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A sliding mode observer for linear systems with unknown time-varying delay

Alexandre Seuret, Thierry Floquet, Jean-Pierre Richard and Sarah K. Spurgeon

Abstract

The design of observers for linear systems with unknown, time-varying, bounded delays (on the state and on the input) still constitutes an open problem. In this paper, we show how to solve it for a class of systems by combining the sliding mode observer approach with an adequate choice of a Lyapunov-Krasovskii functional. This result provides workable conditions in terms of rank assumptions and LMI conditions. The dynamic properties of the observer are also analyzed. A 4th-order example is proposed to study the feasibility.


1. Introduction

State observation is an important issue for both linear and nonlinear systems. This work considers the observation problem in the case of linear systems with unknown delay. Several authors proposed observers for delay systems (see, e.g., [21, 20]). Most of them, as it is pointed out in [20], consider that the value of the delay (mainly constant) can be involved in the observer realization. Concretely, this means that the delay is known or measured. Likewise, what is defined as “observers without internal delay” [4, 5, 11] involves the output knowledge at the present and delayed instants. Besides, in [16] was designed a finite-dimensional observer (thus, without delay) since it was constructed just for the finite set of unstable or poorly damped modes of the delay system. However, the determination of these modes, here again, requires the delay knowledge.

Yet, in concrete applications (for instance teleoperation, or networked systems), the delay invariance and delay knowledge remain assumptions coming more from the identification and analysis limits than from technical facts. There are presently only few results in which the observer does not assume the delay knowledge [2, 3, 7, 14, 26]. These interesting approaches consider linear systems and guarantee an $H_{\infty}$ performance for the filtering error. Those approaches are based on i.o.d stability techniques (independent of the delay). So it should be interesting to reduce the probable conservatism of such results by taking into account the information on a delay upper-bound.

In this paper, we propose a method to solve the problem of the observation of linear systems with unknown time delays by combining some results on sliding mode observers (see, e.g., [1, 8, 9, 12, 19]) with an adequate choice of a Lyapunov-Krasovskii functional. The observer dynamical properties will also be discussed. For the sake of simplicity, the unknown time delay $h(t)$ is assumed to be the same for the state and the input. In order to reduce the conservatism of the worked out conditions, it is supposed to have a known upper bound $h_m$ so that:

$$0 \leq h(t) \leq h_m, \forall t \in \mathbb{R}_+.$$ 

Throughout the article, the notation $P > 0$ for $P \in \mathbb{R}^{n \times n}$ means that $P$ is a symmetric and positive definite matrix. $[A_1 | A_2 | \ldots | A_n]$ is the concatenated matrix with matrices $A_i$. $I_n$ represents the $n \times n$ identity matrix. Finally, $\text{Sym}(P) = (P + P^T)$.

2. Problem statement

Let us consider the linear time-invariant system with state and input delay:

$$\begin{align*}
\dot{x}(t) &= A x(t) + A_p x(t-h(t)) + B u(t) + B_h u(t-h(t)) + D \zeta(t) \\
y(t) &= C x(t) \\
x(s) &= \phi(s), \quad \forall s \in [-h_m,0]
\end{align*}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^q$ are the state vector, the input vector and the measurement vector, respectively. $\zeta \in \mathbb{R}^r$ is an unknown and bounded perturbation that satisfies:

$$||\zeta(t)|| \leq \alpha_1(t,y,u),$$

where $\alpha_1$ is a known scalar function. $\phi \in C^0([-h_m,0], \mathbb{R}^n)$ is the vector of initial conditions. It is assumed that
A. $A$, $B$, $B_h$, $C$ and $D$ are constant known matrices of appropriate dimensions. The following structural assumptions are required for the design of the observer:

A1. $\text{rank}(C[A_k|B_h|D]) = \text{rank}([A_k|B_h|D]) \triangleq p$, $p < q \leq n$.

A2. The invariant zeros of $(A, [A_k|B_h|D], C)$ lie in $\mathbb{C}^-$. 

