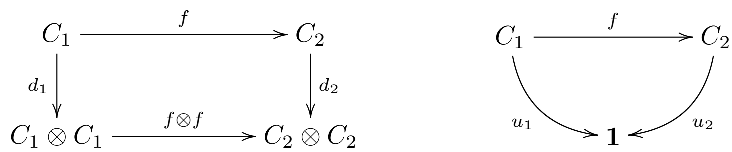
and with the unit u in the case of weakening

$$\Gamma \otimes !A \otimes \Delta \xrightarrow{u} \Gamma \otimes \Delta \xrightarrow{\pi} B.$$

Besides, linear logic is generally interpreted in a *symmetric* monoidal category, and one requires that the comonoid $!A$ is commutative, this meaning that the following equality holds:

$$A \xrightarrow{d} A \otimes A \xrightarrow{symmetry} A \otimes A \quad = \quad A \xrightarrow{d} A \otimes A\,.$$

When linear logic was introduced by Jean-Yves Girard, twenty years ago, it was soon realized by Robert Seely and others that the multiplicative fragment of the logic should be interpreted in a $*$-autonomous category, or at least, a symmetric monoidal closed category $\mathscr{C}$; and that the category should have finite products in order to interpret the additive fragment of the logic, see [10]. A more difficult question was to understand what categorical properties of the exponential modality " ! " were exactly required, in order to define a model of propositional linear logic – that is, including the multiplicative, additive and exponential components of the logic. However, Yves Lafont found in his PhD thesis [6] a simple way to define a model of linear logic. Recall that a comonoid morphism between two comonoids (C_1, d_1, u_1) and (C_2, d_2, u_2) is defined as a morphism $f : C_1 \to C_2$ such that the two diagrams



commute. One says that the commutative comonoid $!A$ is freely generated by an object A when there exists a morphism

$$\varepsilon \quad : \quad !A \quad \to \quad A$$

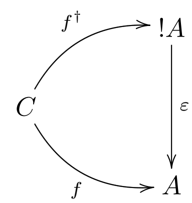
such that for every morphism

$$f \quad : \quad C \quad \to \quad A$$

from a commutative comonoid C to the object A, there exists a unique comonoid morphism

$$f^{\dagger} \quad : \quad C \quad \to \quad !A$$

such that the diagram



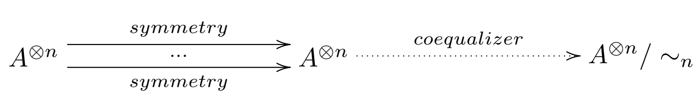
(2)

commutes. From a logical point of view, $!A$ is the weakest comonoid that implies A. Lafont noticed that the existence of a free commutative comonoid $!A$ for every object A of a symmetric monoidal closed category $\mathscr{C}$ induces automatically a model of propositional linear logic. Recall however that this is not the only way to construct a model of linear logic. A folklore example is the coherence space model, which admits two alternative interpretations of the exponential modality: the original one, formulated by Girard [3] where the coherence space $!A$ is defined as a space of *cliques*, and the free construction, where $!A$ is defined as a space of *multicliques* (cliques with multiplicity) of the original coherence space A.

In this paper, we explain how to construct the free commutative comonoid in the symmetric monoidal categories $\mathscr{C}$ typically encountered in the semantics of linear logic. Our starting point is the well-known formula defining the *symmetric algebra*

$$SA \quad = \quad \bigoplus_{n \in \mathbb{N}} \quad A^{\otimes n} \ / \ \sim_n \tag{3}$$

generated by a vector space A. Recall that the formula (3) computes the free commutative monoid associated to the object A in the category of vector spaces over a given field $\Bbbk$. The group Σ_n of permutations on $\{1, \ldots, n\}$ acts on the vector space $A^{\otimes n}$, and the vector space $A^{\otimes n}/\sim_n$ of equivalence classes (or orbits) modulo the group action is defined as the coequalizer of the $n!$ symmetries



in the category of vector spaces. Since a comonoid in the category $\mathscr{C}$ is the same thing as a monoid in the opposite category $\mathscr{C}^{op}$, it is tempting to apply the *dual* formula to (3) in order to define the free commutative comonoid $!A$ generated by an object A in the monoidal category $\mathscr{C}$. Although the idea is extremely naive, it is surprisingly close to the solution... Indeed, one significant aspect of our work is to establish that the equalizer A^n of the $n!$ symmetries

