Optimal transportation problems with free Dirichlet regions

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Abstract. A Dirichlet region for an optimal mass transportation problem is, roughly speaking, a zone in which the transportation cost is vanishing. We study the optimal transportation problem with an unknown Dirichlet region Σ which varies in the class of closed connected subsets having prescribed 1-dimensional Hausdorff measure. We show the existence of an optimal Σ_{opt} and study some of its geometrical properties. We also present numerical computations which show the shape of Σ_{opt} in some model examples.

Contents

1. Introduction

Optimal mass transportation problems received a lot of attention in the last years, among all, also for extensive connections with other fields such as shape optimization, fluid mechanics, partial differential equations, geometric measure theory (see [5, 4, 12, 13, 14]). Given two nonnegative measures f^+ and f^- over $\mathbf{R}^N$ the problem consists in the optimization of the cost of transporting f^+ into f^- by

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means of a *transport map* $T : \mathbf{R}^N \to \mathbf{R}^N$. More precisely, we say that T transports f^+ into f^- and call T a *transport map*, if $T_\# f^+ = f^-$ where $T_\#$ is the push-forward operator, that is

$$f^+\big(T^{-1}(B)\big) = f^-(B) \qquad \text{for every Borel set } B \subset \mathbf{R}^N. \tag{1}$$

Clearly, in this way, in order to have a nonempty class of admissible transport maps, we have to require that the two measures f^+ and f^- have the same mass.

For every transport map T the cost of transporting f^+ to f^- is defined by

$$J(T) = \int c\big(x, T(x)\big)\, df^+(x), \tag{2}$$

where $c(x,y)$ is a given continuous nonnegative function. The problem of optimal mass transportation is then

$$\min\big\{J(T) \ : \ T \text{ transports } f^+ \text{ into } f^-\big\}. \tag{3}$$

Usually the cost density $c(x,y)$ is taken as a function of the Euclidean distance. In particular, one often chooses $c(x,y) = |x-y|^p$. The case $p = 1$ is then the classical Monge transportation problem and is related to several problems in shape optimization theory (see [4]). The case $p = 2$ is also widely studied for its applications in fluid mechanics, while the case $p < 1$, or more generally when c is a concave function, seems to be the most realistic for several applications, and has been studied in [7]. We quote as general surveys on mass transportation problems the book [12] as well as the monographs [6, 5, 1, 13, 14], where the reader may find all the details that here, for the sake of brevity, will be omitted.

When the measures f^+ and f^- may concentrate on lower dimensional sets, it may happen that no admissible transport maps exist. This is for instance the case, even if $N = 1$, when $f^+ = 2\delta_0$ and $f^- = \delta_1 + \delta_{-1}$. For this reason it is convenient to consider, the relaxed formulation of the problem due to Kantorovich, which uses instead of transport maps T, the so called *transport plans*, which are nonnegative Borel measures γ on the product space $\mathbf{R}^N \times \mathbf{R}^N$ such that

$$\pi^+_\# \gamma = f^+, \qquad \pi^-_\# \gamma = f^-,$$

where π^+ and π^- are the projections of $\mathbf{R}^N \times \mathbf{R}^N$ on the first and second factors respectively. It is easy to see (cf. [1]) that a transport map T always induces a transport plan γ given by $\gamma = (\mathrm{Id} \times T)_\# f^+$. Conversely, every transport plan γ which is concentrated on a γ-measurable graph Γ is induced by a suitable transport map T. The cost of a transport plan γ is simply given by

$$J(\gamma) = \int c(x,y)\, d\gamma(x,y)\ , \tag{4}$$

so that the optimal mass transportation problem becomes

$$\min\big\{J(\gamma) \ : \ \gamma \text{ transport plan of } f^+ \text{ into } f^-\big\}. \tag{5}$$

Again we notice for the class of admissible transport plans to be nonempty, the measures f^+ and f^- must have the same mass.

Most often one considers the optimal mass transportation problem (5) constrained to a set $K \subset \mathbf{R}^N$. The latter represents a region to which the transportation process is confined. This simply means that all the geodesic paths along which the mass is carried have to remain into K. In what follows we assume that $K = \bar{\Omega}$ is the closure of a smooth connected bounded open set $\Omega \subset \mathbf{R}^N$. The function $c(x, y)$, which measures the cost to carry the mass from the point x to the point y, has then to take into account that, unless Ω is convex, the shape of Ω modifies the length of geodesic paths. In particular, the cost function of the form $c(|x-y|)$ has to be replaced by $c\big(d_\Omega(x,y)\big)$ where d_Ω is the *geodesic distance* on Ω given by the formula

$$d_\Omega(x,y) := \inf\Big\{ \int_0^1 |\alpha'(t)|\,dt \;:\; \alpha \in \mathrm{Lip}([0,1];\bar{\Omega}),\ \alpha(0)=x,\ \alpha(1)=y \Big\}.$$

Furthermore, in applications (see for instance [4] for the relations with shape optimization problems) one often considers the presence of the so-called Dirichlet region $\Sigma \subset \bar{\Omega}$ (in the sequel assumed to be a closed set), which represents the zone where the cost of transportation vanishes. Heuristically it means that you are allowed to transport mass free of charge "along Σ". More formally, it means that the presence of Σ modifies the distance which governs the optimal mass transportation problem. In fact, setting

$$d_{\Omega,\Sigma}(x,y) \quad := \quad \inf\{d_\Omega(x,y) \wedge (d_\Omega(x,\xi_1) + d_\Omega(y,\xi_2)) \;:\; \xi_1,\xi_2 \in \Sigma\}$$

we obtain a semi-distance on $\bar{\Omega}$ which does not count the paths that both start and end in Σ. We also generalize the notion of a transport plan for the case of the presence of a nonempty Dirichlet region $\Sigma \subset \bar{\Omega}$, saying that a Borel measure γ over $\bar{\Omega} \times \bar{\Omega}$ is a transport plan of f^+ into f^-, if

$$\pi^+_\# \gamma - \pi^-_\# \gamma = f^+ - f^- \text{ on } \bar{\Omega} \setminus \Sigma.$$

Plugging now the above semi-distance instead of d_Ω into the problem (5), we obtain the new optimal mass transportation problem [4]

$$\min\Big\{ \int \phi\big(d_{\Omega,\Sigma}(x,y)\big)\,d\gamma(x,y) \;:\; \gamma \text{ transport plan of } f^+ \text{ into } f^- \Big\}. \tag{6}$$

Note that for the latter problem, in view of the generalized definition of a transport plan, it is not necessary to require that f^+ and f^- have the same mass.

In this paper we are studying the optimization problem of finding the "best possible" Dirichlet region $\Sigma \subset \bar{\Omega}$ subject to certain constraints. Namely, we will call $MK(\Sigma)$ the minimum of the problem (6) and we will study the minimization of MK with respect to Σ. In the case $f^+ := \mathcal{L}^N \llcorner \Omega$ and $f^- = 0$ the functional MK reduces to the average distance functional

$$MK(\Sigma) = \int_\Omega \mathrm{dist}_\Omega(x,\Sigma)\,dx,$$

where $\mathrm{dist}_\Omega(x,\Sigma) := \inf_{y\in\Sigma} d_\Omega(x,y)$.

The natural constraints for Σ are as follows: Σ can vary in the class of closed subsets of $\bar{\Omega}$ with prescribed length (i.e. Hausdorff $\mathcal{H}^1$ measure) and with prescribed finite number of connected components. In fact, it is clear that if either of the constraints is dropped, then the infimum of this optimization problem in Σ is trivially zero. When $f^+ := \mathcal{L}^N \llcorner \Omega$, $f^- = 0$ (i.e. MK is just the average distance functional) and the length constraint on Σ is zero, the above problem turns out to be the problem of optimal location of a finite number of points in a set $\bar{\Omega}$. The latter has a lot of applications in economics and urban planning, but despite being extensively studied recently, still lacks a complete understanding of the qualitative properties of solutions. For a recent survey on this problem we refer the reader to [9]. In this paper we focus our attention on the case of nonzero length constraint, and, just for the sake of simplicity, the set Σ is required to be connected (it will be clear from our results that allowing a finite number of connected components will not change the qualitative properties of each of the components). We will show that an optimal Σ_{opt} exists and we study some geometrical properties of Σ_{opt} in the simplest situation when MK is reduced to the average distance functional. We also present some numerical computations which show the shape of Σ_{opt} in some model examples in order to justify some of our conjectures which still lack a rigorous proof.

