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A sliding mode observer for linear systems with unknown time varying delay

Alexandre Seuret, Thierry Floquet, Jean-Pierre Richard and Sarah K. Spurgeon ‡

Abstract

The design of observers for linear systems with unknown, time-varying, bounded delays (on the state and on the input) still constitutes an open problem. In this paper, we show how to solve it for a class of systems by combining the sliding mode observer approach with an adequate choice of a Lyapunov-Krasovskii functional. This result provides workable conditions in terms of rank assumptions and LMI conditions. The dynamic properties of the observer are also analyzed. A 4th-order example is proposed to study the feasibility.

Sliding Mode Observer, Time-Delay Systems, Unknown Delay, Linear Matrix Inequalities.

1. Introduction

State observation is an important issue for both linear and nonlinear systems. This work considers the observation problem in the case of linear systems with unknown delay. Several authors proposed observers for delay systems (see, e.g., [21, 20]). Most of them, as it is pointed out in [20], consider that the value of the delay (mainly constant) can be involved in the observer realization. Concretely, this means that the delay is known or measured. Likewise, what is defined as "observers without internal delay" [4, 5, 11] involves the output knowledge at the present and delayed instants. Besides, in [16] was designed a finite-dimensional observer (thus, without delay) since it was constructed just for the finite set of unstable or poorly damped modes of the delay system. However, the determination of these modes, here again, requires the delay knowledge.

Yet, in concrete applications (for instance teleoperation, or networked systems), the delay invariance and delay knowledge remain assumptions coming more from the identification and analysis limits than from technical facts. There are presently only few results in which the observer does not assume the delay knowledge [2, 3, 7, 14, 26]. These interesting approaches consider linear systems and guarantee an H_∞ performance for the filtering error. Those approaches are based on i.o.d stability techniques (independent of the delay). So it should be interesting to reduce the probable conservatism of such results by taking into account the information on a delay upper-bound.

In this paper, we propose a method to solve the problem of the observation of linear systems with unknown time delays by combining some results on sliding mode observers (see, e.g., [1, 8, 9, 12, 19]) with an adequate choice of a Lyapunov-Krasovskii functional. The observer dynamical properties will also be discussed. For the sake of simplicity, the unknown time delay $h(t)$ is assumed to be the same for the state and the input. In order to reduce the conservatism of the worked out conditions, it is supposed to have a known upper bound h_m so that:

$$0 \leq h(t) \leq h_m, \ \forall t \in \mathbb{R}_+.$$

Throughout the article, the notation $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is a symmetric and positive definite matrix. $[A_1|A_2|...|A_n]$ is the concatenated matrix with matrices A_i. I_n represents the $n \times n$ identity matrix. Finally, $Sym\{P\} = (P + P^T)$.

2. Problem statement

Let us consider the linear time-invariant system with state and input delay:

$$\left\{\begin{array}{ll} \dot{x}(t) = & Ax(t) + A_h x(t-h(t)) \\ & +Bu(t) + B_h u(t-h(t)) + D\zeta(t) \\ y(t) = & Cx(t) \\ x(s) = & \phi(s), \quad \forall\, s \in [-h_m, 0] \end{array}\right. \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^q$ are the state vector, the input vector and the measurement vector, respectively. $\zeta \in \mathbb{R}^r$ is an unknown and bounded perturbation that satisfies:

$$\|\zeta(t)\| \leq \alpha_1(t, y, u), \tag{2}$$

where α_1 is a known scalar function. $\phi \in C^0([-h_m, 0], \mathbb{R}^n)$ is the vector of initial conditions. It is assumed that

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A, A_h, B, B_h, C and D are constant known matrices of appropriate dimensions. The following structural assumptions are required for the design of the observer:

A1. $\text{rank}(C[A_h|B_h|D]) = \text{rank}([A_h|B_h|D]) \triangleq p$,

A2. $p < q \leq n$,

A3. The invariant zeros of $(A, [A_h|B_h|D], C)$ lie in $\mathbb{C}^-$.

Under those assumptions and using the same linear change of coordinates as in [10], Chapter 6, the system can be transformed into:

$$\left\{\begin{array}{ll} \dot{x}_1(t) = & A_{11}x_1(t)+A_{12}x_2(t)+B_1u(t), \\ \dot{x}_2(t) = & A_{21}x_1(t)+A_{22}x_2(t)+B_2u(t) \\ & +G_1x_1(t-h(t))+G_2x_2(t-h(t)) \\ & +G_uu(t-h(t))+D_1\zeta(t), \\ y(t) = & Tx_2(t), \end{array}\right. \tag{3}$$

where $x_1 \in \mathbb{R}^{n-q}$, $x_2 \in \mathbb{R}^q$ and where G_1, G_2, G_u, D_1 and A_{21} are defined by:

$$G_1 = \left[\begin{array}{c} 0 \\ \bar{G}_1 \end{array}\right], G_2 = \left[\begin{array}{c} 0 \\ \bar{G}_2 \end{array}\right], G_u = \left[\begin{array}{c} 0 \\ \bar{G}_u \end{array}\right],$$
$$D_1 = \left[\begin{array}{c} 0 \\ \bar{D}_1 \end{array}\right], A_{21} = \left[\begin{array}{c} A_{211} \\ A_{212} \end{array}\right],$$

with $\bar{G}_1 \in \mathbb{R}^{p\times(n-q)}$, $\bar{G}_2 \in \mathbb{R}^{p\times q}$, $\bar{G}_u \in \mathbb{R}^{p\times m}$, $\bar{D}_1 \in \mathbb{R}^{p\times r}$, $A_{211} \in \mathbb{R}^{(q-p)\times(n-q)}$, $A_{212} \in \mathbb{R}^{p\times(n-q)}$ and T an orthogonal matrix involved in the change of coordinates given in [10].