Under those assumptions and using the same linear change of coordinates as in [10], Chapter 6, the system can be transformed into:

$$
\begin{align*}
\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \\
\dot{x}_2(t) &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) \\
&+ G_1x_1(t - h(t)) + G_2x_2(t - h(t)) + G_uu(t - h(t)) + D_1\xi(t), \\
y(t) &= Tx_2(t),
\end{align*}
$$

where $x_1 \in \mathbb{R}^{n-q}$, $x_2 \in \mathbb{R}^q$ and where $G_1$, $G_2$, $G_u$, $D_1$ and $A_{21}$ are defined by:

$$
G_1 = \begin{bmatrix} 0 \\ G_1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ G_2 \end{bmatrix}, \quad G_u = \begin{bmatrix} 0 \\ G_u \end{bmatrix}, \quad D_1 = \begin{bmatrix} A_{211} \\ A_{212} \end{bmatrix}, \quad A_{21} = \begin{bmatrix} A_{211} \\ A_{212} \end{bmatrix},
$$

with $G_1 \in \mathbb{R}^{p \times (n-q)}$, $G_2 \in \mathbb{R}^{p \times q}$, $G_u \in \mathbb{R}^{p \times m}$, $D_1 \in \mathbb{R}^{p \times r}$, $A_{21} \in \mathbb{R}^{(q-p) \times (n-q)}$, $A_{211} \in \mathbb{R}^{p \times p}$, $A_{212} \in \mathbb{R}^{p \times (n-q)}$ and $T$ an orthogonal matrix involved in the change of coordinates given in [10].

Under conditions $A$, the system can be decomposed in two subsystems. $A_1$ implies that the unmeasurable state $x_1$ is not affected by the delayed terms and the perturbations. $A_3$ ensures that the pair $(A_1, A_{211})$ is at least detectable.

In this article, the following lemma will be used:

**Lemma 1 [15]** For any matrices $A$, $P_0 > 0$ and $P_1 > 0$, the inequality

$$
ATP_1A - P_0 < 0,
$$

is equivalent to the existence of a matrix $Y$ such that:

$$
\begin{bmatrix} -P_0 & A^TY^T \\ YA & -Y - Y^T + P_1 \end{bmatrix} < 0.
$$

3. Observer design

Let us define the following sliding mode observer:

$$
\begin{align*}
\dot{\hat{x}}_1(t) &= A_{11}\hat{x}_1(t) + A_{12}\hat{x}_2(t) + B_1\hat{u}(t), \\
\dot{\hat{x}}_2(t) &= A_{21}\hat{x}_1(t) + A_{22}\hat{x}_2(t) + B_2\hat{u}(t) \\
&+ (LT^TG_1T)\dot{x}_1(t) + G_2\hat{x}_2(t - \hat{h}) + G_u\hat{u}(t - \hat{h}) \\
&+ G_1\hat{x}_1(t - \hat{h}) + G_2\hat{x}_2(t - \hat{h}) + G_u\hat{u}(t - \hat{h}) \\
&- G_1L_1(x_1(t - \hat{h}) - \hat{x}_1(t - \hat{h})) - T^T\nu(t), \\
\dot{\hat{y}}(t) &= Tx_2(t)
\end{align*}
$$

where the linear gain $G_1$ is an Hurwitz matrix and $L$ has the form $[\bar{L} \ 0]$ with $\bar{L} \in \mathbb{R}^{(n-q) \times (q-p)}$. The computed delay $\hat{h} \leq h_m$ is an implemented value that can be chosen according to the parameters of the system. It could also be time-varying. For instance $\hat{h}$ could be equal to some expected nominal estimation of the time-varying delay. The discontinuous injection term $v$ is given by:

$$
v(t) = \begin{cases} -P(t, y, u) |_{t \geq y(t) - \hat{y}(t)} & \text{if } y(t) - \hat{y}(t) \neq 0, \\
0 & \text{otherwise.}
\end{cases}
$$

where $P > 0$, $P \in \mathbb{R}^{p \times p}$ and where $p$ is a nonlinear positive gain yet to be defined. Note that the non delayed terms depending on $x_2$ are known because $x_2(t) = T^T y(t)$.