2. Existence of optimal sets

Let $l \geq 0$ be fixed and let Ω be a bounded connected subset of $\mathbf{R}^N$ with a Lipschitz boundary. We also fix two nonegative measures f^+ and f^- on $\bar{\Omega}$ and consider the optimization problem

$$\min \ \left\{MK\,(\Sigma) : \Sigma \subset \bar{\Omega} \text{ closed, connected, } \mathcal{H}^1(\Sigma) \leq l\right\} \tag{7}$$

where the functional MK is defined in the introduction as the minimum value of problem (6). We have the following existence result.

Theorem 2.1. *Let the function ϕ appearing in (6) be continuous. Then the problem (7) admits a solution.*

PROOF: Let a sequence $\{\Sigma_\nu\}_{\nu=1}^{\infty}$ of closed connected subsets of $\bar{\Omega}$ be a minimizing sequence for the functional MK, satisfying $\mathcal{H}^1(\Sigma_\nu) \leq l$ for all $\nu \in \mathbf{N}$. According to the Blaschke theorem (theorem 4.4.6 of [3]) one has $\Sigma_\nu \to \Sigma$ in the sense of Hausdorff convergence up to a subsequence (not relabeled), while $\Sigma \subset \bar{\Omega}$ is still closed and connected. Moreover, in view of the Golab theorem (theorem 4.4.7 of [3]) one also has $\mathcal{H}^1(\Sigma) \leq l$. Observing now that the Hausdorff convergence implies $d_\Omega(x, \Sigma_\nu) \to d_\Omega(x, \Sigma)$ for all $x \in \bar{\Omega}$, we obtain

$$d_{\Omega,\Sigma_\nu}(x,y) \to d_{\Omega,\Sigma}(x,y)$$

for all $(x,y) \in \bar{\Omega} \times \bar{\Omega}$. Moreover, since all d_{Ω,Σ_ν} are Lipschitz-continuous for the Euclidean distance with the same Lipschitz constant, then the convergence is actually uniform.

Let now γ_ν be the respective optimal transport plans, i.e.

$$MK(\Sigma_\nu) = \int_{\bar\Omega\times\bar\Omega} \phi(d_{\Omega,\Sigma_\nu}(x,y))\,d\gamma_\nu(x,y),$$

while

$$\pi^+_\#\gamma_\nu - \pi^-_\#\gamma_\nu = f^+ - f^- \text{ on } \bar\Omega\setminus\Sigma_\nu.$$

The sequence $\{\gamma_\nu(\bar\Omega)\}_{\nu=1}^\infty$ can be assumed bounded, and hence up to a subsequence (again not relabeled) $\gamma_\nu \rightharpoonup \gamma$ $*$-weakly in the sense of measures, where γ is some positive Borel measure over $\bar\Omega$. Clearly then

$$\pi^+_\#\gamma - \pi^-_\#\gamma = f^+ - f^- \text{ on } \bar\Omega\setminus\Sigma.$$

In fact, for every $\psi \in C_0(\bar\Omega\setminus\Sigma)$ one has

$$\int_{\bar\Omega} \psi\, d(\pi^+_\#\gamma - \pi^-_\#\gamma) = \lim_\nu \int_{\bar\Omega} \psi\, d(\pi^+_\#\gamma_\nu - \pi^-_\#\gamma_\nu) = \int_{\bar\Omega} \psi\, d(f^+ - f^-),$$

since every function with compact support in $\bar\Omega\setminus\Sigma$ has also compact support in $\bar\Omega\setminus\Sigma_\nu$ for sufficiently large $\nu\in\mathbf{N}$ (this follows from the convergence $\Sigma_\nu\to\Sigma$ in the sense of Hausdorff). At last it remains to observe that

$$MK(\Sigma) \le \int_{\bar\Omega\times\bar\Omega} \phi(d_{\Omega,\Sigma}(x,y))\,d\gamma(x,y) = \lim_\nu \int_{\bar\Omega\times\bar\Omega} \phi(d_{\Omega,\Sigma_\nu}(x,y))\,d\gamma_\nu(x,y),$$

which shows that Σ is a minimizer of the problem. □

Remark 2.2. A word-to-word restating of this proof shows even a formally slightly more general result, namely, that the functional MK attains a minimum even over a class of closed subsets $\Sigma\subset\bar\Omega$ with $\mathcal{H}^1(\Sigma)\le l$ and having fixed prescribed number of connected components. The latter case includes also the situation $l=0$ but the number of connected components is greater than one. If $f^+ := \mathcal{L}^N \llcorner \Omega$ and $f^- = 0$, the functional MK reduces to the average distance functional, and (7) is just the problem of optimal location of a finite number of points in $\bar\Omega$ (see [9]).

3. Qualitative properties of optimal sets

In this section we consider some qualitative properties every optimal solution Σ_{opt} to the problem (7) has to fulfill. We present here some problems together with some conjectures which we believe are true. We also present some numerical approximations of Σ_{opt} in different particular situations as well as proofs of some results which, though sometimes weaker than our expectations, still induce to think that the conjectures we formulate most probably hold true.

When formulating the problems below we assume that $\Omega\subset\mathbf{R}^N$ is a bounded connected open set with Lipschitz boundary and that the measures f^+ and f^- are absolutely continuous with respect to the Lebesgue measure and different from each other. For simplicity we consider only the case of the cost function $\phi(t)=t$, though most of the questions below could also be raised for more general cost functionals.

Problem 3.1 (Regularity). *Study the regularity properties of the solutions Σ_{opt}. We actually expect that Σ_{opt} are piecewise smooth, i.e. made of a finite number of smooth curves connected through a finite number of singular points.*

Problem 3.2 (Absence of loops). *Study the topological properties of the solutions Σ_{opt}. We expect that Σ_{opt} do not form closed loops in Ω. When the dimension N is equal to 2 this can be expressed by saying that $\mathbf{R}^2 \setminus \Sigma_{opt}$ is connected.*

Problem 3.3 (Triple points). *Study the nature of the singular points mentioned in Problem 3.1. We expect that they can only be triple points, that is points where three curves meet, with angles of 120 degrees.*

Problem 3.4 (Distance from the boundary). *Study the cases when the optimal solutions Σ_{opt} do not touch the boundary $\partial\Omega$. We expect that this occurs at least when Ω is convex.*

Problem 3.5 (Behavior for small lengths). *Study the asymptotic behaviour of Σ_{opt} as $l \to 0$. In particular we expect that for l small enough Σ_{opt} is a smooth curve without singular points. More in general, it would be interesting to obtain an estimate of the number of singular points in terms of the length of Σ_{opt}.*

Problem 3.6 (Behavior for large lengths). *Study the asymptotic behaviour of Σ_{opt} as $l \to +\infty$. It is not difficult to see that the value V_l of the optimization problem (7) vanishes as $l \to +\infty$. It would be interesting to evaluate the order of the vanishing quantity V_l. Moreover, once we estimate that $V_l = O(l^{-\beta})$ for some $\beta > 0$ as $l \to +\infty$, it is interesting to study the Γ-limit as $l \to +\infty$ of the rescaled functionals*

$$G_l(\Sigma) = \frac{1}{l^\beta} MK(\Sigma)$$

with respect to the convergence

$$\Sigma_l \to \lambda \quad \Leftrightarrow \quad \frac{1}{l}\mathcal{H}^1 \llcorner \Sigma_l \to \lambda \quad \textit{weakly* in the sense of measures.}$$

In particular, if Σ_l are optimal configurations of length l, it is interesting to study the asymptotic behaviour of Σ_l as well as their limit λ in the sense above.

We now start to develop the program introduced above by considering the simpler situation when Ω is convex, f^+ is the Lebesgue measure on Ω, $f^- = 0$. Then the functional $MK(\Sigma)$ reduces to the average distance functional and d_Ω to the Euclidean distance, so that problem (7) becomes

$$\min\left\{\int_\Omega \operatorname{dist}(x,\Sigma)\,dx : \Sigma \subset \bar{\Omega} \text{ closed, connected, } \mathcal{H}^1(\Sigma) \le l\right\}. \tag{8}$$

3.1. The problem in a unit disk of $\mathbf{R}^2$

We start by considering the case when Ω is the unit disc in $\mathbf{R}^2$. The first guess for Σ_{opt} in the case l is suffuciently small could be a circumference centered at the origin, which is however ruled out by theorem 3.10. A second guess for Σ_{opt}, always in the case of a sufficiently small l, would be a segment centered at the origin. This is again excluded by proposition below, the proof of which can be found in Appendix A.

Proposition 3.7. *There exists $l_0 > 0$ such that for all $l \leq l_0$ the centered segment of length l is not optimal for problem (8).*

We will now discuss the numerical approximations of the optimal set in a unit disc of $\mathbf{R}^2$. We will limit ourselves to presenting the results and ideas of the methods rather than the technical aspects of our algorithms. The latter will be described in details in [10]. Two cases will be subject of our numerical study:

(i) Σ is a set consisting of finite prescribed number of points (location problem);
(ii) Σ is a compact connected set with prescribed length.