Under conditions A, the system can be decomposed in two subsystems. $A1$ implies that the unmeasurable state x_1 is not affected by the delayed terms and the perturbations. $A3$ ensures that the pair (A_{11}, A_{211}) is at least detectable.

In this article, the following lemma will be used:

Lemma 1 *[15] For any matrices A, $P_0 > 0$ and $P_1 > 0$, the inequality*

$$A^TP_1A - P_0 < 0,$$

is equivalent to the existence of a matrix Y such that:

$$\left[\begin{array}{cc} -P_0 & A^TY^T \\ YA & -Y-Y^T+P_1 \end{array}\right] < 0.$$

3. Observer design

Let us define the following sliding mode observer:

$$\left\{\begin{array}{ll} \dot{\hat{x}}_1(t) = & A_{11}\hat{x}_1(t)+A_{12}x_2(t)+B_1u(t) \\ & +\left(LT^TG_lT-A_{11}L\right)(x_2(t)-\hat{x}_2(t))+LT^T\nu(t) \\ \dot{\hat{x}}_2(t) = & A_{21}\hat{x}_1(t)+A_{22}x_2(t)+B_2u(t) \\ & -\left(A_{21}L+T^TG_lT\right)(x_2(t)-\hat{x}_2(t)) \\ & +G_1\hat{x}_1(t-\hat{h})+G_2x_2(t-\hat{h})+G_uu(t-\hat{h}) \\ & -G_1L\left(x_2(t-\hat{h})-\hat{x}_2(t-\hat{h})\right)-T^T\nu(t) \\ \hat{y}(t) = & T\hat{x}_2(t) \end{array}\right. \tag{4}$$

where the linear gain G_l is an Hurwitz matrix and L has the form $\left[\begin{array}{cc} \bar{L} & 0 \end{array}\right]$ with $\bar{L} \in \mathbb{R}^{(n-q)\times(q-p)}$. The computed delay $\hat{h} \leq h_m$ is an implemented value that can be chosen according to the parameters of the system. It could also be time-varying. For instance $\hat{h}$ could be equal to some expected nominal estimation of the time-varying delay. The discontinuous injection term ν is given by:

$$\nu(t) = \left\{\begin{array}{cc} -\rho(t,y,u)\frac{P_y(y(t)-\hat{y}(t))}{\|P_y(y(t)-\hat{y}(t))\|} & \text{if} \quad y(t)-\hat{y}(t) \neq 0, \\ 0 & \text{otherwise.} \end{array}\right. \tag{5}$$

where $P_y > 0$, $P_y \in \mathbb{R}^{p\times p}$ and where ρ is a nonlinear positive gain yet to be defined. Note that the non delayed terms depending on x_2 are known because $x_2(t) = T^Ty(t)$. Defining the state estimation errors as $e_1 = x_1(t) - \hat{x}_1(t)$ and $e_2 = x_2(t) - \hat{x}_2(t)$, one obtains:

$$\left\{\begin{array}{ll} \dot{e}_1(t) = & A_{11}e_1(t) - L\left(T^TG_lTe_2(t)+T^T\nu(t)\right) \\ & +A_{11}Le_2(t), \\ \dot{e}_2(t) = & A_{21}e_1(t)+G_1e_1(t-h(t))+D_1\zeta(t) \\ & +T^T\nu(t)+\xi_0(t)+(T^TG_lT+A_{21}L)e_2(t) \\ & +G_1Le_2(t-\hat{h}), \end{array}\right. \tag{6}$$

where $\xi_0 : \mathbb{R} \longmapsto \mathbb{R}^p$ is given by:

$$\begin{array}{ll} \xi_0(t) = & G_1(\hat{x}_1(t-h(t))-\hat{x}_1(t-\hat{h})) \\ & +G_2(x_2(t-h(t))-x_2(t-\hat{h})) \\ & +G_u(u(t-h(t))-u(t-\hat{h})). \end{array}$$

Let us introduce the change of coordinates $\left[\begin{array}{c} \bar{e}_1 \\ \bar{e}_2 \end{array}\right] = T_L\left[\begin{array}{c} e_1 \\ e_2 \end{array}\right]$ with $T_L = \left[\begin{array}{cc} I_{n-q} & L \\ 0 & T \end{array}\right]$. Using the fact that $LG_1 = LG_2 = LG_u = LD_1 = 0$, one obtains:

$$\left\{\begin{array}{ll} \dot{\bar{e}}_1(t) = & (A_{11}+LA_{21})\bar{e}_1(t), \\ \dot{\bar{e}}_2(t) = & TA_{21}\bar{e}_1(t)+TG_1\bar{e}_1(t-h(t)) \\ & +G_l\bar{e}_2(t)+\nu(t)+T\xi(t)+TD_1\zeta(t), \end{array}\right. \tag{7}$$

with

$$\begin{array}{ll} \xi(t) = & G_1(\hat{x}_1(t-h(t))-\hat{x}_1(t-\hat{h})) \\ & +G_2(x_2(t-h(t))-x_2(t-\hat{h})) \\ & -G_1L(e_2(t-h(t))-e_2(t-\hat{h})) \\ & +G_u(u(t-h(t))-u(t-\hat{h})), \end{array}$$

that can be rewritten as:

$$\xi(t) = \left[\begin{array}{cccc} G_1 & G_2 & -G_1L & G_u \end{array}\right]\int_{t-\hat{h}}^{t-h(t)}\left[\begin{array}{c} \dot{\hat{x}}_1(s) \\ \dot{x}_2(s) \\ \dot{e}_2(s) \\ \dot{u}(s) \end{array}\right]ds. \tag{8}$$