Defining the state estimation errors as $e_1 = x_1(t) - \hat{x}_1(t)$ and $e_2 = x_2(t) - \hat{x}_2(t)$, one obtains:

$$
\begin{align*}
\dot{e}_1(t) &= A_{11}e_1(t) - L(T^TG_1Te_2(t) + T^Tv(t)) \\
\dot{e}_2(t) &= A_{21}e_1(t) + G_1e_1(t - h(t)) + D_1\xi(t) \\
&+ T^Tv(t) + \xi_0(t) + (T^TG_1T + A_{21}L)e_2(t) \\
&+ G_1Le_2(t - \hat{h}),
\end{align*}
$$

where $\xi_0 : \mathbb{R} \rightarrow \mathbb{R}^p$ is given by:

$$
\xi_0(t) = G_1(x_1(t - h(t)) - \hat{x}_1(t - \hat{h})) + G_2(x_2(t - h(t)) - x_2(t - \hat{h})) + G_u(u(t - h(t)) - u(t - \hat{h})).
$$

Let us introduce the change of coordinates $\begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} = T_t \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ with $T_t = \begin{bmatrix} I_{n-q} & L \\ 0 & T \end{bmatrix}$. Using the fact that $L_1G_1 = L_2G_2 = L_3G_u = LD_1 = 0$, one obtains:

$$
\begin{align*}
\dot{\hat{\xi}}_1(t) &= (A_{11} + 0)\hat{\xi}_1(t), \\
\dot{\hat{\xi}}_2(t) &= T\hat{\xi}_1(t) + T_1G_1\hat{\xi}_1(t - h(t)) + G_1\hat{\xi}_2(t) + v(t) + T_2\xi\dot{c}(t) + T_1D_1\xi(t),
\end{align*}
$$

with

$$
\xi = G_1(x_1(t - h(t)) - \hat{x}_1(t - \hat{h})) + G_2(x_2(t - h(t)) - x_2(t - \hat{h})) + G_u(u(t - h(t)) - u(t - \hat{h})),
$$

that can be rewritten as:

$$
\xi(t) = \int_{t - \hat{h}}^{t} \begin{bmatrix} G_1 & G_2 - G_1L & G_u \end{bmatrix} \begin{bmatrix} \hat{x}_1(s) \\ \hat{x}_2(s) \\ \hat{u}(s) \end{bmatrix} ds.
$$
The function $\xi$ only depends on the known variables $\xi_1$, $x_2$, $e_2$ and $u$ and on the unknown delay $h(t)$. One can then assume that there exists a known scalar function $\alpha_2$ such that:

$$\|\xi(t)\| \leq \alpha_2(t, \xi_1, x_2, e_2, u).$$  

(9)

**Remark 1** Note that the nearest the available estimation $h$ of $h(t)$, the smallest the bound $\alpha_2$ (indeed $h = h(t)$ implies $\xi = 0$). This means that an available information on the delay size order allows for reducing the observer gain. Furthermore, as it will be seen hereafter, a value $h_m$ of the upper-bound of the admissible time delays will be available via an LMI formulation.

It is now possible to define more precisely the "discontinuous gain" $\rho$, using the same technique for the design of the sliding mode control law in [13]:

$$\rho(t, y, u) = \|D_1\|\alpha_1(t, y, u) + \alpha_2(t, \xi_1, x_2, e_2, u) + \gamma,$$  

(10)

with $\gamma$ a positive real number. Then the following result holds:

**Theorem 1** Under the conditions $A$ and (9) and for any Hurwitz matrix $G_1$, the system (7) is asymptotically stable for all delay $h(t) \leq h_m$ if there exist symmetric positive definite matrices $P_1$ and $R_1 \in \mathbb{R}^{(n-q)\times(n-q)}$, $P_2 \in \mathbb{R}^{n\times n}$, a symmetric matrix $Z_2 \in \mathbb{R}^{n\times n}$ and a matrix $W \in \mathbb{R}^{(n-q)\times(q-p)}$ such that the following LMI conditions are satisfied:

$$\begin{bmatrix}
\Psi_0 \\
A_1^T P_1 + A_1^T W^T (A_2 + G_1) \bar{P}_1 P_2 \\
-2P_1 + h_m R_1 \\
* \\
G_1^T P_2 + P_1 G_1 + h_m Z_2
\end{bmatrix} < 0,$$  

(11)

and

$$\begin{bmatrix}
R_1 \\
(T G_1)^T P_2 \\
Z_2
\end{bmatrix} \geq 0,$$  

(12)

where

$$\Psi_0 = A_{11}^T P_1 + P_1 A_{11} + A_{21}^T W^T + W A_{211}.$$  

The observer gain $L$ is then given by $L = P_1^{-1} W$.