For each of the above constraints we use a different numerical approach.

3.1.1. Optimal location of a finite number of points Given $n \in \mathbf{N}$, we are looking for an optimal n-point set Σ_n, which minimizes the quantity (8). If $n = 1$, it is not difficult to prove that the only minimizer is the center of the disk. When $n \geq 2$ is not too large we could guess that the optimal set Σ_n is given by the vertices of a centered regular polygon.

To approximate numerically Σ_n for $n > 1$, we use the classical finite difference method. As underlined in [8], this is a reasonable way to solve design optimization problems with few parameters. We present in figure 1 two pictures obtained by this process. The first one shows that for $n = 5$ the optimal set Σ_5 seems to be distributed on a regular centered polygon. The same situation occurs for $n = 2$, 3, 4. The second image represents the case of $n = 6$, and one can observe that the center of the disk is one of the optimal locations.

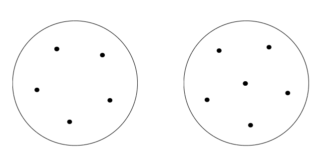

Figure 1. Optimal locations of 5 and 6 points in a disk.

3.1.2. OPTIMAL COMPACT CONNECTED SET WITH PRESCRIBED LENGTH We are now looking for an optimal set Σ_{opt} among all compact connected sets of the disk with one-dimensional Hausdorff measure not exceeding l. This situation is definitely more complex from a numerical point of view than the previous one. The main problem is to build a process which is able to identify the topology of Σ_l. Moreover, it is intuitively clear that this problem has a great number of local minima. Last but not least, the length constraint is rather difficult to handle numerically. Considering these difficulties, it seems natural to use an Evolutionary Algorithm (EAs) with an adaptive penalty method. We will not present here the theory of EAs but we refer the interested reader to [11] for an introduction to Adaptive methods in EAs. Further numerical details like the representation of Σ_{opt}, the cost function, the adaptive penalty and different test cases will be described more accurately in [10].

In figure 2 one can see the values $V(l)$ obtained by the chosen numerical method as a function of length constraint l. They are compared to numerical evaluations of (8) for some simple sets like a circumference, a regular cross (of two perpendicular intervals), a regular trisection (i.e. 3 equal intervals joined at one of their endpoints at the angle of 120 degrees each) and a segment, all centered at the origin. In this figure

$\circ$	stands for the cirumference centered at the origin,
$+$	stands for the centered perpendicular cross,
$*$	stands for the regular trisection,
- -	stands for the segment,
—	stands for the numerical approximations of optimal sets.

In the same figure we present a graph of $lV(l)$ as a function of l (in fact, in theorem 3.16 we will show that the quantity $lV(l)$ for the optimal set is bounded as $l \to +\infty$).

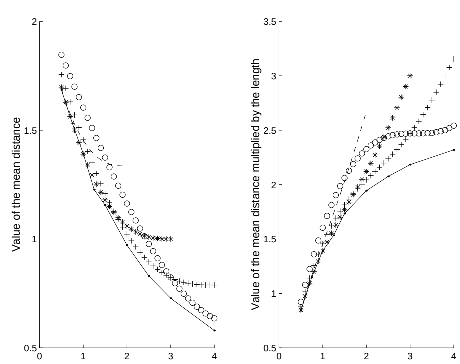

FIGURE 2

In figures 3–5 the numerical approximations of optimal sets in a disc for different lengths are shown. Perhaps somewhat unexpected is the fact that the number of singular points does not seem to be an increasing function of the prescribed length. More numerical computations are presented in Appendix B.

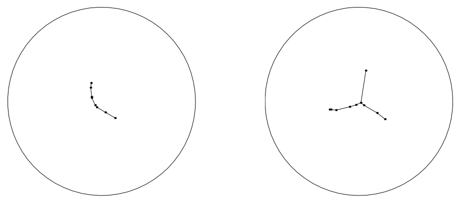

FIGURE 3. Optimal sets of length 0.5 and 1 in a unit disk

FIGURE 4. Optimal sets of length 1.25 and 1.5 in a unit disk

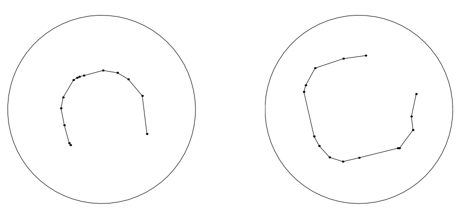

FIGURE 5. Optimal sets of length 2 and 3 in a unit disk

3.2. Singular points

From now on we present some simple results on the qualitiative properties of the optimal set Σ_{opt} in a generic closed convex set $\bar{\Omega}$. The results we present here do not completely answer the questions raised at the beginning of this section, but rather indicate what kind of strong results one can reasonably expect.

We start from a very simple proposition which gives a partial answer to Problem 3.3, restricting, roughly speaking, the possible singularities of the optimal set to only triple points, i.e. points where three curves meet with angles of 120 degrees. In order not to overburden the paper with technical details we just formulate the proposition in its simplest way, namely, we assert that a cross can never be an optimal set in $\bar{\Omega} \subset \mathbf{R}^2$. We call a cross the union of two mutually perpendicular closed intervals intersecting in a point which is internal for both.

Proposition 3.8. *Let $N = 2$. Then a cross is not an optimal set.*

PROOF: Assume the contrary, i.e. that Σ_{opt} is a cross, and assume without loss of generality that its center (i.e. the intersection point of two intervals) is the origin of coordinate system.

STEP 1. For every sufficiently small $\varepsilon > 0$ the set $D_\varepsilon := \Sigma_{opt} \cap \partial B_\varepsilon(0)$ consists of exactly 4 points. Denote by $S_4(D_\varepsilon) \subset B_\varepsilon(0)$ a set of minimum length in the ball $B_\varepsilon(0)$ which connects the all the four points of D_ε as in figure 6 (we will call it a *Steiner connection* of these points)as in Figure 6. Observe now that

$$\mathcal{H}^1(\Sigma_{opt} \cap B_\varepsilon(0)) - \mathcal{H}^1(S_4(D_\varepsilon)) \geq C\varepsilon$$

for some $C > 0$ (here and below the value of the constant C may vary from line to line). In fact, to show this estimate, it is enough to prove ist rescaled version

$$\mathcal{H}^1((1/\varepsilon)\Sigma_{opt} \cap B_1(0)) - \mathcal{H}^1(S_4(D_1)) \geq C,$$

which follows from the direct computation

$$\mathcal{H}^1((1/\varepsilon)\Sigma_{opt} \cap B_1(0)) = 4 \text{ and } \mathcal{H}^1(S_4(D_1)) = \sqrt{2}(\sqrt{3}+1) < 4.$$

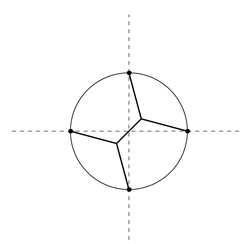

FIGURE 6. Steiner connection

STEP 2. Let now Σ_ε be Σ_{opt} outside of $B_\varepsilon(0)$ and $S_4(D_\varepsilon)$ inside $B_\varepsilon(0)$. Clearly then

$$\text{dist}\,(x, \Sigma_\varepsilon) \leq \text{dist}\,(x, \Sigma_{opt}) + 2\varepsilon$$

for all $x \in \bar{\Omega}$. Moreover, let Λ_ε be the set of points whose projection on Σ_{opt} is different from the projection on Σ_ε. It is easy to see that Λ_ε is contained inside a square centered at the origin of size ε. Therefore

$$\int_\Omega \text{dist}\,(x, \Sigma_\varepsilon)\,dx \leq \int_\Omega \text{dist}\,(x, \Sigma_{opt})\,dx + C\varepsilon^2 \tag{9}$$

According to Step 1, we have however at least the additional length $C\varepsilon$ to use in order to decrease the functional. This is achieved by using lemma 3.12 below. Namely, according to this lemma we can attach a segment of this length to Σ_ε obtaining a set Σ'_ε so that

$$\int_\Omega \operatorname{dist}(x, \Sigma'_\varepsilon)\, dx \le \int_\Omega \operatorname{dist}(x, \Sigma_\varepsilon)\, dx - C\varepsilon^{3/2}. \tag{10}$$

Now, in view of (9) and (10), one gets for sufficiently small $\varepsilon > 0$ the estimate

$$\int_\Omega \operatorname{dist}(x, \Sigma'_\varepsilon)\, dx \le \int_\Omega \operatorname{dist}(x, \Sigma_{opt})\, dx - C\varepsilon^{3/2},$$

for some $C > 0$, which contradicts the optimality of Σ_{opt}. □

Remark 3.9. Arguments similar to the ones used in the above proof allow to show that our conjecture stated in Problem 3.3 is true at least if Σ_{opt} is piecewise sufficiently smooth.

