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Paulo Carrillo Rouse, Bertrand Monthubert

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An index theorem for manifolds with boundary

BY PAULO CARRILLO ROUSE AND BERTRAND MONTUBERT

Abstract

In [2] II.5, Connes gives a proof of the Atiyah-Singer index theorem for closed manifolds by using deformation groupoids and appropriate actions of these on $\mathbb{R}^N$. Following these ideas, we prove an index theorem for manifolds with boundary.

Résumé


Version française abrégée

Dans [2], II.5, Alain Connes donna une preuve du théorème d’Atiyah-Singer pour une variété fermée entièrement fondée sur l’utilisation de groupoïdes, grâce à une action du groupoïde tangent de la variété sur $\mathbb{R}^N$. L’idée centrale est de remplacer des groupoïdes qui ne sont pas (Morita) équivalents à des espaces, par des groupoïdes obtenus par produit croisé et qui possèdent cette propriété, ce qui permet ensuite de donner une formule.

Si $X$ est une variété à bord $\partial X$, nous construisons le groupoïde $T_bX := (adG_{\partial X} \times \mathbb{R}) \cup \partial TX$ en recollant $adG_{\partial X} \times \mathbb{R}$ avec $TX$ le long de leur bord commun $T\partial X \times \partial X \times (0, 1)$ est le groupoïde adiabatique). Nous le recollons alors avec le groupoïde tangent de l’intérieur de $X$, $T_G = TX \cup \hat{X} \times \hat{X} \times (0, 1) : T_G := T_bX \cup \hat{T}_G$. Il existe un homomorphisme $T_G \xrightarrow{h} \mathbb{R}^N$ induit par un plongement de $X$ dans $\mathbb{R}^{N-1} \times \mathbb{R}_+$, tel que $\partial X$ se plonge dans $\mathbb{R}^{N-1} \times \mathbb{R}_+ \times \{0\}$ et $\hat{X}$ se plonge dans $\mathbb{R}^{N-1} \times \mathbb{R}^*_+$. Le produit croisé de $T_G$ par $h$ (noté $(T_G)_h)$ est un groupoïde propre dont les groupes d’isotropie sont triviaux, il est donc Morita-équivalent à son espace d’orbites.

Soit $V(\hat{X})$ le fibré normal de $\hat{X}$ dans $\mathbb{R}^N$, et $V(\partial X)$ le fibré normal de $\partial X$ dans $\mathbb{R}^{N-1}$; soit enfin $V(X) = V(\hat{X}) \cup V(\partial X)$. En notant $\mathcal{D}_0 = V(\partial X) \times \{0\} \cup \mathbb{R}^{N-1} \times \{0, 1\}$ et $\mathcal{D}_\circ = V(X) \times \{0\} \cup \mathbb{R}^N \times (0, 1]$ les déformations au cône normal, on construit les espaces $\mathcal{D} := V(X) \cup \mathcal{D}_0$ et $\mathcal{B} := \mathcal{D}_0 \cup \mathcal{D}_\circ$.

Proposition 0.1. Le groupoïde $(T_G)_h$ est Morita équivalent à l’espace $\mathcal{D}$.

Soit

$$ind_T = (e_1)_* \circ (e_0)^{-1} : K^0(T_bX) \longrightarrow K^0(\hat{X} \times \hat{X}) \approx \mathbb{Z}.$$

Définition 0.1 (Indice topologique pour une variété à bord). Soit $X$ une variété à bord. L’indice topologique de $X$ est le morphisme

$$ind_T : K^0(T_bX) \longrightarrow \mathbb{Z}$$

défini comme la composition des trois morphismes suivants.
(1) L’isomorphisme de Connes-Thom $CT_0$ suivi de l’équivalence de Morita $\mathcal{M}_0$ :

$$K^0(T_bX) \xrightarrow{CT_0} K^0((T_bX)_{h_0}) \xrightarrow{\delta_{h_0}} K^0(\mathcal{B}_h),$$

où $(T_bX)_{h_0}$ est le produit croisé de $T_bX$ par $h_0$ (l’homomorphisme $h$ en $t = 0$).