$$A^n \xrightarrow{\;equalizer\;} A^{\otimes n} \underset{symmetry}{\overset{symmetry}{\rightrightarrows}} A^{\otimes n} \tag{4}$$

exists in several distinctive models of linear logic, and provides there the n-th layer of the free commutative comonoid $!A$ generated by the object A. This principle will be nicely illustrated in Section 3 by the equalizer A^n in the category of coherence spaces, which contains the multicliques of cardinality n in the coherence space A ; and in Section 4 by the equalizer A^n in the category of Conway games, which defines the game where Opponent may open up to n copies of the game A, one after the other, in a sequential order.

Of course, the construction of the free exponential modality does not stop here: one still needs to combine the layers A^n together in order to define $!A$ properly. One obvious solution is to apply the dual of formula (3) and to define $!A$ as the infinite cartesian product

$$!A \quad = \quad \mathop{\&}_{n\in\mathbb{N}} \quad A^n. \tag{5}$$

This formula works perfectly well for symmetric monoidal categories $\mathscr{C}$ where the infinite product commutes with the tensor product, in the sense that the canonical morphism

$$X \quad \otimes \quad \left(\mathop{\&}_{n\in\mathbb{N}} \quad A^n \right) \quad \rightarrow \quad \mathop{\&}_{n\in\mathbb{N}} \; (\; X \; \otimes \; A^n \;) \tag{6}$$

is an isomorphism. This useful algebraic degeneracy is not entirely uncommon: it typically happens in the relational model of linear logic, where the free exponential $!A$ is defined according to formula (5) as the set of finite multisets of A, each equalizer A^n describing the set of multisets of cardinality n.

On the other hand, the formula (5) is far too optimistic in general, and does not work when one considers the familiar models of linear logic based either on coherence spaces, or on sequential games. It is quite instructive to apply the formula to the category of Conway games: it defines a game $!A$ where the first move by Opponent selects a component A^n, and thus decides the number n of copies of the game A played subsequently. This departs from the free commutative comonoid $!A$ which we shall examine in Section 4, where Opponent is allowed to open a new copy of the game A at any point of the interaction.

So, there remains to understand how the various layers A^n should be combined together inside $!A$ in order to perform this particular copy policy. One well-inspired temptation is to ask that every layer A^n is "glued" inside the next layer A^{n+1} in order to allow the computation to transit from one layer to the next in the course of interaction. One simple way to perform this "glueing" is to introduce the notion of (co)pointed (or affine) object. By pointed object in a monoidal category $\mathscr{C}$, one means a pair (A, u) consisting of an object A and of a morphism $u : A \rightarrow \mathbf{1}$ to the monoidal unit. So, a pointed object is the same thing as a comonoid, without a comultiplication. It is folklore that the category $\mathscr{C}_\bullet$ of pointed objects and pointed morphisms (defined in the expected way) is symmetric monoidal, and moreover *affine* in the sense that its monoidal unit $\mathbf{1}$ is terminal.

The main purpose of this paper is to compute (in Section 2) the free commutative comonoid $!A$ of the category $\mathscr{C}$ as a sequential limit of equalizers. The construction is excessively simple and works every time the sequential limit exists in the category $\mathscr{C}$, and commutes with the tensor product. We establish that the category of coherence spaces (in Section 3) and the category of Conway games (in Section 4) fulfill these hypotheses. This establishes that despite their difference in style, the free exponential modalities are defined in *exactly* the same way in the two models. We then clarify (in Section 5) the topological reasons why neither formula (5) nor the sequential limit of equalizers formulated below (9) define the free exponential modality in the finiteness space model of linear logic recently introduced by Thomas Ehrhard [2].

2 The sequential limit construction

Before stating the general proposition, we present the construction in three steps.