3.3. Absence of loops

We present here a result giving a partial answer to Problem 3.2.

Theorem 3.10. *Let $N = 2$. Then for $\mathcal{H}^1$-a.e. $x \in \Sigma_{opt}$ and for all sufficiently small $\varepsilon > 0$ the set $\Sigma_{opt} \setminus B_\varepsilon(x)$ is disconnected.*

PROOF: Suppose the contrary, namely that the set

$$A := \{x \in \Sigma_{opt} : \exists \{\varepsilon_\nu\}_{\nu=1}^\infty, \varepsilon_\nu \searrow 0 \text{ such that } \Sigma_{opt} \setminus B_{\varepsilon_\nu}(x) \text{ is connected}\}$$

has positive length, i.e. $\mathcal{H}^1(A) > 0$. Further on we omit the reference to the index ν writing always ε instead of ε_ν. Let $\tilde{A} \subset A$ stand for the set of density points of Σ_{opt}, i.e. such that for every $x \in \tilde{A}$ one has

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{H}^1(\Sigma_{opt} \cap B_\varepsilon(x))}{2\varepsilon} = 1.$$

Since Σ_{opt} is $(\mathcal{H}^1, 1)$-rectifiable, then $\mathcal{H}^1(\tilde{A}) = \mathcal{H}^1(A)$ by Besicovitch-Marstrand-Mattila theorem (theorem 2.63 from [2]).

For every $x \in \tilde{A}$ and $\varepsilon > 0$ let $T(x, \varepsilon)$ stand for the union of transport rays of the Monge-Kantorovich problem of transporting $\mathcal{L}^N \llcorner \Omega$ to its projection over Σ which end at $\Sigma_{opt} \cap B_\varepsilon(x)$. Set $\Sigma_\varepsilon(x) := \Sigma_{opt} \setminus B_\varepsilon(x)$. Clearly, according to our assumption, $\Sigma_\varepsilon(x)$ is still closed and connected, and, of course, satisfies the length constraint since $\mathcal{H}^1(\Sigma_\varepsilon(x)) \le \mathcal{H}^1(\Sigma_{opt})$. The following estimate is valid

$$\begin{aligned}
&\int_\Omega \operatorname{dist}(z, \Sigma_\varepsilon(x))\, dz = \\
&\qquad \int_{\Omega \setminus T(x,\varepsilon)} \operatorname{dist}(z, \Sigma_\varepsilon(x))\, dz + \int_{T(x,\varepsilon)} \operatorname{dist}(z, \Sigma_\varepsilon(x))\, dz \le \\
&\qquad \int_{\Omega \setminus T(x,\varepsilon)} \operatorname{dist}(z, \Sigma_{opt})\, dz + \int_{T(x,\varepsilon)} (\operatorname{dist}(z, \Sigma_{opt}) + \varepsilon)\, dz = \\
&\qquad \int_\Omega \operatorname{dist}(z, \Sigma_{opt})\, dz + \varepsilon \mathcal{L}^N(T(x, \varepsilon)).
\end{aligned}$$

One also has

$$\mathcal{L}^N(T(x,\varepsilon)) = \psi(B_\varepsilon(x)),$$

where ψ stands for the projection of $\mathcal{L}^N$ to Σ_{opt}. But

$$\limsup_{\varepsilon\to 0^+} \frac{\psi(B_\varepsilon(x))}{\varepsilon} < +\infty$$

for $\mathcal{H}^1$-a.e. $x \in \tilde{A}$, which implies

$$\mathcal{L}^N(T(x,\varepsilon)) \le C\varepsilon$$

for some $C = C(x) > 0$ depending on x and for sufficiently small $\varepsilon > 0$. Summing up, for $\mathcal{H}^1$-a.e. $x \in \tilde{A}$ one has

$$\int_\Omega \operatorname{dist}(z, \Sigma_\varepsilon(x))\, dz \le \int_\Omega \operatorname{dist}(z, \Sigma_{opt})\, dz + C\varepsilon^2 \tag{11}$$

once $\varepsilon > 0$ is sufficiently small.

Let now

$$l(\varepsilon) := \mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1(\Sigma_\varepsilon(x)),$$

where $x \in \tilde{A}$ is such that (11) holds. Since we chose $x \in \tilde{A}$, one has $l(\varepsilon) = C\varepsilon + o(\varepsilon)$ for $\varepsilon \to 0$ for some $C = C(x) > 0$. Applying now lemma 3.12 with $\Sigma_\varepsilon(x)$ instead of Σ and $l(\varepsilon)$ instead of ε to find a closed connected set $\Sigma' \subset \bar{\Omega}$ satisfying $\mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma) + l(\varepsilon)$ and

$$\int_\Omega \operatorname{dist}(x, \Sigma')\, dx \le \int_\Omega \operatorname{dist}(x, \Sigma_\varepsilon(x))\, dx - C\varepsilon^{3/2}$$

for some $C > 0$ as $\varepsilon \to 0^+$. Using (11), one arrives at the estimate

$$\int_\Omega \operatorname{dist}(x, \Sigma')\, dx \le \int_\Omega \operatorname{dist}(z, \Sigma_{opt})\, d(z) - C\varepsilon^{3/2}$$

for sufficiently small $\varepsilon > 0$, which gives a contradiction with the optimality of Σ_{opt}.
□

Remark 3.11. For a piecewise smooth Σ_{opt} the conclusion of theorem 3.10 implies the absense of loops indicated in Problem 3.2. In fact in this case $\mathbf{R}^2 \setminus \Sigma_{opt}$ is connected.

Lemma 3.12. *Let $N = 2$ and $\Sigma \subset \bar{\Omega}$ be a compact connected set such that $\Sigma \cap \Omega \neq \emptyset$. Then there is a constant $C > 0$ such that for all $\varepsilon > 0$ there is a segment S_ε of length ε such that the following conditions hold:*

- *the set $\Sigma_\varepsilon := \Sigma \cup S_\varepsilon$ is connected;*
- *the inequality*

$$\int_\Omega \mathit{dist}\,(z, \Sigma_\varepsilon)\, dz \le \int_\Omega \mathit{dist}\,(z, \Sigma)\, dz - C\varepsilon^{3/2}$$

is true for sufficiently small $\varepsilon > 0$.

PROOF: Let $A \in \Sigma \cap \Omega$ and $O \in \Omega \backslash \Sigma$ such that A is a projection of O on Σ. Without loss of generality consider the coordinate axes to be positioned so that O is the origin of the coordinate system and the following conditions are satisfied:

- $B_1(0) \subset \Omega$;
- $\Sigma \cap B_1(0) = \emptyset$ where $B_1(0)$ stands for the unit open disc;
- $A := (0, -1) \in \Sigma \cap \bar{B}_1(0)$.

Let $A_\varepsilon := (0, -1+\varepsilon)$, $S_\varepsilon := [A, A_\varepsilon]$ stand for the closed interval with endpoints A and A_ε and $\Sigma_\varepsilon := \Sigma \cup S_\varepsilon$. Consider the set

$$\Lambda_\varepsilon := \{z = (x,y) \in B_1(0) \,:\, y \leq 0,\ d(z, A_\varepsilon) - \text{dist}\,(z, \partial B_1(0)) \leq -\varepsilon/4\}.$$

If we are able to prove that $\mathcal{L}^N(\Lambda_\varepsilon) \geq C\varepsilon^{1/2}$ for some $C > 0$, then the proof will be finished. Indeed, for $z \in \Lambda_\varepsilon$ one has

$$\text{dist}\,(z, \Sigma_\varepsilon) \leq d(z, A_\varepsilon) \leq \text{dist}\,(z, \partial B_1(0)) - \varepsilon/4 \leq \text{dist}\,(z, \Sigma) - \varepsilon/4,$$

and hence

$$\int_\Omega \text{dist}\,(z, \Sigma_\varepsilon)\, dz - \int_\Omega \text{dist}\,(z, \Sigma)\, dz \leq -\varepsilon \mathcal{L}^N(\Lambda_\varepsilon)/2 \leq -C\varepsilon^{3/2}$$