(2) Le morphisme indice de l’espace de déformation $\mathcal{B} : K^0(\mathcal{B}_h) \xrightarrow{(e_0),} K^0(\mathcal{B}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)$

(3) Le morphisme de périodicité de Bott : $K^0(\mathbb{R}^N) \xrightarrow{Bott} \mathbb{Z}$.

**Theorem 0.2.** Pour toute variété à bord, on a l’égalité

$$\text{ind}_X^N = \text{ind}_X^N.$$

1. Actions of $\mathbb{R}^N$

All the groupoids considered here will be continuous family groupoids. Hence we can consider their convolution and $C^*$-algebras without any problem. If $G$ is such a groupoid, we will denote by $K^0(G)$ the K-theory group of its $C^*$-algebra (unless explicitly written otherwise). This is consistent with the usual notation when $G$ is a space (a groupoid made only of units). In the sequel, given a smooth manifold $N$, we will denote by $\alpha^k G_N : T N \times \{0\} \bigsqcup N \times N \times \mathbb{R}^* \rightarrow N \times \mathbb{R}$, the deformation to normal cone of $N$ in $N \times N$ (for complete details about this deformation functor see [3]). At each time, we will need to restrict it to some interval, e.g. $[0, 1]$ gives the tangent groupoid, and $[0, 1]$ gives the adiabatic groupoid.

Let $G \cong M$ be a groupoid and $h : G \rightarrow \mathbb{R}^N$ a (smooth or continuous) homomorphism of groupoids, $(\mathbb{R}^N$ as an additive group). Connes defined the semi-direct product groupoid $G_h = G \times \mathbb{R}^N \cong M \times \mathbb{R}^N$ [2], II.5) with structure maps $t(\gamma, X) = (t(\gamma), X)$, $s(\gamma, X) = (s(\gamma), X + h(\gamma))$ and product $(\gamma, X) \circ (\eta, X + h(\gamma)) = (\gamma \circ \eta, X)$ for composable arrows.

At the level of $C^*$-algebras, $C^*(G_h)$ can be seen as the crossed product algebra $C^*(G) \rtimes \mathbb{R}^N$ where $\mathbb{R}^N$ acts on $C^*(G)$ by automorphisms by the formula: $\alpha_X(f)(\gamma) = e^{i(X \cdot h(\gamma))} f(\gamma)$, $\forall f \in C_c(G)$, (see [2], proposition II.5.7 for details). In particular, in the case $N$ is even, we have a Connes-Thom isomorphism in K-theory $K^0(G) \xrightarrow{\cong} K^0(G_h)$ [2], II.C).

Using this groupoid, Connes gives a conceptual, simple proof of the Atiyah-Singer Index theorem for closed smooth manifolds. Let $M$ be a smooth manifold, $G_M = M \times M$ its groupoid, and consider the tangent groupoid $T_G M$. It is well known that the index morphism provided by this deformation groupoid is precisely the analytic index of Atiyah-Singer [2]. In other words, the analytic index of $M$ is the morphism

(1) $K^0(TM) \xrightarrow{(e_0)^{-1}} K^0(TG_M) \xrightarrow{(e_1)_*} K^0(M \times M) = K^0(\mathcal{C}(L^2(M))) \approx \mathbb{Z},$

where $e_t$ are the obvious evaluation algebra morphisms at $t$. As discussed by Connes, if the groupoids appearing in this interpretation of the index were equivalent to spaces then we would immediately have a geometric interpretation of the index. Now, $M \times M$ is equivalent to a point (hence to a space), but the other fundamental groupoid playing a role is not, that is, $TM$ is a groupoid whose fibers
are the groups $T_2 M$, which are not equivalent (as groupoids) to a space. The idea of Connes is to use an appropriate action of the tangent groupoid in some $\mathbb{R}^N$ in order to translate the index (via a Thom isomorphism) in an index associated to a deformation groupoid which will be equivalent to some space.