The function ξ only depends on the known variables $\hat{x}_1$, x_2, e_2 and u and on the unknown delay $h(t)$. One can then assume that there exists a known scalar function α_2 such that:

$$\|\xi(t)\| \leq \alpha_2(t,\hat{x}_1,x_2,e_2,u). \quad (9)$$

Remark 1 *Note that the nearest the available estimation $\hat{h}$ of $h(t)$, the smallest the bound α_2 (indeed $\hat{h} = h(t)$ implies $\xi = 0$). This means that an available information on the delay size order allows for reducing the observer gain. Furthermore, as it will be seen hereafter, a value h_m of the upper-bound of the admissible time delays will be available via an LMI formulation.*

It is now possible to define more precisely the "discontinuous gain" ρ, using the same technique for the design of the sliding mode control law in [13]:

$$\rho(t,y,u) = \|D_1\|\alpha_1(t,y,u) + \alpha_2(t,\hat{x}_1,x_2,e_2,u) + \gamma, \quad (10)$$

with γ a positive real number. Then the following result holds:

Theorem 1 *Under the conditions A and (9) and for any Hurwitz matrix G_l, the system (7) is asymptotically stable for all delay $h(t) \leq h_m$ if there exist symmetric positive definite matrices P_1 and $R_1 \in \mathbb{R}^{(n-q)\times(n-q)}$, $P_2 \in \mathbb{R}^{q\times q}$, a symmetric matrix $Z_2 \in \mathbb{R}^{q\times q}$ and a matrix $W \in \mathbb{R}^{(n-q)\times(q-p)}$ such that the following LMI conditions are satisfied:*

$$\begin{bmatrix} \psi_0 & A_{11}^T P_1 + A_{211}^T W^T & (A_{21}+G_1)^T T^T P_2 \\ & -2P_1 + h_m R_1 & 0 \\ & * & G_l^T P_2 + P_2 G_l + h_m Z_2 \end{bmatrix} < 0, \quad (11)$$

$$\begin{bmatrix} R_1 & (TG_1)^T P_2 \\ P_2 T G_1 & Z_2 \end{bmatrix} \geq 0, \quad (12)$$

where $\psi_0 = A_{11}^T P_1 + P_1 A_{11} + A_{211}^T W^T + W A_{211}$.

The observer gain $\bar{L}$ is then given by $\bar{L} = P_1^{-1} W$.

Consider the candidate for a Lyapunov-Krasovskii functional:

$$\begin{aligned} V(t) = \; & \bar{e}_1^T(t) P_1 \bar{e}_1(t) + \bar{e}_2^T(t) P_2 \bar{e}_2(t) \\ & + \int_{-h_m}^{0} \int_{t+\theta}^{t} \dot{\bar{e}}_1^T(s) R_1 \dot{\bar{e}}_1(s) ds d\theta. \end{aligned} \quad (13)$$

Using the following transformation:

$$\bar{e}_1(t-h(t)) = \bar{e}_1(t) - \int_{t-h(t)}^{t} \dot{\bar{e}}_1(s) ds, \quad (14)$$

and differentiating (13) along the trajectories of (7), one gets:

$$\begin{aligned} \dot{V}(t) = \; & \bar{e}_1^T(t)[Sym\{(A_{11}+LA_{21})^T P_1\}]\bar{e}_1(t) \\ & + \bar{e}_2^T(t)[G_l^T P_2 + P_2 G_l]\bar{e}_2(t) \\ & + Sym\left\{\bar{e}_2^T(t) P_2 T (A_{21}+G_1)\bar{e}_1(t)\right\} \\ & - 2\rho(t,y,u)\|P_2\bar{e}_2(t)\| + h_m \dot{\bar{e}}_1^T(t) R_1 \dot{\bar{e}}_1(t) \\ & + \eta_1(t) + \eta_2(t) - \int_{t-h_m}^{t} \dot{\bar{e}}_1^T(s) R_1 \dot{\bar{e}}_1(s) ds, \end{aligned} \quad (15)$$

where

$$\begin{aligned} \eta_1(t) &= -2\bar{e}_2^T(t) P_2 T G_1 \int_{t-h(t)}^{t} \dot{\bar{e}}_1(s) ds, \\ \eta_2(t) &= 2\bar{e}_2^T(t) P_2 T \left[D_1 \zeta(t) + \xi(t)\right]. \end{aligned}$$

The LMI condition (12) implies that for any vector X:

$$X^T \begin{bmatrix} R_1 & (TG_1)^T P_2 \\ P_2 T G_1 & Z_2 \end{bmatrix} X \geq 0,$$

Developing this relation for $X = \begin{bmatrix} \dot{\bar{e}}_1(s) \\ \bar{e}_2(t) \end{bmatrix}$, one has:

$$-2\bar{e}_2(t) P_2 G_1 \dot{\bar{e}}_1(s) \leq \bar{e}_2(t)^T Z_2 \bar{e}_2(t) + \dot{\bar{e}}_1^T(s) R_1 \dot{\bar{e}}_1(s).$$