It remains therefore to estimate $\mathcal{L}^N(\Lambda_\varepsilon)$ from below. Let $0 < k < 1/2$ and

$$\Pi_{k,\varepsilon} := \left\{ z = (x,y) \in B_1(0) \,:\, |y + 1/2| \leq (1 - 4k^2)^{1/2}/2,\ |x| \leq k\varepsilon^{1/2} \right\}.$$

For every $(x,y) \in \Pi_{k,\varepsilon}$, we have

$$\begin{aligned} d(z, A_\varepsilon) - \text{dist}\,(z, \partial B_1(0)) &\leq \left(x^2 + (y+1-\varepsilon)^2\right)^{1/2} - 1 + \left(x^2 + y^2\right)^{1/2} \\ &\leq \left(k^2\varepsilon + (y+1-\varepsilon)^2\right)^{1/2} - 1 + \left(k^2\varepsilon + y^2\right)^{1/2} \\ &\leq -1 + |y| + k^2\varepsilon/2|y| + (y+1) + \\ &\quad\ \varepsilon\left(k^2 - 2(y+1)\right)/2(y+1) + \alpha\varepsilon^2 \\ &= \varepsilon\left(k^2/(y+1) - 2 + k^2/|y|\right)/2 + \alpha\varepsilon^2, \end{aligned}$$

where $\alpha = \alpha(k) > 0$ is some constant. It is easy to verify that since $z \in \Pi_{k,\varepsilon}$, then

$$k^2/(y+1) - 2 + k^2/|y| \leq -1,$$

and hence

$$d(z, A_\varepsilon) - \text{dist}\,(z, \partial B_1(0)) \leq -\varepsilon/4$$

for sufficiently small $\varepsilon > 0$, which means $\Pi_{k,\varepsilon} \subset \Lambda_\varepsilon$ for such ε. Thus

$$\mathcal{L}^N(\Lambda_\varepsilon) \geq \mathcal{L}^N(\Pi_{k,\varepsilon}) \geq C\varepsilon^{1/2},$$

for sufficiently small $\varepsilon > 0$, which concludes the proof. □

3.4. The optimal set and the boundary

We are able to give now a partial answer to Problem 3.4. It only says that the intersection of Σ_{opt} with $\partial\Omega$ cannot have a positive $\mathcal{H}^1$ measure. Moreover, it is only proven when $N = 2$ and for $\Omega \subset \mathbf{R}^2$ convex, with sufficiently regular boundary having everywhere positive curvature.

Theorem 3.13. *Let $\Omega \subset \mathbf{R}^2$ be a convex set with a C^2 boundary having everyhere positive curvature and Σ_{opt} be a solution to problem (8). Then $\mathcal{H}^1(\Sigma_{opt} \cap \partial\Omega) = 0$.*

PROOF: Suppose the contrary, i.e. $\mathcal{H}^1(\Sigma_{opt} \cap \partial\Omega) =: \alpha > 0$. Consider then for every $\varepsilon > 0$ the set

$$\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) > \varepsilon\}$$

and let p_ε stand for the projection map on the closed convex set $\bar{\Omega}_\varepsilon$, namely,

$$p_\varepsilon(x) := x + (\varepsilon - d(x))^+ \nabla d(x),$$

where $d(x) := \mathrm{dist}\,(x, \partial\Omega)$ and $(\cdot)^+$ stands for the positive part function. Let $\Sigma_\varepsilon := p_\varepsilon\,(\Sigma_{opt})$. Clearly, Σ_ε is still connected and compact.

STEP 1. Let us estimate from below

$$\mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1(\Sigma_\varepsilon).$$

This difference is nonnegative since p_ε is Lipschitz continuous with constant one. To make a more precise estimate, we note that

$$\nabla p_\varepsilon(x) := Id + (\varepsilon - d(x))^+ \nabla^2 d(x) - \nabla d(x) \otimes \nabla d(x), \text{ for } 0 \le d(x) < \varepsilon,$$

where Id stands for the identity matrix. In particular, for $x \in \partial\Omega$ one has

$$\nabla p_\varepsilon(x) := T + \varepsilon \nabla^2 d(x) \text{ where } T := Id - \nabla d(x) \otimes \nabla d(x).$$

A simple calculation shows that $(Tz, z) = |z|^2 - |z_\nu|^2 \le |z|^2$, where z_ν stands for the projection of z on the direction of the normal $\nu := \nabla d(x)$ to the boundary $\partial\Omega$ at the point x. Since we assumed $\partial\Omega$ to be of class C^2, then $\nabla^2 d$ is continuous over $\partial\Omega$, and, moreover, since the curvature of $\partial\Omega$ is supposed to be strictly positive, then the matrix $-\nabla^2 d(x)$ is positive definite along the directions tangential to $\partial\Omega$. Namely, there is a $K > 0$ such that $-(\nabla^2 d(x) z_\tau, z_\tau) \ge K|z_\tau|^2$ for all $x \in \partial\Omega$, where z_τ stands for the projection of z on the tangent space to the boundary $\partial\Omega$ at the point x. Also, clearly $\nabla p_\varepsilon(x) z_\nu = 0$ for every $z \in \mathbf{R}^N$. Summing up, we have

$$|\nabla p_\varepsilon(x) z| = |\nabla p_\varepsilon(x) z_\tau| \le (1 - K\varepsilon)|z_\tau| \le (1 - K\varepsilon)|z|$$

for all $x \in \partial\Omega$.

Without loss of generality we may suppose that $\partial\Omega$ is parametrized by a curve and let γ: $[0, L] \to \partial\Omega$ be the arc-length parametrization of $\partial\Omega$. Then, setting $e := \Sigma_{opt} \cap \partial\Omega$, we get by the area formula

$$\begin{aligned}\mathcal{H}^1(p_\varepsilon(e)) &= \int_{\gamma^{-1}(e)} |\nabla p_\varepsilon(\gamma(t))\gamma'(t)|\, dt \leq \\ &(1-K\varepsilon)\int_{\gamma^{-1}(e)} |\gamma'(t)|\, dt = \\ &(1-K\varepsilon)\mathcal{H}^1(e).\end{aligned}$$

Hence we finally arrive at the estimate

$$\mathcal{H}^1(\Sigma_{opt}) - \mathcal{H}^1(\Sigma_\varepsilon) \geq \mathcal{H}^1(e) - \mathcal{H}^1(p_\varepsilon(e)) \geq K\alpha\varepsilon.$$

STEP 2. Clearly,

$$\int_\Omega \mathrm{dist}\,(z, \Sigma_\varepsilon)\, dz \geq \int_\Omega \mathrm{dist}\,(z, \Sigma_{opt})\, dz.$$

We now estimate more precisely the difference between the two integrals. For this purpose consider a point of $z \in \Omega$ for which

$$\mathrm{dist}\ (z, \Sigma_\varepsilon) > \mathrm{dist}\ (z, \Sigma_{opt})\,. \tag{12}$$

Denote by z_0 an arbitrary projection of z to Σ_{opt}. Then obviously $z_0 \in \bar{\Omega} \setminus \Omega_\varepsilon$. We claim first that $z \in \bar{\Omega}\backslash\Omega_\varepsilon$. In fact, suppose the contrary, and let $z_{0\varepsilon}$ stand for the projection fo z_0 to $\partial\Omega_\varepsilon$. Since Ω_ε is convex, then z belongs to a half-plane bounded by a tangent line to Ω_ε at $z_{0\varepsilon}$, and hence also to the half-plane bounded by a line passing through the center of the segment $[z_0, z_{0\varepsilon}]$ perpendicular to the latter (see figure 7). This implies $d(z, z_{0\varepsilon}) < d(z, z_0)$ and therefore

$$\mathrm{dist}\,(z, \Sigma_\varepsilon) \leq d(z, z_{0\varepsilon}) < d(z, z_0) = \mathrm{dist}\,(z, \Sigma_{opt})$$

which contradicts the assumption (12).

Let now z_ε stand for a projection of z on $\partial\Omega_\varepsilon$. One has then

$$\begin{aligned}\mathrm{dist}\,(z, \Sigma_\varepsilon) - \mathrm{dist}\,(z, \Sigma_{opt}) &= d(z, z_\varepsilon) - d(z, z_0) \\ &\leq d\,(z, z_\varepsilon) \leq \varepsilon.\end{aligned}$$

With the above estimate we get

$$\begin{aligned}\int_\Omega \mathrm{dist}\,(z, \Sigma_\varepsilon)\, dz - \int_\Omega \mathrm{dist}\,(z, \Sigma_{opt})\, dz &\leq \int_{\Omega\backslash\Omega_\varepsilon} (\mathrm{dist}\,(z, \Sigma_\varepsilon) - \mathrm{dist}\,(z, \Sigma_{opt}))\, dz \\ &\leq \varepsilon\mathcal{L}^2\,(\Omega\backslash\Omega_\varepsilon) \\ &= C\varepsilon^2 + o(\varepsilon^2)\end{aligned} \tag{13}$$

for some $C \geq 0$ as $\varepsilon \to 0^+$.