Consider the candidate for a Lyapunov-Krasovskii functional:

$$V(t) = e_1^T(t) P_1 e_1(t) + e_2^T(t) P_2 e_2(t) + \int_{t-h(t)}^{t} \int_{0}^{h(t)} e_1^T(s) R_1 e_1(s) ds ds.$$  

(13)

Using the following transformation:

$$\hat{e}_1(t-h(t)) = \hat{e}_1(t) - \int_{t-h(t)}^{t} \hat{e}_1(s) ds,$$  

(14)

and differentiating (13) along the trajectories of (7), one gets:

$$V(t) = e_1^T(t) \underbrace{\{\text{Sym} \{ (A_{11} + L A_{211})^T P_1 \} \}}_{\Psi_1} e_1(t)$$  

+ \int_{t-h(t)}^{t} \int_{0}^{h(t)} e_1^T(s) R_1 e_1(s) ds ds,$$  

(15)

where

$$\eta_1(t) = -2 e_1^T(t) P_1 T G_1 \int_{t-h(t)}^{t} \hat{e}_1(s) ds,$$  

and

$$\eta_2(t) = 2 e_2^T(t) P_2 T [D_1 \xi(t) + \xi(t)].$$  

The LMI condition (12) implies that for any vector $X):

$$X^T \begin{bmatrix}
P_1 & (T G_1)^T P_2 \\
P_2 T G_1 & Z_2
\end{bmatrix} X \geq 0,$$  

Developing this relation for $X = \begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \end{bmatrix}$, one has:

$$-2 e_2^T(t) P_2 G_1 \hat{e}_1(t) \leq e_2^T (t) Z_2 e_2(t) + e_1^T(t) \hat{e}_2(t).$$  

(16)

By integrating this inequality with respect to the $s$ variable, one can upperbound $\eta_1(t)$:

$$\eta_1(t) \leq \int_{t-h(t)}^{t} e_2^T(s) Z_2 e_2(t) ds + \int_{t-h(t)}^{t} e_1^T(s) R_1 \hat{e}_1(s) ds,$$  

(17)

Under the definition (10) of $\rho$ and since $T$ is an orthogonal matrix:

$$\eta_2(t) = 2 \rho(t, y, u) \|P_2 e_2(t)\| \leq -2 \|P_2 e_2(t)\|.$$  

(18)

Taking into account (16), (17) and that $\hat{e}_1(t) = (A_{11} + L A_{211}) \hat{e}_1(t)$, $V$ can be upperbounded as follows:

$$V(t) \leq \underbrace{\text{Sym} \{ e_1^T(t) P_1 (A_{11} + L A_{211}) \hat{e}_1(t) + e_2^T(t) P_2 (A_{21} + G_1) \hat{e}_1(t) \}}_{\Psi_1}$$  

$$+ \underbrace{h_m e_1^T(t) (A_{11} + L A_{211})^T R_1 (A_{11} + L A_{211}) \hat{e}_1(t) \}}_{\Psi_3}$$  

$$+ \underbrace{h_m e_2^T(t) Z_2 \hat{e}_2(t) - \gamma \|P_2 e_2(t)\| \}}_{\Psi_0}.$$  

(19)

This can be rewritten as:

$$V(t) \leq \begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \end{bmatrix} \Psi \begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \end{bmatrix} - \gamma \|P_2 e_2(t)\|,$$  

(20)

with

$$\Psi = \begin{bmatrix}
\Psi_0 \\
A_{11}^T P_1 + P_1 A_{11} + A_{21}^T W^T + W A_{211} \\
(21)

(22)

and

$$\Psi_3 = (A_{11} + L A_{211})^T P_1 + P_1 (A_{11} + L A_{211})$$  

$$+ h_m (A_{11} + L A_{211})^T R_1 (A_{11} + L A_{211}).$$  

One can note that (19) is not a LMI condition since there are some nonlinear terms in the first row and the first column. Nevertheless, this problem can be transformed into an LMI condition using Lemma 1:

$$\begin{bmatrix}
\Psi_0 \\
\Psi_1 (211) T^T P_2 \bar{Y} + Y \bar{Y}^T + h_m R_1 \\
* \\
\bar{Y}^T \bar{Y} \end{bmatrix} < 0.$$  

(22)

Let us set $Y = P_1$ and define $W = P_2 L$. The LMI conditions of Theorem 1 appear. Thus, if (11) and (12) are satisfied, (20) is also satisfied. This implies that the observation error is asymptotically stable.
4. Dynamic properties of the observer

4.1. Finite time convergence on the sliding manifold

**Corollary 1** Under the observer design of Theorem 1, an ideal sliding motion takes place on $S_0 = \{ \dot{x}_2 = 0 \}$ in finite time.