By integrating this inequality with respect to the s variable, one can upperbound $\eta_1(t)$:

$$\begin{aligned} \eta_1(t) \leq \; & \int_{t-h(t)}^{t} \bar{e}_2^T(t) Z_2 \bar{e}_2(t) ds + \int_{t-h(t)}^{t} \dot{\bar{e}}_1^T(s) R_1 \dot{\bar{e}}_1(s) ds, \\ \eta_1(t) \leq \; & h_m \bar{e}_2^T(t) Z_2 \bar{e}_2(t) + \int_{t-h_m}^{t} \dot{\bar{e}}_1^T(s) R_1 \dot{\bar{e}}_1(s) ds. \end{aligned} \quad (16)$$

Under the definition (10) of ρ and since T is an orthogonal matrix:

$$\eta_2(t) - 2\rho(t,y,u)\|P_2\bar{e}_2(t)\| \leq -2\gamma\|P_2\bar{e}_2(t)\|. \quad (17)$$

Taking into account (16), (17) and that $\dot{\bar{e}}_1(t) = (A_{11} + \bar{L}A_{211})\bar{e}_1(t)$, $\dot{V}$ can be upperbounded as follows:

$$\begin{aligned} \dot{V}(t) \leq \; & Sym\left\{\bar{e}_1^T(t) P_1 (A_{11}+\bar{L}A_{211})\bar{e}_1(t) \right. \\ & \left. + \bar{e}_2^T(t) P_2 (A_{21}+G_1)\bar{e}_1(t) + \bar{e}_2^T(t) P_2 G_l]\bar{e}_2(t)\right\} \\ & + h_m \bar{e}_1^T(t)(A_{11}+LA_{21})^T R_1 (A_{11}+LA_{21})\bar{e}_1(t) \\ & + h_m \bar{e}_2^T(t) Z_2 \bar{e}_2(t) - 2\gamma\|P_2 e_2(t)\|, \end{aligned}$$

This can be rewritten as:

$$\dot{V}(t) \leq \begin{bmatrix} \bar{e}_1(t) \\ \bar{e}_2(t) \end{bmatrix}^T \Psi \begin{bmatrix} \bar{e}_1(t) \\ \bar{e}_2(t) \end{bmatrix} - 2\gamma\|P_2\bar{e}_2(t)\|, \quad (18)$$

with

$$\Psi = \begin{bmatrix} \psi_1 & (A_{21}+G_1)^T T^T P_2 \\ P_2 T (A_{21}+G_1) & G_l^T P_2 + P_2 G_l + h_m Z_2 \end{bmatrix} \quad (19)$$

and

$$\begin{aligned} \psi_1 = \; & (A_{11}+\bar{L}A_{211})^T P_1 + P_1 (A_{11}+\bar{L}A_{211}) \\ & + h_m (A_{11}+\bar{L}A_{211})^T R_1 (A_{11}+\bar{L}A_{211}) \end{aligned}$$

One can note that (19) is not a LMI condition since there are some nonlinear terms in the first row and the first column. Nevertheless, this problem can be transformed into an LMI condition using Lemma 1:

$$\begin{bmatrix} \psi_0 & (A_{11}+\bar{L}A_{211})^T Y^T & (A_{21}+G_1)^T T^T P_2 \\ * & -Y - Y^T + h_m R_1 & 0 \\ * & * & G_l^T P_2 + P_2 G_l + h_m Z_2 \end{bmatrix} < 0. \quad (20)$$

Let us set $Y = P_1$ and define $W = P_1 \bar{L}$. The LMI conditions of Theorem 1 appear. Thus, if (11) and (12) are satisfied, (20) is also satisfied. This implies that the observation error is asymptotically stable.

4. Dynamic properties of the observer

4.1. Finite time convergence on the sliding manifold

Corollary 1 *Under the observer design of Theorem 1, an ideal sliding motion takes place on $S_0 = \{\bar{e}_2 = 0\}$ in finite time.*

Consider the Lyapunov function:

$$V_2(t) = \bar{e}_2^T(t) P_y \bar{e}_2(t) \tag{21}$$

Differentiating along the trajectories of (7), one obtains:

$$\begin{aligned} \dot{V}_2(t) = \quad & \bar{e}_2^T(t)(G_l^T P_y + P_y G_l)\bar{e}_2(t) + 2\bar{e}_2^T(t) P_y T \left[T^T \nu \right. \\ & \left. + A_{21}\bar{e}_1(t) + G_1 \bar{e}_1(t-h(t)) + D_1 \zeta(t) + \xi(t)\right]. \end{aligned}$$

Using the fact that G_l is Hurwitz and (5), one can write the following upper bound for $\dot{V}_2(t)$:

$$\dot{V}_2(t) \leq 2\|P_y \bar{e}_2(t)\| \left[\|A_{21}\bar{e}_1(t) + G_1 \bar{e}_1(t-h(t))\| - \gamma\right].$$

From Theorem 1, the error e_1 is asymptotically stable. Thus, there exist an instant t_0 and a positive scalar δ such that : $\forall t \geq t_0, \quad \|A_{21}\bar{e}_1(t) + G_1\bar{e}_1(t-h(t))\| \leq \gamma - \delta$

This leads to

$$\begin{aligned} \forall t \geq t_0, \quad \dot{V}_2(t) \quad & \leq -2\delta \|P_y \bar{e}_2(t)\| \\ & \leq -2\delta \sqrt{\lambda_{min}(P_y)} \sqrt{V_2(t)}. \end{aligned}$$

where $\lambda_{min}(P_y)$ is the smallest eigenvalue of P_y. Integrating the last differential inequation, it follows that an ideal sliding motion takes place on S_0 in finite time.