STEP 3. Consider $O \in \Omega_\varepsilon\backslash\Sigma_\varepsilon$ and its arbitrary projection O_ε to Σ_ε. We define

$$\Sigma'_\varepsilon := \Sigma_\varepsilon \cup S_\varepsilon$$

where S_ε is a segment of length $\mathcal{H}^1(\Sigma_{opt})- \mathcal{H}^1(\Sigma_\varepsilon)$ starting at $p_\varepsilon\,(O)$ and pointing to O. In view of lemma 3.12 one has

$$\int_\Omega \mathrm{dist}\,(z, \Sigma'_\varepsilon)\, dz \leq \int_\Omega \mathrm{dist}\,(z, \Sigma_\varepsilon)\, dz - C\varepsilon^{3/2} \tag{14}$$

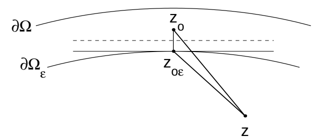

FIGURE 7

for some $C > 0$. Combining (13) and (14), we have

$$\int_\Omega \operatorname{dist}(z, \Sigma'_\varepsilon)\, dz \le \int_\Omega \operatorname{dist}(z, \Sigma_{opt})\, dz - C\varepsilon^{3/2}$$

for some $C > 0$ when $\varepsilon \to 0^+$, which contradicts the optimality of Σ_{opt}. □

Now we prove a result which in a sense is much stronger, namely, that when the length of the optimal set is sufficiently small, it must stay away from the boundary $\partial\Omega$. The result will be proven for a generic space dimension N.

Theorem 3.14. *There exist $l_0 > 0$ and $d_0 > 0$ which depend only on Ω and on N such that for all $l < l_0$ the optimal set Σ_{opt} solving the problem (7) satisfies dist $(\Sigma_{opt}, \partial\Omega) > d_0$. In particular, $\Sigma_{opt} \cap \partial\Omega = \emptyset$.*

PROOF: Consider the functionals F_l defined over compact connected subsets $\Sigma \subset \bar{\Omega}$ according to the formula

$$F_l(\Sigma) := \begin{cases} \displaystyle\int_\Omega \operatorname{dist}(x, \Sigma)\, dx, & \text{if } \mathcal{H}^1(\Sigma) \le l, \\ +\infty, & \text{otherwise.} \end{cases}$$

As $l \to 0$, these functionals Γ^--converge to a functional

$$F_0(\Sigma) := \begin{cases} \displaystyle\int_\Omega |x - P|\, dx, & \text{if } \Sigma = \{P\} \text{ consists of one point,} \\ +\infty, & \text{otherwise.} \end{cases}$$

In fact, if $\mathcal{H}^1(\Sigma_\nu) = l_\nu \searrow 0$ and $\Sigma_\nu \to \Sigma$ in the sense of Hausdorff, then $\mathcal{H}^1(\Sigma) = 0$ according to the Golab theorem, and hence Σ consists of a single point, $\Sigma = \{P\}$, while

$$F_0(\Sigma) = \lim_\nu F_{l_\nu}(\Sigma_\nu).$$

Supposing now that the assertion to be proven is false, we would have a sequence of optimal sets $\Sigma_\nu \subset \bar{\Omega}$ such that $\mathcal{H}^1(\Sigma_\nu) \to 0$ and dist $(\Sigma_\nu, \partial\Omega) \to 0$. Up to a subsequence we may assume Σ_ν to be converging in Hausdorff sense to some compact connected set Σ consisting of a single point, i.e. $\Sigma = \{P\}$. Moreover, clearly, $P \in \partial\Omega$. The above Γ-convergence implies that P is optimal in the sense that it minimizes the distance functional $\int_\Omega |x-Q|\,dx$ among all $Q \in \bar{\Omega}$. According to lemma 3.15 below such a point should belong to Ω, which is a contradiction. □

Lemma 3.15. *Consider the optimal location problem for a single point in a convex set $\bar{\Omega} \subset \mathbf{R}^N$, i.e. a problem of finding a point $P \in \bar{\Omega}$ which provides the minimum of*

$$\inf\left\{\int_\Omega |x-Q|\,dx\ :\ Q \in \bar{\Omega}\right\}.$$

Then $P \notin \partial\Omega$.

PROOF: Let $P \in \partial\Omega$ be an arbitrary point of $\partial\Omega$. We will show that it is never optimal. In fact, without loss of generality assume that P is the origin of the coordinate system $(x_1, \dots, x_N)$ and that the first $N-1$ coordinates $(x_1, \dots, x_{N-1})$ are in the supporting hyperplane of Ω at P. Let the x_N axis be directed so that $x_N > 0$ for all $x \in \Omega$. Consider

$$F(x) := \int_\Omega |z-x|\,dz.$$

Then for each $i = 1, \dots, N$ one has

$$\frac{\partial F}{\partial x_i}(0, \dots, 0) = -\int_\Omega \frac{z_i}{|z-P|}\,dz < 0$$

showing the claim. □

3.5. Asymptotic estimates

We claim the following result on the asymptotic behaviour of the minimum value of the functional MK as the prescribed length tends to infinity, which gives a partial answer to Problem 3.4. Denote

$$V(l) := \int_\Omega \mathrm{dist}\,(x, \Sigma_{opt})\,dx$$

when $\mathcal{H}^1(\Sigma_{opt}) = l$.

Theorem 3.16. *Let $N > 1$. Then*

$$c \le V(l) l^{1/(N-1)} \le C$$

for some positive constants c and C which depend only on N and Ω.

To prove the lower estimate announced in the above theorem, we need the following lemma.

Lemma 3.17. *Let $N > 1$ and $Q \subset \mathbf{R}^N$ be a cube. Suppose that Q is divided by a uniform grid parallel to the edges into small subcubes with the side $\varepsilon > 0$. Let $\Sigma \subset \mathbf{R}^N$ be a Lipschitz curve of length l and k be the number of subcubes which have nonempty intersection with Σ. Then one has*

$$k \le c_1 l/\varepsilon + c_2$$

for some positive constants c_1 and c_2 which do not depend on ε and l.

PROOF: Note that to intersect all the cubes of the union of $2^N + 1$ cubes one needs a curve of length at least ε. In fact, in such a union there are two cubes, the distance between which is at least ε. Therefore, since the curve of length l connects k cubes, one has

$$l \geq [k/(2^N + 1)]\varepsilon,$$

where $[\cdot]$ stands for the integer part of the number. The above estimate shows the statement. □

PROOF OF THEOREM 3.16: The proof will be achieved in two steps.

STEP 1. Let $Q \subset \Omega$ be a cube. Divide Q by a uniform grid into small subcubes with the side $\varepsilon > 0$ (a dyadic decomposition is a particular example). Without loss of generality we may consider Σ_{opt} to be parametrized as a Lipschitz curve of length at most $2l$. Then

$$\int_\Omega \operatorname{dist}(x, \Sigma_{opt})\, dx \geq \int_Q \operatorname{dist}(x, \Sigma_{opt})\, dx.$$

The latter integral is estimated as follows. For each of the subcubes $Q_\varepsilon \subset Q$ which do not intersect Σ_{opt} one has

$$\int_{Q_\varepsilon} \operatorname{dist}(x, \Sigma_{opt})\, dx \geq \int_{\alpha Q_\varepsilon} \operatorname{dist}(x, \Sigma_{opt})\, dx \geq \\ \mathcal{L}^N(\alpha Q_\varepsilon)(1-\alpha)\varepsilon = \mathcal{L}^N(Q_\varepsilon)\alpha^N(1-\alpha)\varepsilon$$

for all $\alpha \in [0,1]$. Hence, maximizing the last expression in α, one gets

$$\int_{Q_\varepsilon} \operatorname{dist}(x, \Sigma_{opt})\, dx \geq C\varepsilon^{N+1}$$

for some $C > 0$. Let k' be a number of subcubes Q_ε not intersecting Σ_{opt}. Then

$$\int_Q \operatorname{dist}(x, \Sigma_{opt})\, dx \geq Ck'\varepsilon^{N+1}.$$

Using the estimate from the lemma 3.17 and the fact that the total number of subcubes is $C\varepsilon^{-N}$, we arrive therefore at the estimate

$$\int_\Omega \operatorname{dist}(x, \Sigma_{opt})\, dx \geq \int_Q \operatorname{dist}(x, \Sigma_{opt})\, dx \geq c_1\varepsilon - c_2\varepsilon^N l - c_3\varepsilon^{N+1}$$

for some positive constants c_1, c_2 and c_3 independent of l and ε. Since the latter estimate is valid for all $\varepsilon > 0$, we may plug in $\varepsilon := Cl^{1/(1-N)}$ for some positive constant C, which is at the moment unknown. We see then that with the choice $0 < C < (c_1/c_2)^{1/(N-1)}$ the latter estimate becomes the desired lower bound

$$\int_\Omega \operatorname{dist}(x, \Sigma_{opt})\, dx \geq cl^{1/(1-N)}$$

for some $c > 0$ and for sufficiently large l.