First step. We make the mild hypothesis that the object A of the monoidal category $\mathscr{C}$ generates a free pointed object $(A_\bullet, u)$ in the affine category $\mathscr{C}_\bullet$. This typically happens when the forgetful functor $\mathscr{C}_\bullet \to \mathscr{C}$ has a right adjoint. Informally speaking, the purpose of the pointed object $A_\bullet$ is to describe one copy of the object A, or none... Note that this free pointed object is usually quite easy to define: in the case of coherence spaces, it is the space $A_\bullet = A \,\&\, \mathbf{1}$ obtained by adding a point to the web of A ; in the case of Conway games, it is the game $A_\bullet = A$ itself, at least when the category is restricted to Opponent-starting games.

Second step. The object $A^{\leq n}$ is then defined as the equalizer $(A_\bullet)^n$ of the diagram

$$A^{\leq n} \xrightarrow{\;equalizer\;} A_\bullet^{\otimes n} \underset{symmetry}{\overset{symmetry}{\rightrightarrows}} A_\bullet^{\otimes n} \qquad (7)$$

in the category $\mathscr{C}$. The purpose of $A^{\leq n}$ is to describe all the layers A^k at the same time, for $k \leq n$. Typically, the object $A^{\leq n}$ computed in the category of coherence spaces is the space of all multicliques in A of cardinality less than or equal to n.

Third step. We take advantage of the existence of a canonical morphism $A^{\leq n} \longleftarrow A^{\leq n+1}$ induced by the unit $u : A_\bullet \to \mathbf{1}$ of the pointed object $A_\bullet$, and define the object A^∞ as the sequential limit of the sequence

$$1 \longleftarrow A^{\leq 1} \longleftarrow A^{\leq 2} \longleftarrow \cdots \longleftarrow A^{\leq n} \longleftarrow A^{\leq n+1} \longleftarrow \cdots \qquad (8)$$

with limiting cone defined by projection maps

$$A^\infty \xrightarrow{\;projection\;} A^{\leq n}.$$

The 2-dimensional study of algebraic theories and PROPs recently performed by Melliès and Tabareau [8] ensures that this recipe in three steps defines the free commutative comonoid $!A$ as the sequential limit A^∞... when the object A satisfies the following limit properties in the category $\mathscr{C}$.

Proposition 1. *Consider an object A in a symmetric monoidal category $\mathscr{C}$. Suppose that the object A generates a free pointed object $(A_\bullet, u)$. Suppose moreover that the equalizer (7) and the sequential limit (8) exist and commute with the tensor product, in the sense that*

$$X \otimes A^{\leq n} \xrightarrow{X\otimes \ equalizer} A_\bullet^{\otimes n} \underset{X\otimes \ symmetry}{\overset{X\otimes \ symmetry}{\rightrightarrows}} X \otimes A_\bullet^{\otimes n}$$

defines an equalizer diagram, and the family of maps

$$X \otimes A^{\infty} \xrightarrow{X\otimes \ projection} X \otimes A^{\leq n}$$

defines a limiting cone, for every object X of the category $\mathscr{C}$. In that case, the free commutative comonoid $!A$ coincides with the sequential limit A^∞.

The proof of Proposition 1 is based on two observations. The first observation is that the category $\mathscr{C}_\bullet$ coincides with the slice category $\mathscr{C} \downarrow \mathbf{1}$, this implying that the forgetful functor $\mathscr{C}_\bullet \to \mathscr{C}$ creates limits. Consequently, the limiting process defining the object A^∞ in the category $\mathscr{C}$ may be alternatively carried out in the category $\mathscr{C}_\bullet$. The second observation is that the limiting process defining A^∞ provides a pedestrian way to compute the end formula

$$A^\infty \quad = \quad \int_{n\in Inj^{op}} FinSet(n,1) \otimes (A_\bullet)^{\otimes n} \quad = \quad \int_{n \in Inj^{op}} (A_\bullet)^{\otimes n} \tag{9}$$

in the category $\mathscr{C}_\bullet$. As explained in our work on categorical model theory [8], this end formula provides an explicit computation of the object $Ran_f A_\bullet(1)$, where $Ran_f A_\bullet : FinSet^{op} \to \mathscr{C}_\bullet$ denotes the right Kan extension of the pointed object $A_\bullet : Inj^{op} \to \mathscr{C}_\bullet$ along the change of basis $f : Inj^{op} \to FinSet^{op}$ going from the theory of pointed objects to the theory of commutative comonoids. Recall that the category Inj has finite ordinals $[n] = \{0, \ldots, n-1\}$ as objects, and injections $[p] \to [q]$ as morphisms, whereas the category $FinSet$ has functions $[p] \to [q]$ as morphisms. Proposition 1 says that the end formula defines the free commutative comonoid when the end exists and commutes with the tensor product.