4.2. Exponential stability

In this part, the observer convergence is improved by giving a criteria of exponential convergence. Exponential stability properties could be an interesting way to characterize the convergence rate of the observers. As in [18, 22], for some given rate $\alpha > 0$, a system (7) is said to be $\alpha-$stable, or "exponentially stable with the rate α", if there exists a scalar $\beta \geq 1$ such that the solution $e(t;t_0,\phi)$ of (7), with any initial function ϕ, satisfies:

$$|e(t,t_0,\phi)| \leq \beta |\phi| e^{-\alpha(t-t_0)}. \tag{22}$$

In spite of the unknown and variable delay, the following Theorem ensures that the observer dynamics is $\alpha-$stable.

Theorem 2 *Under conditions A and (9), the system (7) is $\alpha-$stable for any delay $h(t) \leq h_m$ if there exist symmetric positive definite matrices P_1, R_1 and $R_2 \in \mathbb{R}^{(n-q)\times(n-q)}$, $P_2 \in \mathbb{R}^{q\times q}$, a symmetric matrix $Z_2 \in \mathbb{R}^{q\times q}$ and a matrix $W \in \mathbb{R}^{(n-q)\times(q-p)}$ such that the following LMI conditions are satisfied:*

$$\left[\begin{array}{ccccc} \psi_1^\alpha & A_{11}^T P_1 + A_{211}^T W^T + \alpha P_1 & (A_{21} + b_0 G_1)^T T^T P_2 & & \\ & -2P_1 + 2h_m R_1 & 0 & & \\ & * & Y^T + Y + 2\alpha P_2 + h_m Z_2 & & \\ & * & * & & \\ & * & * & & \\ & 0 & 0 & \multirow{4}{*}{$\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right] < 0,$} & \\ & 0 & 0 & & \\ & b_m P_2 T G_1 & h_m b_m P_2 T G_1 & & \\ & -R_1 & 0 & & \\ & & -h_m R_2 & & \end{array}\right. \tag{23}$$

$$\left[\begin{array}{cc} R_1 & b_0 (TG_1)^T P_2 \\ b_0 P_2 T G_1 & Z_2 \end{array}\right] \geq 0. \tag{24}$$

where

$$\begin{aligned} \psi_1^\alpha \quad &= \quad A_{11}^T P_1 + P_1 A_{11} + 2\alpha P_1 \\ & \quad + A_{211}^T W^T + W A_{211} + R_2 \\ b_0 \quad &= \quad (1 + e^{\alpha h_m})/2, \;\; b_m = (-1 + e^{\alpha h_m})/2 \end{aligned}$$

The observer gains are given by $\bar{L} = P_1^{-1} W$ et $G_l = P_2^{-1} Y$.

Let us introduce the new variable $\bar{e}_i^\alpha(t) = e^{\alpha t}\bar{e}_i(t)$ in(7). Then, the asymptotic convergence of $\bar{e}^\alpha$ implies that $\bar{e}$ is $\alpha-$stable. Equation (7) becomes:

$$\left\{\begin{array}{ll} \dot{\bar{e}}_1^\alpha(t) = & (A_{11} + LA_{21} + \alpha I_{n-p})\bar{e}_1^\alpha(t), \\ \dot{\bar{e}}_2^\alpha(t) = & TA_{21}\bar{e}_1^\alpha(t) + (\nu + T\xi(t) + TD_1\zeta(t))e^{\alpha t} \\ & + e^{\alpha h(t)} TG_1 \bar{e}_1^\alpha(t-h(t)) + (G_l + \alpha I_p)\bar{e}_2^\alpha(t), \end{array}\right. \tag{25}$$

Note that $e^{\alpha h(t)} = b_0 + \Delta(t) b_m$, where $\Delta(t)$ is an unknown scalar function satisfying $\|\Delta(t)\| \leq 1$. Consider the following candidate for a Lyapunov-Krasovskii functional:

$$\begin{aligned} V^\alpha(t) = \quad & \bar{e}_1^{\alpha T}(t) P_1 \bar{e}_1^\alpha(t) + \bar{e}_2^{\alpha T}(t) P_2 \bar{e}_2^\alpha(t) \\ & + 2\int_{-h_m}^0 \int_{t+\theta}^t \dot{\bar{e}}_1^{\alpha T}(s) R_1 \dot{\bar{e}}_1^\alpha(s) ds d\theta. \end{aligned} \tag{26}$$