STEP 2. The upper estimate will immediately follow from the construction of a particular Σ with $\mathcal{H}^1(\Sigma) \le Cl$ and such that

$$\int_\Omega \operatorname{dist}(x,\Sigma)\,dx \le Cl^{1/(1-N)}, \tag{15}$$

where C denotes a positive constant different in different occasions. For this purpose consider a $(N-1)$-dimensional hyperplane π intersecting Ω by some open set T. Consider a uniform grid in T parallel to the coordinate axes in the hyperplane, with the size of a cell equal to ε. Clearly, the total length of this grid is less or equal than C/ε. Let Σ stand for the union of this grid with all the line segments perpendicular to π, passing through the nodes of T and staying in Ω. Since the total length of all such line segments is bounded from above by C/ε^{N-1}, for small ε we have $\mathcal{H}^1(\Sigma) \le C/\varepsilon^{N-1}$. Since now by construction $\operatorname{dist}(x,\Sigma) \le C\varepsilon$ for all $x \in \bar{\Omega}$, where C is independent of x, then

$$\int_\Omega \operatorname{dist}(x,\Sigma)\,dx \le C\varepsilon.$$

The estimate (15) follows now by setting $l := 1/\varepsilon^{N-1}$. □

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Appendix A. Proof of proposition 3.7

In order to prove proposition 3.7, we consider the deformation of the segment S_l described in figure 8. Namely, for all $\varepsilon > 0$, we define the polygonal line L_ε with vertices

$$(-l/2+\delta(\varepsilon),0),(0,\varepsilon) \text{ and } (l/2-\delta(\varepsilon),0), \text{ where } \delta(\varepsilon) = l/2 - \sqrt{l^2/4-\varepsilon^2}.$$

Obviously, $\mathcal{H}^1(L_\varepsilon) = \mathcal{H}^1(S_l)$ for all $\varepsilon > 0$.

We will now prove that for ε small enough,

$$\int_D \operatorname{dist}(z,L_\varepsilon)dz < \int_D \operatorname{dist}(z,S_l)dz.$$

Unfortunately, it is clear that the function $f : [-\eta,\eta] \to \mathrm{R}$ defined by

$$f(\varepsilon) := \int_D \operatorname{dist}(z,L_\varepsilon)dz,$$

is an even function with respect to ε. So we must expect its first derivative (once it is established that f is differentiable) in $\varepsilon = 0$ to be zero. Therefore, to prove

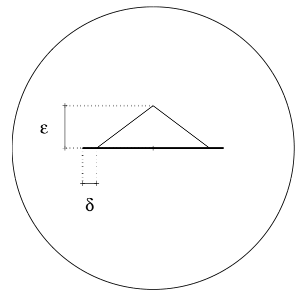

FIGURE 8. The deformation of the segment

our assumption, we will need to compute the asymptotic expansion of f in 0^+ up to the order 2. We assume until the end of the proof D is the half-disc.

We consider the partition of the half-disc defined by figure 9.

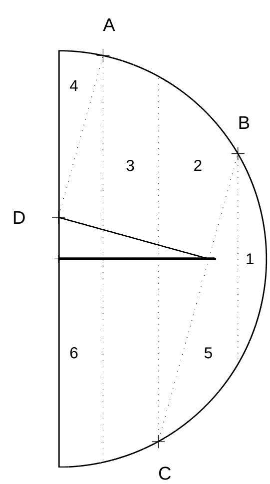

FIGURE 9. Partition of the half-disk

For each region D_i which of course depend of ε, we will compute the asymptotic expansion of $\int_{D_i(\varepsilon)} \mathrm{dist}(z, L_\varepsilon)dz$ in ε as $\varepsilon \to 0^+$.

STEP 1. We calculate the derivatives of first order with respect to ε of the above integrals. For this purpose we first consider the following asymptotic expansion in $\varepsilon = 0^+$ (the points A, B, C, D are defined in figure 9):

$$\begin{aligned}
x_A(\varepsilon) &= 2\varepsilon/l - 2\varepsilon^2/l + o\left(\varepsilon^2\right), \\
x_B(\varepsilon) &= l/2 + 2(1 - l^2/4)^{1/2}/l\varepsilon - 3\varepsilon^2/l + o\left(\varepsilon^2\right), \\
x_C(\varepsilon) &= l/2 - 2(1 - l^2/4)^{1/2}/l\varepsilon - 3\varepsilon^2/l + o\left(\varepsilon^2\right), \\
y_A(\varepsilon) &= 1 - 2\varepsilon^2/l^2 + o\left(\varepsilon^2\right), \\
y_B(\varepsilon) &= \left(1 - l^2/4\right)^{1/2} - \varepsilon + o(\varepsilon), \\
y_C(\varepsilon) &= -\left(1 - l^2/4\right)^{1/2} - \varepsilon + o(\varepsilon).
\end{aligned}$$

Furthermore, in each of the regions D_i it is possible to evaluate explicitly the distance $d(x, y, \varepsilon)$ from the point (x, y) to the polygonal line L_ε, namely

$$d(x,y,\varepsilon) := \begin{cases} \left((x - l/2 + \delta(\varepsilon))^2 + y^2\right)^{1/2}, & (x,y) \in D_1 \cup D_5, \\ \left(x^2 + (y-\varepsilon)^2\right)^{1/2}, & (x,y) \in D_4, \\ \left(\frac{(x\varepsilon + (y-\varepsilon)(l/2 - \delta(\varepsilon)))^2}{\left(\varepsilon^2 + (l/2 - \delta(\varepsilon))^2\right)}\right)^{1/2}, & (x,y) \in D_2 \cup D_3 \cup D_6. \end{cases}$$

With these notations we can compute the first derivative of the following integrals

$$\begin{aligned}
\frac{d}{d\varepsilon} \int_{D_1(\varepsilon)} d(x,y,0)\,dxdy \bigg|_{\varepsilon=0+} &= -2\left(1 - l^2/4\right)^{3/2}/l, \\
\frac{d}{d\varepsilon} \int_{D_2(\varepsilon)} d(x,y,0)\,dxdy \bigg|_{\varepsilon=0+} &= 2\left(1 - l^2/4\right)^{3/2}/l, \\
\frac{d}{d\varepsilon} \int_{D_3(\varepsilon)} d(x,y,0)\,dxdy \bigg|_{\varepsilon=0+} &= -2/l - 2\left(1 - l^2/4\right)^{3/2}/l, \\
\frac{d}{d\varepsilon} \int_{D_4(\varepsilon)} d(x,y,0)\,dxdy \bigg|_{\varepsilon=0+} &= 2/3l, \\
\frac{d}{d\varepsilon} \int_{D_5(\varepsilon)} d(x,y,0)\,dxdy \bigg|_{\varepsilon=0+} &= 2\left(1 - l^2/4\right)^{3/2}/l, \\
\frac{d}{d\varepsilon} \int_{D_6(\varepsilon)} d(x,y,0)\,dxdy \bigg|_{\varepsilon=0+} &= 4/3l.
\end{aligned}$$

Analogous calculus gives

$$\frac{d}{d\varepsilon} d(x,y,\varepsilon)\bigg|_{\varepsilon=0+} = \begin{cases} 0, & (x,y) \in D_1 \cup D_5, \\ -y/\sqrt{x^2+y^2}, & (x,y) \in D_4, \\ 2(x - l/2)\operatorname{sign} y/l, & (x,y) \in D_2 \cup D_3 \cup D_6. \end{cases}$$

Then we have

$$
\begin{aligned}
f(\varepsilon) &= \int_D d(z, L_\varepsilon) dz = \sum_{i=1}^{6} \int_{D_i(\varepsilon)} \operatorname{dist}(z, L_\varepsilon) dz \\
&= \sum_{i=1}^{6} \int_{D_i(0)} d(z,0)\, dz + \varepsilon \sum_{i=1}^{6} \int_{D_i(0)} \left.\frac{d}{d\varepsilon} d(z,\varepsilon)\right|_{\varepsilon=0+} dz + \\
&\quad \varepsilon \sum_{i=1}^{6} \frac{d}{d\varepsilon} \left.\int_{D_i(\varepsilon)} d(z,0)\, dz\right|_{\varepsilon=0+} + o(\varepsilon) \\
&= \int_D \operatorname{dist}(z, S_l) dz + o(\varepsilon).
\end{aligned}
$$

As expected, f is differentiable in 0^+ and its first derivative in this point is zero.