2. Groupoids and Manifolds with boundary

Let $X$ be a manifold with boundary $\partial X$. We denote, as usual, $\overset{\circ}{X}$ the interior which is a smooth manifold. Let $X_\partial$ be the smooth manifold obtained by glueing $\overset{\circ}{X}$ with $\partial X \times [0, 1)$ along their common boundary, $\partial X \sim \partial X \times \{0\}$. Set $TX := TX_\partial|_X$, and consider the smooth manifold $\mathcal{T}_b X := (\text{ad}G_{\partial X} \times \mathbb{R}) \bigcup \overset{\circ}{X} TX$ obtained by glueing $\text{ad}G_{\partial X} \times \mathbb{R}$ and $\overset{\circ}{X} TX$ along their common boundary $\partial \overset{\circ}{X} \times \mathbb{R}$ ($\text{ad}G_{\partial X} = T\partial X \cup \partial X \times \partial X \times (0, 1)$ is the adiabatic groupoid). Now, we have a continuous family groupoid over $X_\partial$: $\mathcal{T}_b X \rightrightarrows X_\partial$. As a groupoid it is the union of the groupoids $\mathcal{T}_b G_{\partial X} \times \mathbb{R} \rightrightarrows \partial X \times [0, 1)$ and $TX \rightrightarrows X$. For the topology, it is very easy to see that all the groupoid structures are compatible with the gluings we considered.

We are going to consider a deformation groupoid $\mathcal{T}G_X h$. This will be a natural generalisation of the Connes tangent groupoid of a smooth manifold, to the case with boundary. The space of arrows $\mathcal{T}G_X := \mathcal{T}_b X \bigcup \mathcal{T}G_\overset{\circ}{X}$ is obtained by glueing at $\overset{\circ}{X}$ ($TX \times \{0\} \subset \mathcal{T}G_\overset{\circ}{X}$ is closed). The space of units $X_{\partial 0} := X_\partial \bigcup \overset{\circ}{X} \times [0, 1]$ is obtained by glueing $\overset{\circ}{X} \sim \overset{\circ}{X} \times \{0\}$ ($\overset{\circ}{X} \times \{0\} \subset \overset{\circ}{X} \times [0, 1]$ is closed). Using the groupoid structures of $\mathcal{T}_b X \rightrightarrows X_\partial$ and of $\mathcal{T}G_\overset{\circ}{X} \rightrightarrows \overset{\circ}{X} \times [0, 1]$, we have a continuous family groupoid $\mathcal{T}G_X \rightrightarrows X_{\partial 0}$. Again, all the groupoid structures are compatible with the considered gluings.

To define a homomorphism $\mathcal{T}G_X \xrightarrow{h} \mathbb{R}^N$ we will need as in the nonboundary case an appropriate embedding. It is possible to find an embedding $i : X \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+$ such that its restrictions to the interior and to the boundary are (smooth embeddings) of the following form $i_\overset{\circ}{X} : \overset{\circ}{X} \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+$ and $i_\partial : \partial X \hookrightarrow \mathbb{R}^{N-1} \times \{0\}$. We define the homomorphism $h : \mathcal{T}G_X \to \mathbb{R}^N$ as follows.

\[
\begin{cases}
  h(x, X, 0) = d_x i_\overset{\circ}{X}(X) \text{ and } h(x, y, \epsilon) = \frac{\epsilon i_\overset{\circ}{X}(y)}{\epsilon} \text{ on } \mathcal{T}G_\overset{\circ}{X} \\
  h(x, \xi, 0, \lambda) = (d_x i_\overset{\circ}{X}(\xi), \lambda) \text{ and } h(x, y, \epsilon, \lambda) = \left(\frac{i_\overset{\circ}{X}(y) - i_\overset{\circ}{X}(x)}{\epsilon}, \lambda\right) \text{ on } \mathcal{T}G_{\partial X} \times \mathbb{R} \\
  h(x, Y) = d_x i_\overset{\circ}{X}(X) \text{ on } TX
\end{cases}
\]

Since all these morphisms are compatible with the gluings, one has:

**Proposition 2.1.** With the formulas defined above, $h : \mathcal{T}G_X \to \mathbb{R}^N$ defines a homomorphism of continuous family groupoids.