Consider the Lyapunov function:

$$V_2(t) = \bar{e}_2^T(t)P_2\bar{e}_2(t) \quad (21)$$

Differentiating along the trajectories of (7), one obtains:

$$\dot{V}_2(t) = -\bar{e}_2^T(t)(G^T_2P_2 + P_2G_1)\bar{e}_2(t) + 2\bar{e}_2^T(t)P_2 [T^T Y + A_21\dot{e}_1(t) + G_1\dot{e}_1(t-h(t))] + D_1\dot{\xi}(t) + \bar{\xi}(t)].$$

Using the fact that $G_1$ is Hurwitz and (5), one can write the following upper bound for $V_2(t)$:

$$V_2(t) \leq 2\|P_2\bar{e}_2(t)\| [\|A_21\dot{e}_1(t) + G_1\dot{e}_1(t-h(t))\| \leq \gamma - \delta].$$

From Theorem 1, the error $e_1$ is asymptotically stable. Thus, there exist an instant $t_0$ and a positive scalar $\delta$ such that:

$$\forall t \geq t_0, \quad \|A_21\dot{e}_1(t) + G_1\dot{e}_1(t-h(t))\| \leq \gamma - \delta$$

This leads to:

$$\forall t \geq t_0, \quad \dot{V}_2(t) \leq -2\delta\|P_2\bar{e}_2(t)\| \leq -2\delta\sqrt{\lambda_{\min}(P_2)}\sqrt{V_2(t)}.$$

where $\lambda_{\min}(P_2)$ is the smallest eigenvalue of $P_2$. Integrating the last differential inequation, it follows that an ideal sliding motion takes place on $S_0$ in finite time.

4.2. Exponential stability

In this part, the observer convergence is improved by giving a criteria of exponential convergence. Exponential stability properties could be an interesting way to characterize the convergence rate of the observers. As in [18, 22], for some given rate $\alpha > 0$, a system (7) is said to be $\alpha$-stable, or “exponentially stable with the rate $\alpha$”, if there exists a scalar $\beta \geq 1$ such that the solution $e(t; t_0, \phi)$ of (7), with any initial function $\phi$, satisfies:

$$|e(t; t_0, \phi)| \leq \beta \|e^{-\alpha(t-t_0)}.$$  \hspace{1cm} (22)

In spite of the unknown and variable delay, the following Theorem ensures that the observer dynamics is $\alpha$-stable.

**Theorem 2** Under conditions A and (9), the system (7) is $\alpha$-stable for any delay $h(t) \leq h_m$ if there exist symmetric positive definite matrices $P_1$, $R_1$ and $R_2 \in \mathbb{R}^{(n-q) \times (n-q)}$, $P_2 \in \mathbb{R}^{q \times q}$, a symmetric matrix $Z_2 \in \mathbb{R}^{q \times q}$ and a matrix $W \in \mathbb{R}^{(n-q) \times (q-p)}$ such that the following LMI conditions are satisfied:

$$\begin{bmatrix}
\psi \ & \ A_1^T P_1 A_1 + A_1^T W^T + \alpha P_1 - 2P_2 + h_m R_1 \\
0 \ & \ (A_2 + b_0 G_1)^T T^T P_2 + Y + 2a P_2 + h_m Z_2 \\
\end{bmatrix} \leq 0,$$

where

$$\psi = \begin{bmatrix}
\lambda_1 & \alpha \ & \ * \\
\ * \ & \ * \ & \ * \\
\ * \ & \ * \ & \ * \\
0 \ & \ 0 \ & \ b_0 P_2 T G_1 \\
0 \ & \ h_m b_0 P_2 T G_1 & \ -R_1 \\
-2 R_1 & \ b_0 (T G_1)^T P_2 & \ Z_2 \\
\end{bmatrix} \geq 0. \quad (24)$$