Differentiating (26) along (25), one gets:

$$\begin{aligned} \dot{V}^\alpha(t) = \quad & \bar{e}_1^{\alpha T}(t)[Sym\{P_1(A_{11} + \bar{L}A_{211} + \alpha I_{n-p})\}]\bar{e}_1^\alpha(t) \\ & + 2\bar{e}_2^{\alpha T}(t) P_2 T (A_{21} + b_0 G_1)\bar{e}_2^\alpha(t) \\ & + \eta_1^\alpha(t) + \eta_2^\alpha(t) + \eta_3^\alpha(t) + \eta_4^\alpha(t) \\ & + \bar{e}_2^{\alpha T}(t)[P_2(G_l + \alpha I_p) + (G_l + \alpha I_p)^T P_2]\bar{e}_2^\alpha(t) \\ & + 2h_m \dot{e}_1^{\alpha T} R_1 \dot{e}_1^\alpha(t) - 2\int_{t-h(t)}^t \dot{e}_1^{\alpha T}(s) R_1 \dot{e}_1^\alpha(s) ds, \end{aligned} \tag{27}$$

$$\begin{aligned} \eta_1^\alpha(t) = \quad & 2\bar{e}_2^{\alpha T}(t) P_2 b_0 T G_1 \int_{t-h(t)}^t \dot{e}_1^\alpha(s) ds, \\ \eta_2^\alpha(t) = \quad & 2\bar{e}_2^{\alpha T}(t) P_2 b_m \Delta(t) T G_1 e_1^\alpha(t), \\ \eta_3^\alpha(t) = \quad & 2\bar{e}_2^{\alpha T}(t) P_2 b_m \Delta(t) T G_1 \int_{t-h(t)}^t \dot{e}_1^\alpha(s) ds, \\ \eta_4^\alpha(t) = \quad & 2\bar{e}_2^{\alpha T}(t) P_2 \left(\nu + T\xi(t) + TD_1\zeta(t)\right) e^{\alpha t}. \end{aligned}$$

Following the lines of Theorem 1, (24) gives a majoration of η_1:

$$\eta_1^\alpha(t) \leq h_m \bar{e}_2^{\alpha T}(t) Z_2 \bar{e}_2^\alpha(t) + \int_{t-h_m}^t \dot{\bar{e}}_1^{\alpha T}(s) R_1 \dot{\bar{e}}_1^\alpha(s) ds. \tag{28}$$

For any $n \times n$ matrix $R > 0$ and for any vectors $a \in R^n$ and $b \in R^n$:

$$\pm 2a^T b \leq a^T R^{-1} a + b^T R b \tag{29}$$

Let us apply (29) for $\eta_2^\alpha(t)$ with:

$$\begin{aligned} a^T &= \bar{e}_2^{\alpha T}(t) P_2 b_m T G_1 \Delta(t) \\ b &= e_1^\alpha(t) \\ R &= R_2 \end{aligned}$$

One gets:

$$\begin{aligned} \eta_2^\alpha(t) \leq \; & \bar{e}_2^{\alpha T}(t) b_m P_2 T G_1 R_2^{-1} b_m (TG_1)^T P_2^T \bar{e}_2^\alpha(t) \\ & + \bar{e}_1^{\alpha T}(t) R_2 \bar{e}_1^\alpha(t). \end{aligned} \tag{30}$$

Using again (29) for $\eta_3^\alpha(t)$ with:

$$\begin{aligned} a^T &= \bar{e}_2^{\alpha T}(t) P_2 b_m T G_1 \Delta(t) \\ b &= \dot{e}_1^\alpha(s) \\ R &= R_1 \end{aligned}$$

one has:

$$\begin{aligned} \eta_3^\alpha(t) \leq \; & h_m \bar{e}_2^{\alpha T}(t) b_m P_2 T G_1 R_1^{-1} b_m (TG_1)^T P_2^T \bar{e}_2^\alpha(t) \\ & + \int_{t-h_m}^{t} \bar{e}_1^{\alpha T}(s) R_1 \bar{e}_1^\alpha(s) ds. \end{aligned} \tag{31}$$

With the discontinuous output injection ν defined in (5) and (10), one has:

$$\eta_4^\alpha(t) \leq -2\gamma \| P_2 \bar{e}_2(t) \|. \tag{32}$$

Then, combining (28-32) with (27) leads to:

$$\dot{V}^\alpha(t) \leq \begin{bmatrix} \bar{e}_1(t) \\ \bar{e}_2(t) \end{bmatrix}^T \Psi^\alpha \begin{bmatrix} \bar{e}_1(t) \\ \bar{e}_2(t) \end{bmatrix} - 2\gamma \| P_2 \bar{e}_2(t) \|, \tag{33}$$

with:

$$\Psi^\alpha = \begin{bmatrix} \psi_{11}^\alpha & (A_{21}+G_1)^T T^T P_2 \\ * & \psi_{22}^\alpha \end{bmatrix},$$

$$\begin{aligned} \psi_{11}^\alpha = \; & Sym\{P_1(A_{11} + \bar{L}A_{211} + \alpha I_{n-p})\} + (A_{11} \\ & + \bar{L}A_{211} + \alpha I_{n-p})^T R_1 (A_{11} + \bar{L}A_{211} + \alpha I_{n-p}), \\ \psi_{22}^\alpha = \; & G_l^T P_2 + P_2 G_l + 2\alpha P_2 + h_m Z_2 \\ & + h_m b_m P_2 T G_1 R_2^{-1} (TG_1)^T P_2 b_m \\ & + b_m P_2 T G_1 R_1^{-1} (TG_1)^T P_2 b_m. \end{aligned}$$

Again, applying Lemma 1 as in Theorem 1 yields:

$$\begin{bmatrix} \psi_1^\alpha & A_{11}^T P_1 + A_{211}^T W^T + \alpha P_1 & (A_{21}+G_1)^T T^T P_2 \\ & -2P_1 + 2h_m R_1 & 0 \\ & * & \psi_{22}^\alpha \end{bmatrix} < 0, \tag{34}$$

Set $Y = P_2 G_l$. Then using the Schur complement, one finds the LMI condition (23). Thus, if (23) and (24) are satisfied, the time derivative of (26) is negative definite.