The action of $\mathcal{T}G_X$ on $\mathbb{R}^N$ defined by $h$ is free because $i$ is an immersion. It is not necessarily proper (in the case of Connes [2] II.5 it is since M was supposed closed), however we can prove the following:

**Proposition 2.2.** The groupoid $(\mathcal{T}G_X) h$ is a proper groupoid with trivial isotropy groups.

Notice that the groupoid $G_h$ is not the action groupoid (if not, the properness of the action would be equivalent to the properness of the groupoid). It is very important that the units of the groupoid $G_h$ be the units of $G$ times $\mathbb{R}^N$. 
As an immediate consequence of the propositions above, the groupoid \((\mathcal{T}G_X)_h\) is Morita equivalent to its space of orbits. Let us specify this space.

Let \(V(X)\) be the total space of the normal bundle of \(\tilde{X}\) in \(\mathbb{R}^N\). Similarly, let \(V(\partial X)\) be the total space of the normal bundle of \(\partial X\) in \(\mathbb{R}^{N-1}\). Observe that they have the same fiber vector dimension. In fact, their union \(V(X) = V(\tilde{X}) \cup V(\partial X)\), is a vector bundle over \(X\), the normal bundle of \(X\) in \(\mathbb{R}^N\).

Take \(\mathcal{B}_0 = V(\partial X) \times \{0\} [\mathbb{R}^{N-1} \times \{0, 1\}\) the deformation to the normal cone associated to the embedding \(\partial X \xrightarrow{i_0} \mathbb{R}^{N-1}\). We consider the space \(\mathcal{B}_0 := V(X) \cup_{\mathcal{B}_3} \mathcal{B}_3\) glued over their common boundary \(V(\partial X) \sim V(\partial X) \times \{0\}\). On the other hand, take \(\mathcal{B}_o = V(\tilde{X}) \times \{0\}[\mathbb{R}^N \times \{0, 1\}\) the deformation to the normal cone associated to the embedding \(\tilde{X} \xrightarrow{i_0} \mathbb{R}^N\). We consider the space \(\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_3\) glued over \(V(\tilde{X})\) by the identity map.

**Proposition 2.3.** The space of orbits of the groupoid \((\mathcal{T}G_X)_h\) is \(\mathcal{B}\).

We can give the explicit homeomorphism. The orbit space of \((\mathcal{T}G_X)_h\) is a quotient of \(X_{\partial} \times \mathbb{R}^N\). To define a map \(\Psi : X_{\partial} \times \mathbb{R}^N \to \mathcal{B}\) it is enough to define it for each component of \(X_{\partial}\). Let

\[
\Psi : \begin{cases}
\partial X \times (0, 1) \times \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}^{N-1} \times \{0, 1\} \to V(\partial X) \\
\Psi(a, t, \xi, \lambda) := (\frac{a-x(a)}{\lambda}, \xi)
\end{cases}
\]

\[
\begin{cases}
\tilde{X} \times (0, 1) \times \mathbb{R}^N \to \mathbb{R}^N \times \{0, 1\} \\
\Psi(x, t, X) := (\frac{X-x(t)}{t}, X)
\end{cases}
\]

where \(X\) denotes the class in \(V_\partial(\partial X) := \mathbb{R}^{N-1}/T_{i_0(x)}\partial X\) (resp. \(\tilde{X}\) denotes the class in \(V_\partial(\tilde{X}) := \mathbb{R}^N/T_{i_0(x)}\tilde{X}\)). This gives a continuous map \(\Psi : X_{\partial} \times \mathbb{R}^N \to \mathcal{B}\) that passes to the quotient into a homeomorphism \(\overline{\Psi} : (X_{\partial} \times \mathbb{R}^N)/\sim \to \mathcal{B}\), where \((X_{\partial} \times \mathbb{R}^N)/\sim\) is the orbit space of the groupoid \((\mathcal{T}G_X)_h\).

3. The index theorem for manifolds with boundary

Deformation groupoids induce index morphisms. The groupoid \(\mathcal{T}G_X\) is naturally parametrized by the closed interval \([0, 1]\). Its algebra comes equipped with evaluations to the algebra of \(\mathcal{T}_{\partial} M\) (at \(t=0\)) and to the algebra of \(\tilde{X} \times \tilde{X}\) (for \(t \neq 0\)). We have a short exact sequence of C*-algebras

\[
0 \to C^*(\tilde{X} \times \tilde{X} \times (0, 1]) \to C^*(\mathcal{T}G_X) \oset{\epsilon_0}{\to} C^*(\mathcal{T}_{\partial} M) \to 0
\]

where the algebra \(C^*(\tilde{X} \times \tilde{X} \times (0, 1])\) is contractible. Hence applying the K-theory functor to this sequence we obtain an index morphism

\[
\ind_{\tilde{X}}(\epsilon_1) : K^0(\mathcal{T}_{\partial} M) \to K^0(\tilde{X} \times \tilde{X}) \cong \mathbb{Z}.
\]