Let us introduce the new variable $\bar{e}_2^a(t) = e^{\alpha t} \bar{e}_2(t)$ in (7). Then, the asymptotic convergence of $\bar{e}_2^a$ implies that $\bar{e}$ is $\alpha$-stable. Equation (7) becomes:

$$\begin{cases}
\bar{e}_2^a(t) = (A_1 + L A_{21} + \alpha a h_{-p})\bar{e}_2^a(t), \\
\bar{e}_2^a(t) = (\alpha (A_{21} + L A_{21} + \alpha a h_{-p} - \rho) \bar{e}_2^a(t) + T \bar{e}_2^a(t) + (\nu + T \xi(t)) e^{\alpha t} + e^{\alpha (t-h(t))} T G_1 \bar{e}_2^a(t-h(t)) + (G_1 + \alpha G_1) \bar{e}_2^a(t), \\
\end{cases} \quad (25)$$

Note that $e^{\alpha (t-h(t))} = b_0 + \Delta(t) b_m$, where $\Delta(t)$ is an unknown scalar function satisfying $\|\Delta(t)\| \leq 1$. Consider the following candidate for a Lyapunov-Krasovskii functional:

$$V^a(t) = e_1^a(t) P_1 e_1^a(t) + e_2^a(t) P_2 e_2^a(t) + 2\int_{t-h(t)}^{t} e_1^a(s) R_1 e_1^a(s) ds$$

Differentiating (26) along (25), one gets:

$$\dot{V}^a(t) = e_1^a(t) P_1 e_1^a(t) + e_2^a(t) P_2 e_2^a(t) + 2\int_{t-h(t)}^{t} e_1^a(s) R_1 e_1^a(s) ds.$$ \hspace{1cm} (27)

where

$$\begin{align*}
\eta_1^a(t) &= 2 e_2^a(t) P_2 b_0 T G_1 \bar{e}_2^a(s) ds, \\
\eta_2^a(t) &= 2 e_2^a(t) P_2 b_0 \Delta(t) T G_1 e_1^a(s) ds, \\
\eta_3^a(t) &= 2 e_2^a(t) P_2 b_0 \Delta(t) T G_1 \int_{t-h(t)}^{t} \bar{e}_2^a(s) ds, \\
\eta_4^a(t) &= 2 e_2^a(t) P_2 (v + T \xi(t) + T D_1 \xi(t)) e^{\alpha t}. \\
\end{align*}$$

Following the lines of Theorem 1, (24) gives a majoration of $\eta_1^a$:

$$\eta_1^a(t) \leq h_m e_1^a(t) Z_2 e_2^a(t) + \int_{t-h(t)}^{t} e_1^a(s) R_1 e_1^a(s) ds.$$  \hspace{1cm} (28)
For any \( n \times n \) matrix \( R > 0 \) and for any vectors \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R}^n \):

\[
\pm 2a^T b \leq a^TR^{-1}a + b^TRb
\]  

(29)

Let us apply (29) for \( \eta_2^a(t) \) with:

\[
a^T = \dot{\eta}_2^a(t)P_2b_0TG_1\Delta(t)
\]

\[
b = \dot{\eta}_2^a(t)
\]

\[
R = R_2
\]

One gets:

\[
\eta_2^a(t) \leq \dot{\eta}_2^a(t)P_2b_0TG_1\Delta(t) + \dot{\eta}_2^a(t)R_2\dot{\eta}_2^a(t).
\]

(30)

Using again (29) for \( \eta_3^a(t) \) with:

\[
a^T = \dot{\eta}_2^a(t)P_2b_0TG_1\Delta(t)
\]

\[
b = \dot{\eta}_2^a(s)
\]

\[
R = R_1
\]

one has:

\[
\eta_3^a(t) \leq \int_{h_m}^{h_m+s} (\dot{\eta}_2^a(t))P_2b_0TG_1\Delta(t)
\]

\[+ \int_{h_m}^{h_m+s} (\dot{\eta}_2^a(s))R_1\dot{\eta}_2^a(s)ds.
\]

(31)

With the discontinuous output injection \( v \) defined in (5) and (10), one has:

\[
\eta_3^a(t) \leq -2\gamma\|P_2\dot{\eta}_2^a(t)\|.
\]

(32)