3 Coherence spaces

In this section, we compute the free exponential modality in the category of coherence spaces defined by Jean-Yves Girard [3]. A *coherence space* $E = (|E|, \frown)$ consists of a set $|E|$ called its *web*, and of a binary reflexive and symmetric relation $\frown$ over E. A *clique* of E is a set X of pairwise coherent elements of the web:

$$\forall e_1, e_2 \in X, \qquad e_1 \frown e_2.$$

We do not recall here the definition of the category **Coh** of coherence spaces. Just remember that a morphism $R : E \to E'$ in **Coh** is a clique of the coherence space $E \multimap E'$, so in particular, R is a relation on the web $|E| \times |E'|$.

It is easy to see that the tensor product does not commute with cartesian products: simply observe that the canonical morphism

$$A \otimes (\mathbf{1} \,\&\, \mathbf{1}) \quad \rightarrow \quad (A \otimes \mathbf{1}) \,\&\, (A \otimes \mathbf{1})$$

is not an isomorphism. This explains why formula (5) does not work, and why the construction of the free exponential modality requires a sequential limit, along the line described in the introduction.

First step: compute the free affine object. Computing the free pointed (or affine) object on a coherence space E is easy, because the category **Coh** has cartesian products: it is simply given by the formula

$$E_\bullet = E \,\&\, 1.$$

It is useful to think of $E \& 1$ has the space of multicliques of E with *at most* one element: the very first layer of the construction of the free exponential modality. Indeed, the unique element of 1 may be seen as the empty clique, while every element e of E may be seen as the singleton clique $\{e\}$. Recall that a multiclique of E is just a multiset on $|E|$ whose underlying set is a clique of E.

Second step: compute the symmetric tensor power $E^{\leq n}$. It is not difficult to see that the equalizer $E^{\leq n}$ of the symmetries

$$(E \,\&\, \mathbf{1})^{\otimes n} \; \underset{symmetry}{\overset{symmetry}{\underset{\cdots}{\rightrightarrows}}} \; (E \,\&\, \mathbf{1})^{\otimes n}$$

is given by the set of multicliques of E with at most n elements, two multicliques being coherent iff their union is still a multiclique. As explained in the introduction, one also needs to check that the tensor product commutes with those equalizers. Consider a cone

$$\begin{array}{ccc} & Y & \\ {}^{R}\swarrow & & \searrow^{R'} \\ X \otimes (E \,\&\, \mathbf{1})^{\otimes n} & \underset{X \otimes symmetry}{\overset{X \otimes symmetry}{\underset{\cdots}{\rightrightarrows}}} & X \otimes (E \,\&\, \mathbf{1})^{\otimes n} \end{array} \tag{10}$$

First, observe that $R = R'$ because one may choose the identity among the $n!$ symmetries. Next, we show that the morphism R factors uniquely through the morphism

$$X \otimes E^{\leq n} \xrightarrow{\;X \otimes \; equalizer\;} X \otimes (E \,\&\, \mathbf{1})^{\otimes n}$$

To that purpose, one defines the relation

$$R^{\leq n} \; : \; Y \;\longrightarrow\; X \otimes E^{\leq n} \qquad \text{by} \qquad y \; R^{\leq n} \; (x, \mu) \quad \text{iff} \quad y \; R \; (x, u)$$

where μ is a multiset of $|E|$ of cardinal less than n, and u is any word of length n whose letters with multiplicity in $|E \,\&\, \mathbf{1}| = |E| \sqcup \{*\}$ define the multiset μ. Remark that the fact that R equalizes the symmetries implies that any u' defining the same multiset μ will also be in the relation: $y \; R \; (x, u')$. We let the reader check that the definition is correct, that it defines a clique $R^{\leq n}$ of $Y \multimap (X \otimes E^{\leq n})$, and that it is the unique way to factor R through (10).