STEP 2. We have now to calculate the derivatives of second order. Estimates similar to those made in Step 1 give

$$
\begin{aligned}
\frac{d^2}{d\varepsilon^2} \left.\int_{D_1(\varepsilon)} d(x,y,0)\, dxdy\right|_{\varepsilon=0+} &= -10\left(l^2/4-1\right)/l, \\
\frac{d^2}{d\varepsilon^2} \left.\int_{D_2(\varepsilon)} d(x,y,0)\, dxdy\right|_{\varepsilon=0+} &= -4\left(l^2/4-1\right)/l, \\
\frac{d^2}{d\varepsilon^2} \left.\int_{D_3(\varepsilon)} d(x,y,0)\, dxdy\right|_{\varepsilon=0+} &= 2\left(5l^2/4-3\right)/l, \\
\frac{d^2}{d\varepsilon^2} \left.\int_{D_i(\varepsilon)} d(x,y,0)\, dxdy\right|_{\varepsilon=0+} &= -2/l, \qquad i=4,6, \\
\frac{d^2}{d\varepsilon^2} \left.\int_{D_5(\varepsilon)} d(x,y,0)\, dxdy\right|_{\varepsilon=0+} &= 4\left(l^2/4-1\right)/l.
\end{aligned}
$$

and

$$
\left.\frac{d^2}{d\varepsilon^2} d(x,y,\varepsilon)\right|_{\varepsilon=0+} = \begin{cases} \dfrac{x-l/2}{(x^2-xl+y^2l/2+l^2/4)^{1/2}}, & (x,y)\in D_1\cup D_5, \\ x^2/\left(x^2+y^2\right)^{3/2}, & (x,y)\in D_4, \\ -4\operatorname{sign} y/yl^2, & (x,y)\in D_2\cup D_3\cup D_6. \end{cases}
$$

It remains to compute the following terms

$$\begin{aligned}
\frac{d}{d\varepsilon}\int_{D_i(\varepsilon)} \frac{d}{d\varepsilon} d(x,y,\varepsilon)\bigg|_{\varepsilon=0+} dxdy\bigg|_{\varepsilon=0+} &= 0,\ i=1,\ 2,\ 3,\ 5,\\
\frac{d}{d\varepsilon}\int_{D_4(\varepsilon)} \frac{d}{d\varepsilon} d(x,y,\varepsilon)\bigg|_{\varepsilon=0+} dxdy\bigg|_{\varepsilon=0+} &= -1/l,\\
\frac{d}{d\varepsilon}\int_{D_6(\varepsilon)} \frac{d}{d\varepsilon} d(x,y,\varepsilon)\bigg|_{\varepsilon=0+} dxdy\bigg|_{\varepsilon=0+} &= 1/l.
\end{aligned}$$

Summing up, we have

$$\begin{aligned}
f(\varepsilon) &= \int_D \operatorname{dist}(z,L_\varepsilon)dz = \sum_{i=1}^{6}\int_{D_i(\varepsilon)} \operatorname{dist}(z,L_\varepsilon)dz\\
&= \sum_{i=1}^{6}\int_{D_i(0)} d(z,0)\,dz + \varepsilon^2\sum_{i=1}^{6}\frac{d}{d\varepsilon}\int_{D_i(\varepsilon)}\frac{d}{d\varepsilon}d(z,\varepsilon)\bigg|_{\varepsilon=0+} dz\bigg|_{\varepsilon=0+} +\\
&\quad \varepsilon^2\sum_{i=1}^{6}\frac{d^2}{d\varepsilon^2}\int_{D_i(\varepsilon)} d(z,0)\,dz\bigg|_{\varepsilon=0+} +\\
&\quad \varepsilon^2\sum_{i=1}^{6}\int_{D_i(0)}\frac{d^2}{d\varepsilon^2}d(z,\varepsilon)\bigg|_{\varepsilon=0+} dz + o\left(\varepsilon^2\right)\\
&= \int_D \operatorname{dist}(z,S_l)dz + \varepsilon^2\int_{D_1(0)}\frac{d^2}{d\varepsilon^2}d(z,\varepsilon)\bigg|_{\varepsilon=0+} dz +\\
&\quad \varepsilon^2\int_{D_3(0)}\frac{d^2}{d\varepsilon^2}d(z,\varepsilon)\bigg|_{\varepsilon=0+} dz + o\left(\varepsilon^2\right).
\end{aligned}$$

STEP 3. It remains to estimate the sign of the coefficient of ε^2. That is, we have to find the sign of the quantity

$$\begin{aligned}
\alpha_2 := \int_{D_1(0)} \frac{2\,(x-l/2)}{l\,(x^2-xl+l^2/4+y^2)^{1/2}}\,dydx - \int_{D_3(0)}\frac{\operatorname{sign} y}{yl^2/4}\,dydx =\\
\int_{l/2}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,(x-l/2)}{l\,(x^2-xl+l^2/4+y^2)^{1/2}}\,dydx-\\
\int_0^{l/2}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{\operatorname{sign} y}{yl^2/4}\,dydx.
\end{aligned} \tag{16}$$

To conclude the proof, we will show that for l small enough the above quantity is always strictly negative. We will compute its asymptotic expansion in $l=0^+$. On one hand

$$\frac{2\,(x-l/2)}{l\,(x^2-xl+l^2/4+y^2)^{1/2}} = \frac{2x}{(x^2+y^2)^{1/2}}l^{-1} - \frac{y^2}{(x^2+y^2)^{3/2}} - \frac{3}{4}\frac{xy^2}{(x^2+y^2)^{5/2}}l + o\,(l)\,,$$

and again computing derivatives we have

$$\int_{l/2}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,(x-l/2)}{l\,(x^2-xl+l^2/4+y^2)^{1/2}}dydx = \frac{2}{l} - \int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{y^2}{(x^2+y^2)^{3/2}}dydx + o\,(1)\,.$$

And on the other hand,

$$\int_0^{l/2}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{\operatorname{sign} y}{yl^2/4}dydx = \frac{2}{3}\frac{l^2/4-3}{l}.$$

So we have shown that

$$\alpha_2 = -\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{y^2}{(x^2+y^2)^{3/2}}dydx + o\,(1)$$

which concludes the proof. □

Appendix B. More numerical results

Here we present some numerical results for optimal sets in a unit square of $\mathbf{R}^2$ and in a unit ball in $\mathbf{R}^3$ obtained with the use of the same evolutionary algorithms with adaptive penalty method that were employed to get optimal sets in a unit disc of $\mathbf{R}^2$. We see that these results confirm our expectations about the qualitative properties of optimal sets. In fact, the numerical approximations of optimal sets obtained are just unions of finite number of injective curves joined by triple points (i.e. points where three curves meet at an angle of 120 degrees), and they never touch the boundary of the ambient set. Moreover, it seems that if the length of the optimal set is suffciently small, then this set contains no triple points.

FIGURE 10. Optimal sets of length 0.5 and 1 in a unit square

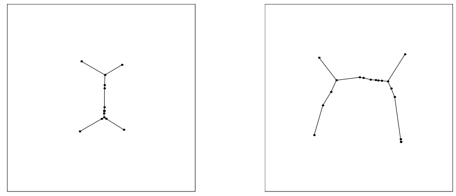

FIGURE 11. Optimal sets of length 1.5 and 2.5 in a unit square

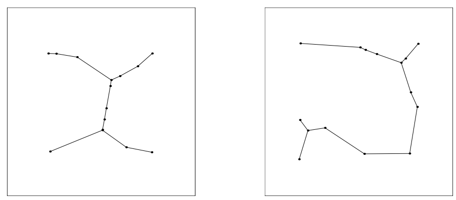

FIGURE 12. Optimal sets of length 3 and 4 in a unit square

FIGURE 13. Optimal sets of length 1 and 2 in the unit ball of $\mathbf{R}^3$

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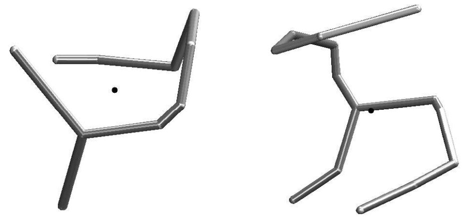

FIGURE 14. Optimal sets of length 3 and 4 in the unit ball of $\mathbf{R}^3$

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