Then, combining (28-32) with (27) leads to:

\[
\Psi^a(t) \leq \left[ \begin{array}{c}
\dot{\eta}_1(t) \\
\dot{\eta}_2(t)
\end{array} \right]^T \Psi^a \left[ \begin{array}{c}
\dot{\eta}_1(t) \\
\dot{\eta}_2(t)
\end{array} \right] - 2\gamma\|P_2\dot{\eta}_2^a(t)\|,
\]

(33)

with:

\[
\Psi^a = \begin{bmatrix}
\psi_{11}^a & (A_{21} + G_1)tT \cdot P_2 \\
0 & \psi_{22}^a
\end{bmatrix},
\]

\[
\psi_{11}^a = \text{Sym} \{P_1(A_{11} + \dot{A}_{211} + \alpha \dot{A}_{n-p}) + (A_{11} + \dot{A}_{211} + \alpha \dot{A}_{n-p})T \cdot R_1(A_{11} + \dot{A}_{211} + \alpha \dot{A}_{n-p})T \}
\]

\[
\psi_{22}^a = G_1^T P_2 + \dot{P}_2 + 2\alpha P + h_m Z_2
\]

\[+ h_m b_0 P_2 G_1 R_1^{-1} (G_1)T P_2 b_m
\]

\[+ b_0 P_2 G_1 R_1^{-1} (G_1)T P_2 b_m.
\]

Again, applying Lemma 1 as in Theorem 1 yields:

\[
\begin{bmatrix}
\psi_{11}^a & (A_{21} + G_1)tT \cdot P_2 \\
0 & \psi_{22}^a
\end{bmatrix} < 0,
\]

(34)

Set \( Y = P_2G_1 \). Then using the Schur complement, one finds the LMI condition (23). Thus, if (23) and (24) are satisfied, the time derivative of (26) is negative definite.

### 4.3. Optimization Problem

This paragraph focusses on the optimization of the exponential decay rate \( \alpha \). The greater \( \alpha \) is, faster the error dynamics converge to the solution \( e(t) = 0 \). The optimization consists in finding the greatest \( \alpha \) (guaranteed speed performance of the application) such that the closed-loop system is \( \alpha \)-stable. This corresponds to the problem:

\[
\max \alpha
\]

subject to (23) and (24) for a given \( h_m \).

Because \( \alpha \) does not appear in a linear form in (23) and (24), this problem is solved by iteratively increasing \( \alpha \) until the LMI conditions become unfeasible.

### 5. Example

Consider the system with time-varying delay (3) with:

\[
A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix},
\]

\[
A_{21} = \begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}, \quad A_{32} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},
\]

\[
G_1 = \begin{bmatrix} 0.1 & 0.21 \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 1 \\ 0.2 & 1 \end{bmatrix},
\]

\[
T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_u = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_1 = B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

The delay is chosen as \( h_m = \frac{h}{12}(1 + \sin(\omega_0 t)) \), with \( h_m = 0.3s \) an frequency \( \omega_0 = 0.5s^{-1} \). The control law is:

\[
u(t) = u_0 \sin(\omega_0 t)
\]

with \( u_0 = 2 \) and \( \omega_0 = 3 \).

Since the system (3) is open loop stable, its dynamics are bounded. Thus the function \( \alpha e(t, \dot{x}_1, x_2, e_2, u) \) could be chosen as a constant \( K = 0.7 \).

The simulation results are given in the following figures. In Figures 1 and 2 are reported the observation errors of the system for \( \alpha = 0 \) and \( \alpha = 2 \).

Figures 3 and 4 show the comparison between the real and observed states, for \( \alpha = 2 \).

It can be noticed that the greater \( \alpha \) is, the faster the error convergence is. Using Theorem 2, the following observer gains for \( \alpha = 2 \) are obtained:

\[
\tilde{L} = \begin{bmatrix} -3.8658 & 1.0722 \\ -8.8160 & -5.8154 \end{bmatrix}, \quad G_t = \begin{bmatrix} -6.0190 & -32.0670 \end{bmatrix}
\]

### 6. Conclusion

The problem of designing observers for linear systems with unknown variable delay on both input and state has
been solved in this article. Delay-dependent LMI conditions have been found to guarantee asymptotic stability of the dynamical error system. In addition, the dynamics properties of the proposed observer can be characterized through finite time and exponential convergence properties.

References

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