Third step: compute the sequential limit

$$E^{\leq 0} = \mathbf{1} \longleftarrow E^{\leq 1} = (E \,\&\, \mathbf{1}) \longleftarrow E^{\leq 2} \longleftarrow E^{\leq 3} \cdots$$

whose arrows are (dualized) inclusions from $E^{\leq n}$ into $E^{\leq n+1}$. Again, it is a basic fact that the limit $!E$ of the diagram is given by the set of all finite multicliques, two multicliques being coherent iff their union is a multiclique. At this point, one needs to check that the sequential limit commutes with the tensor product. Consider a cone

$$\begin{array}{c} Y \xrightarrow{R_0} X \otimes 1, \quad Y \xrightarrow{R_1} X \otimes (E \,\&\, 1), \quad Y \xrightarrow{R_2} X \otimes E^{\leq 2}, \quad Y \xrightarrow{R_3} X \otimes E^{\leq 3} \\ X \otimes 1 \longleftarrow X \otimes (E \,\&\, 1) \longleftarrow X \otimes E^{\leq 2} \longleftarrow X \otimes E^{\leq 3} \cdots \end{array}$$

and define the relation

$$R_\infty \;:\; Y \;\longrightarrow\; X \otimes !E \qquad \text{by} \qquad y \; R_\infty \; (x, \mu) \quad \text{iff} \quad \exists n, \quad y \; R_n \; (x, u)$$

where μ is a multiset of elements of $|E|$ and the element u of the web of $E^{\leq n}$ is any word of length n whose letters with multiplicity in $|E \,\&\, \mathbf{1}| = |E| \sqcup \{*\}$ define the multiset μ. We let the reader check that R_∞ is a clique of $Y \multimap (X \otimes !E)$ and defines the unique way to factor the cone. This concludes the proof that the sequential limit $!E$ defines the free commutative comonoid generated by E in the category **Coh** of coherence spaces.

4 Conway games

In this section, we compute the free exponential modality in the category of Conway games introduced by André Joyal in [4]. One unifying aspect of our approach is that the construction works in exactly the same way as for coherence spaces.

Conway games. A *Conway game* A is an oriented rooted graph (V_A, E_A, λ_A) consisting of (1) a set V_A of vertices called the *positions* of the game; (2) a set $E_A \subset V_A \times V_A$ of edges called the *moves* of the game; (3) a function $\lambda_A : E_A \to \{-1, +1\}$ indicating whether a move is played by Opponent (-1) or by Proponent $(+1)$. We write $\star_A$ for the root of the underlying graph. A Conway game is called *negative* when all the moves starting from its root are played by Opponent.
A *play* $s = m_1 \cdot m_2 \cdot \ldots \cdot m_{k-1} \cdot m_k$ of a Conway game A is a path $s : \star_A \twoheadrightarrow x_k$ starting from the root $\star_A$

$$s : \star_A \xrightarrow{m_1} x_1 \xrightarrow{m_2} \ldots \xrightarrow{m_{k-1}} x_{k-1} \xrightarrow{m_k} x_k$$

Two paths are parallel when they have the same initial and final positions. A play is *alternating* when

$$\forall i \in \{1, \ldots, k-1\}, \qquad \lambda_A(m_{i+1}) = -\lambda_A(m_i).$$

We note Play_A the set of plays of a game A.

Dual. Every Conway game A induces a *dual* game A^* obtained simply by reversing the polarity of moves.

Tensor product. The tensor product $A \otimes B$ of two Conway games A and B is essentially the asynchronous product of the two underlying graphs. More formally, it is defined as:

– $V_{A\otimes B} = V_A \times V_B$,
– its moves are of two kinds :

$$x \otimes y \to \begin{cases} z \otimes y \text{ if } x \to z \text{ in the game } A \\ x \otimes z \text{ if } y \to z \text{ in the game } B, \end{cases}$$

– the polarity of a move in $A \otimes B$ is the same as the polarity of the underlying move in the component A or the component B.

The unique Conway game 1 with a unique position $\star$ and no move is the neutral element of the tensor product. As usual in game semantics, every play s of the game $A \otimes B$ can be seen as the interleaving of a play $s_{|A}$ of the game A and a play $s_{|B}$ of the game B.