4.3. Optimization Problem

This paragraph focusses on the optimization of the exponential decay rate α. The greater α is, faster the error dynamics converge to the solution $e(t) = 0$. The optimization consists in finding the greatest α (guaranteed speed performance of the application) such that the closed-loop system is α-stable. This corresponds to the problem

$$\begin{gathered} \max \alpha \\ \text{subject to (23) and (24) for a given } h_m. \end{gathered}$$

Because α does not appear in a linear form in (23) and (24), this problem is solved by iteratively increasing α until the LMI conditions become unfeasible.

5. Example

Consider the system with time-varying delay (3) with:

$$\begin{gathered} A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, A_{12} = \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix}, \\ A_{21} = \begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}, A_{22} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ G_1 = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.21 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 \\ 0.2 & 1 \end{bmatrix}, \\ T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, G_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D_1 = B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{gathered}$$

The delay is chosen as $h(t) = \frac{h_m}{2}(1 + sin(\omega_1 t))$, with $h_m = 0.3s$ an frequency $\omega_1 = 0.5s^{-1}$. The control law is

$$u(t) = u_0 sin(\omega_2 t)$$

with $u_0 = 2$ and $\omega_2 = 3$.

Since the system (3) is open loop stable, its dynamics are bounded. Thus the function $\alpha_2(t, \hat{x}_1, x_2, e_2, u)$ could be chosen as a constant $K = 0,7$.

The simulation results are given in the following figures. In Figures 1 and 2 are reported the observation errors of the system for $\alpha = 0$ and $\alpha = 2$.

Figures 3 and 4 show the comparison between the real and observed states, for $\alpha = 2$.

It can be noticed that the greater α is, the faster the error convergence is. Using Theorem 2, the following observer gains for $\alpha = 2$ are obtained:

$$\bar{L} = \begin{bmatrix} -3.8658 \\ 1.0722 \end{bmatrix}, G_l = \begin{bmatrix} -8.8160 & -6.0190 \\ -5.8154 & -32.0670 \end{bmatrix}$$

6. Conclusion

The problem of designing observers for linear systems with unknown variable delay on both input and state has

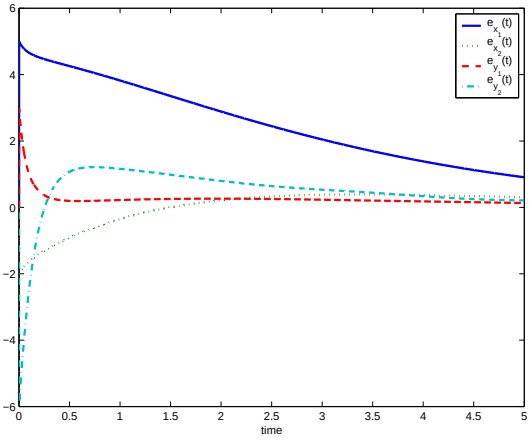


Figure 1. Observation errors for $\alpha = 0$ and $h_m = 0.3$

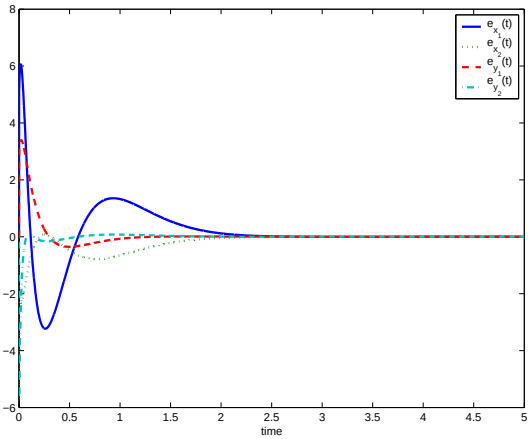


Figure 2. Observation errors for $\alpha = 2$ and $h_m = 0.3$

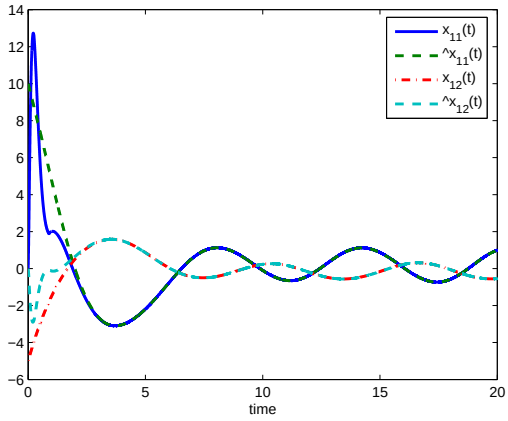


Figure 3. x_1 and $\hat{x}_1$, for $\alpha = 2$,

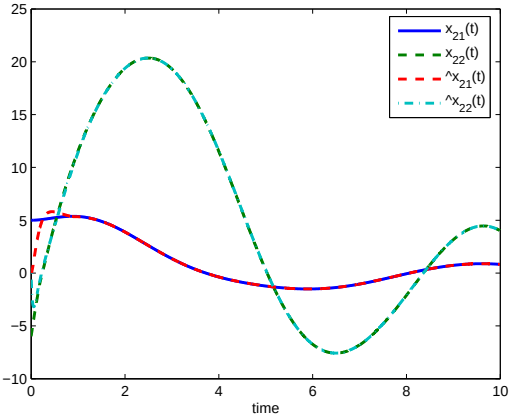


Figure 4. x_2 and $\hat{x}_2$, for $\alpha = 2$

been solved in this article. Delay-dependent LMI conditions have been found to guarantee asymptotic stability of the dynamical error system. In addition, the dynamics properties of the proposed observer can be characterized through finite time and exponential convergence properties.