Strategies. Remark that the definition of a Conway game does not imply that all the plays are alternating. The notion of alternation between Opponent and Proponent only appears at the level of strategies (i.e. programs) and not at the level of games (i.e. types). A *strategy* σ of a Conway game A is defined as a non empty set of *alternating plays* of even length such that (1) every non empty play starts with an Opponent move; (2) σ is closed by even length prefix; (3) σ is *deterministic*, i.e. for all plays s, and for all moves m, n, n',

$$s \cdot m \cdot n \in \sigma \wedge s \cdot m \cdot n' \in \sigma \Rightarrow n = n'.$$

The category of Conway games. The category **Conway** has Conway games as objects, and strategies σ of $A^* \otimes B$ as morphisms $\sigma : A \to B$. The composition is based on the usual "parallel composition plus hiding" technique and the identity is defined by a copycat strategy. The resulting category **Conway** is compact-closed in the sense of [5].

It appears that the category **Conway** does not have finite nor infinite products [9]. For that reason, we compute the free exponential modality in the full subcategory $\mathbf{Conway}^{-}$ of negative Conway games, which is symmetric monoidal closed, and has products. We explain in a later stage how the free construction on the subcategory $\mathbf{Conway}^{-}$ induces a free construction on the whole category.

First step: compute the free affine object. The monoidal unit $\mathbf{1}$ is terminal in the category $\mathbf{Conway}^{-}$. In other words, every negative Conway game may be seen as an affine object in a unique way, by equipping it with the empty strategy $t_A : A \to 1$. In particular, the free affine object $A_\bullet$ is simply A itself.

Second step: compute the symmetric tensor power. A simple argument shows that the equalizer $A^n = A^{\leq n}$ of (7) is the following Conway game:

– the positions of the game A^n are the finite words $w = x_1 \cdots x_n$ of length n, whose letters are positions x_i of the game A, and such that $x_{i+1} = \star_A$ is the root of A whenever $x_i = \star_A$ is the root of A, for every $1 \leq i < n$. The intuition is that the letter x_k in the position $w = x_1 \cdots x_n$ of the game A^n describes the position of the k-th copy of A, and that the $i+1$-th copy of A cannot be opened by Opponent unless all the i-th copy of A has been already opened.
– its root is the word $\star_{A^n} = \star_A \cdots \star_A$ where the n the positions x_k are at the root $\star_A$ of the game A,

– a move $w \to w'$ is a move played in one copy:

$$w_1 \; x \; w_2 \;\to\; w_1 \; y \; w_2$$

where $x \to y$ is a move of the game A. Note that the condition on the positions implies that when a new copy of A is opened (that is, when $x = \star_A$) no position in w_1 is at the root, and all the positions in w_2 are at the root.
– the polarities of moves are inherited from the game A in the obvious way.

Note that A^n may be also seen as the subgame of $A^{\otimes n}$ where the $i+1$-th copy of A is always opened after the i-th copy of A.

Third step: compute the sequential limit. We now consider Diagram (8)

$$A^0 = 1 \longleftarrow A^1 = A \longleftarrow A^2 \longleftarrow A^3 \longleftarrow \cdots$$

whose morphisms are the partial copycat strategies $A^n \leftarrow A^{n+1}$ identifying A^n as the subgame of A^{n+1} where only the first n copies of A are played. The limit of this diagram in the category **Conway**$^{-}$ is the game A^∞ defined in the same way as $A^{\leq n}$ except that its positions $w = x_1 \cdot x_2 \cdots$ are infinite sequences of positions of A, all of them at the root except for a finite prefix $x_1 \cdots x_k$. It is possible to show that A^∞ is indeed the limit of this diagram, and that the tensor product commutes with this limit. From this, we deduce that the sequential limit A^∞ describes the free commutative comonoid in the category **Conway**$^{-}$.

It is nice to observe that the free construction extends to the whole category **Conway** of Conway games. Indeed, one shows easily that every commutative comonoid in the category of Conway games is in fact a negative game. Moreover, the inclusion functor from **Conway**$^{-}$ to **Conway** has a right adjoint, which associates to every Conway game A, the negative Conway game A^{-} obtained by removing all the Proponent moves from the root $\star_A$. By combining these two observations, we obtain that $(A^{-})^\infty$ is the free commutative comonoid generated by a Conway game A in the category **Conway**.

5 Finiteness spaces – an inviting counter-example

In Sections 3 and 4 we have seen how to refine Formula (5) into Formula (9) in order to compute the free exponential modality in the coherence space and the Conway game models. We conclude the paper by explaining why the two formulas do not work in the finiteness space model. Recall that there are two levels of finiteness spaces. On the one hand, *relational* finiteness spaces constitute a refinement of the relational model, while on the other hand *linear* finiteness spaces are linearly topologized vector spaces [7] built on the relational layer. We explain the failure of our two formulas at both levels. We refer the reader to [2] for an introduction to finiteness spaces.

Relational finiteness spaces. Two subsets u, u' of a countable set $\mathbb{E}$ are called orthogonal, denoted by $u \perp u'$, whenever their intersection $u \cap u'$ is finite. The orthogonal of $\mathcal{G} \subseteq \mathcal{P}(\mathbb{E})$ is then defined by $\mathcal{G}^\perp = \{u' \subseteq \mathbb{E} \,|\, \forall u \in \mathcal{G},\; u \perp u'\}$.

A *relational finiteness space* $E = (|E|, \mathcal{F}(E))$ is given by its *web* (a countable set $|E|$) and by a set $\mathcal{F}(E) \subseteq \mathcal{P}(|E|)$ orthogonally closed, i.e. such that $\mathcal{F}(E)^{\perp\perp} = \mathcal{F}(E)$.

The elements of $\mathcal{F}(E)$ (resp. $\mathcal{F}(E)^\perp$) are called *finitary* (resp. *antifinitary*). A finitary relation R between two finiteness spaces E_1 and E_2 is a subset of $|E_1| \times |E_2|$ such that

$$\forall u \in \mathcal{F}(E_1),\ \ R \cdot u \ := \big\{ b \in |E_2| \ \big|\ \exists a \in u,\ (a,b) \in R \big\} \in \mathcal{F}(E_2),$$
$$\forall v' \in \mathcal{F}(E_2)^\perp,\ {}^tR \cdot v' := \big\{ a \in |E_1| \ \big|\ \exists b \in v',\ (a,b) \in R \big\} \in \mathcal{F}(E_1)^\perp.$$

The category **RelFin** of relational finiteness spaces and finitary relations is $*$-autonomous. As such, it provides a model of multiplicative linear logic (MLL).

The exponential modality ! is then defined as follows [2]: given a finiteness space E, the finiteness space $!E$ has its web $|!E| = \mathcal{M}_{\text{fin}}(|E|)$ defined as the set of finite multisets $\mu : |E| \to \mathbb{N}$ and its finiteness structure defined as

$$\mathcal{F}(!E) = \{ M \in \mathcal{M}_{\text{fin}}(|E|) \mid \mathit{\Pi}_E(M) \in \mathcal{F}(E) \},$$

where for every $M \in \mathcal{M}_{\text{fin}}(|E|)$, $\mathit{\Pi}_E(M) \ \stackrel{\text{def}}{=}\ \{ x \in |E| \mid \exists \mu \in M,\ \mu(x) \neq 0 \}$.

Given a finiteness space E, let us compute the finiteness space E^∞ defined by Formula (9). The free pointed space generated by E exists, and is defined as

$$E_\bullet \ \stackrel{\text{def}}{=}\ E \,\&\, 1.$$

The equalizer $E^{\leq n}$ of the $n!$ symmetries exists in **RelFin** and provides the n-th layer of $!E$. Its web $|E^{\leq n}| = \mathcal{M}^{\leq n}_{\text{fin}}(|E|)$ consists of the multisets of cardinality at most n and its finiteness structure is defined as

$$\mathcal{F}(E^{\leq n}) \quad = \quad \{ \quad M_n \subseteq \mathcal{M}^{\leq n}_{\text{fin}}(|E|) \quad | \quad \mathit{\Pi}_E(M_n) \in \mathcal{F}(E) \quad \}.$$

Finally, the limit defined by Formula (9) is given by the finiteness space E^∞ whose web is $|E^\infty| = \mathcal{M}_{\text{fin}}(|E|)$ and whose finiteness structure is

$$\mathcal{F}\left(E^\infty\right) \quad = \quad \left\{ \quad M \in \mathcal{M}_{\text{fin}}(|E|) \quad \middle| \quad \forall n \in \mathbb{N}, \quad \begin{array}{l} M_n = M \cap \mathcal{M}^{\leq n}_{\text{fin}}(|E|), \\ \mathit{\Pi}_E(M_n) \in \mathcal{F}(E). \end{array} \quad \right\}.$$