References

[1] J.-P. Barbot, T. Boukhobza and M. Djemaï, "Sliding mode observer for triangular input form". 35-th IEEE CDC'99, Conference on Decision and Control, Kobe, Japan, 1996.

[2] H. H. Choi, M. J. Chung, "Observer-based H_∞ controller design for state delayed linear systems". Automatica, 32(7), pp. 1073-1075, 1996.

[3] H. H. Choi and M. J. Chung, "Robust Observer-based H_∞ controller design for linear uncertain time-delay systems". Automatica, 33(9), pp. 1749-1752, 1997.

[4] M. Darouach, "Linear functional observers for systems with delays in the state variables". IEEE trans. on Automatic Control, 46(3), pp. 491-497, March 2001.

[5] M. Darouach, P. Pierrot and E. Richard, "Design of reduced-order observers without internal delays". IEEE Trans. on Automatic Control, 44(9), pp. 1711-1713, 1999.

[6] M. Darouach, M. Zasadzinski and S. J. Xu, "Full-order observers for linear systems with unknown inputs". IEEE Trans. on Automatic Control, vol. 39, pp. 606-609, 1994.

[7] C.E. de Souza, R.E. Palhares and P.L.D. Peres, "Robust H_∞ filtering for uncertain linear systems with multiple time-varying state: An LMI approach". 38-th IEEE CDC'99, Conference on Decision and Control, Phoenix, AZ, pp. 2023-2028, 1999.

[8] S. V. Drakunov, "Sliding-mode Observer Based on Equivalent Control Method". 31st IEEE CDC'92, Conference on Decision and Control, Tucson, Arizona, 1992.

[9] C. Edwards and S. K. Spurgeon, "On the development of discontinuous observers". International Journal of Control, 59, pp. 1211-1229, 1994.

[10] Edwards C. and Spurgeon S. K., *Sliding Mode Control: The-*

ory and Applications, Taylor & Francis, 1998.

[11] F. W. Fairmar, Kumar and E. Richard, "Design of reduced-order observers without internal delays". IEEE trans. on Automatic Control, 44(9), pp. 1711-1713, 1999.

[12] T. Floquet, J.-P. Barbot, W. Perruquetti and M. Djemaï, "On the robust fault detection via a sliding mode disturbance observer". International Journal of control, 77(7), pp. 622-629, 2004.

[13] E. Fridman, F. Gouaisbaut, M. Dambrine and J.-P. Richard, "Sliding mode control of systems with time-varying delays via a descriptor approach". Int. J. System Sc. Vol. 34, No.8-9, pp. 553-559, July 2003.

[14] E. Fridman, U. Shaked and L. Xie, "Robust H_∞ Filtering of linear systems with time-varying delay". IEEE Trans. on Automatic Control, vol. 48, pp. 159-165, 2003.

[15] L.-S. Hu, J. Huang and H.-H. Cao, "Robust digital model predicitve control for linear uncertain systems with saturations". IEEE Trans. on Automatic Control , vol. 49(5), pp. 792-796, 2004.

[16] J. Leyva-Ramos and A. E. Pearson, "An asymptotic modal observer for linear autonomous time lag systems". IEEE Trans. on Automatic Control, 40, 1291-1294, 1995.

[17] S.I. Niculescu , "Delay effects on stability", LNCIS series, Vol. 269, Springer, Berlin, 2001.

[18] S-I. Niculescu, C-E. de Souza, L. Dugard and J-M. Dion, "Robust exponential Stability of uncertain systems with time-varying delays", IEEE Trans. on Automatic Control, Vol. 43(5), pp. 743-748, 1998.

[19] W. Perruquetti and J. P. Barbot, *Sliding Mode Control in Engineering*, Ed. Marcel Dekker, 2002.

[20] J.-P. Richard, "Time Delay Systems: An overview of some recent advances and open problems". Automatica 39(10), pp. 1667–1694, 2003.

[21] O. Sename, "New trends in design of observers for time-delay systems". Kybernetica, 37(4), pp. 427-458, 2001.

[22] A. Seuret, M. Dambrine and J.P. Richard, "Robust exponential stabilization for systems with time-varying delays", 5-th IFAC Workshop on Time-Delay Systems, Leuven 2004.

[23] C. P. Tan, and C. Edwards, "An LMI approach for designing sliding mode observers". International Journal of Control, 74(16), pp. 1559–1568, 2001.

[24] Utkin, V. I. *Sliding Modes in Control and Optimization*, Berlin, Germany, Springer-Verlag, 1992.

[25] S. Xu, P. Shi, C. Feng, Y. Guo and Y. Zou, "Robust Observers for a class of uncertain nonlinear stochastic systems with state delay". Nonlinear Dynamics and systems Theory, 4(3), 2004.

[26] Z. Wang, B. Huang and H. Unbehausen, "Robust H_∞ observer design for uncertain time-delay systems:(I) the continuous case". IFAC 14-th world congress, Beijing, China, pp. 231-236, 1999.