Note that the webs of $!E$ and of E^∞ are equal, and coincide in fact with the free exponential in the relational model. However, it is obvious that the finiteness structures of $!E$ and E^∞ do not coincide in general:

$$\mathcal{F}(!E) \quad \subsetneq \quad \mathcal{F}\left(E^\infty\right).$$

In fact, Formula (9) does not work here because the sequential limit (8) does not commute with the tensor product. This phenomenon comes from the fact that an infinite directed union of finitary sets is not necessarily finitary in the finiteness space model – whereas an infinite directed union of cliques is a clique in the coherence space model, this explaining the success of Formula (9) in this model. The interested reader will check that Formula (5) computes the same finiteness space E^∞ as Formula (9) because $E^{\leq n}$ coincides with the cartesian product of E^k for $k \leq n$. We now turn to the topological version of finiteness spaces to understand the topological difference between $!E$ and E^∞.

Linear finiteness spaces. Let $\Bbbk$ be an infinite field endowed with the discrete topology. Every relational finiteness space E generates a vector space, the *linear finiteness space*

$$\Bbbk\langle E \rangle = \big\{ x \in \Bbbk^{|E|} \ \big|\ |x| \in \mathcal{F}(E) \big\},$$

where for any sequence $x \in \Bbbk^{|E|}$, $|x| = \{a \in |E| \mid x_a \neq 0\}$. Endowed with a topology defined with respect to the antifinitary parts, $\Bbbk\langle E\rangle$ is a linearly topologized space [7]. The category **LinFin**, with linear finiteness spaces as objects and linear continuous functions as morphisms, is $*$-autonomous and provides a model of MLL.

We now consider $\Bbbk\langle E^\infty\rangle$ and $\Bbbk\langle !E\rangle$, or more precisely their duals since the functional definition is more intuitive. In **LinFin**, the dual space $\Bbbk\langle E\rangle^\perp = (\Bbbk\langle E\rangle \multimap \Bbbk)$ consists of continuous linear forms and is endowed with the topology of uniform convergence on *linearly compact subspaces*, i.e. subspaces $K \subseteq \Bbbk\langle E\rangle$ that are closed and have a finitary support $|K| \stackrel{\text{def}}{=} \cup_{x\in K}|x|$.

It appears that $\Bbbk\langle E^\infty\rangle^\perp$ is the space of *polynomials*[1]. However, thanks to the Taylor formula shown in [2], the functions in $\Bbbk\langle !E\rangle^\perp$ are *analytic*, i.e. they coincide with the limits of converging sequences of polynomials. Moreover, the topology of $\Bbbk\langle E^\infty\rangle^\perp$ is generated by the subspaces whose restrictions to polynomials of degree at most n are opens. This topology differs from the linearly compact open topology. Therefore, $\Bbbk\langle E^\infty\rangle^\perp$ is topologically different from $\Bbbk\langle !E\rangle^\perp$, which is the *completion* of the space of polynomials, endowed with the linearly compact open topology as shown in [1].

In a word, the dual of $\Bbbk\langle E^\infty\rangle$ gives rise to a simple space of computation, the polynomials. Its topology is related to the local information given at each degree. On the contrary, the dual of the exponential modality $\Bbbk\langle !E\rangle$ gives rise to the richer space of analytic functions, where the Taylor formula makes sense. Its topology is related to a global information which is not reduced to its finite approximations. One main open question in the future is to understand the algebraic nature of this exponential construction, as was achieved here for the coherence space and the Conway game model.

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[1] An *homogeneous polynomial* of degree n is a function $P : \Bbbk\langle E\rangle \to \Bbbk$ which is associated with a symmetric n-linear form $\phi : \Bbbk\langle E\rangle \times \cdots \times \Bbbk\langle E\rangle \to \Bbbk$ which is hypocontinuous (a notion of continuity in between continuity and separate continuity) such that $P(x) = \phi(x, \ldots, x)$. A *polynomial* is then a finite linear combination of homogeneous polynomials.