The morphism \(h : \mathcal{T}G_X \to \mathbb{R}^N\) is by definition also parametrized by \([0, 1]\), i.e., we have morphisms \(h_0 : \mathcal{T}_{\partial} M \to \mathbb{R}^N\) and \(h_t : \tilde{X} \times \tilde{X} \to \mathbb{R}^N\), for \(t \neq 0\). We can consider the associated groupoids, which satisfy the same properties as in proposition \([5]\).
(in fact, for proving such proposition it is better to do it for each $t$, and to check all the compatibilities).

**Définition 3.1.** [Topological index morphism for a manifold with boundary] Let $X$ be a manifold with boundary. The topological index morphism of $X$ is the morphism

$$\text{ind}_t^X : K^0(T_t X) \longrightarrow \mathbb{Z}$$

defined (using an embedding as above) as the composition of the following three morphisms

1. The Connes-Thom isomorphism $CT_0$ followed by the Morita equivalence $\mathcal{M}_0$:

   $$K^0(T_t X) \xrightarrow{CT_0} K^0((T_t X)_h^0) \xrightarrow{\mathcal{M}_0} K^0(\mathcal{B}_\partial)$$

2. The index morphism of the deformation space $\mathcal{B}$:

   $$K^0(\mathcal{B}_\partial) \xrightarrow{(e_0)_*} K^0(\mathcal{B}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)$$

3. The usual Bott periodicity morphism:

   $$K^0(\mathbb{R}^N) \xrightarrow{\text{Bott}} \mathbb{Z}.$$

**Remark 1.** The topological index defined above is a natural generalisation of the topological index theorem defined by Atiyah-Singer. Indeed, in the boundaryless case, they coincide. The index of the deformation space $\mathcal{B}$ is quite easy to understand because we are dealing now with spaces (as groupoids the product is trivial), then the group $K^0(\mathcal{B})$ is the K-theory of the algebra of continuous functions vanishing at infinity $C_0(\mathcal{B})$ and the evaluation maps are completely explicit. In particular, if we identify $\mathcal{B}_\partial$ with an open subset of $\mathbb{R}^N$ (in the natural way), then the morphism (ii) above correspond to the canonical extension of functions of $C_0(\mathcal{B}_\partial)$ to $C_0(\mathbb{R}^N)$.

The following diagram, in which the morphisms $CT$ and $\mathcal{M}$ are the Connes-Thom and Morita isomorphisms respectively, is trivially commutative:

$$
\begin{align*}
K^0(T_t X) & \xrightarrow{(e_0)_*} K^0((T_t X)_h^0) \xrightarrow{(e_1)_*} K^0((\mathcal{B}_\partial)_h^0) \xrightarrow{\mathcal{M}_0} K^0(\mathbb{R}^N) \\
\xrightarrow{CT} & \xrightarrow{CT} \xrightarrow{CT} \xrightarrow{\mathcal{M}} \\
K^0((T_t X)_h^0) & \xrightarrow{(e_0)_*} K^0((T_t X)_h) \xrightarrow{(e_1)_*} K^0((\mathcal{B}_\partial)_h) \xrightarrow{\mathcal{M}_0} K^0(\mathbb{R}^N) \\
\xrightarrow{\mathcal{M}} & \xrightarrow{\mathcal{M}} \xrightarrow{\mathcal{M}} \xrightarrow{\mathcal{M}} \\
K^0(\mathcal{B}_\partial) & \xrightarrow{(e_0)_*} K^0(\mathcal{B}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N).
\end{align*}
$$

The left vertical line gives the first part of the topological index map. The bottom line is the morphism induced by the deformation space $\mathcal{B}$. And the right vertical line is precisely the inverse of the Bott isomorphism $\mathbb{Z} = K^0(\mathcal{B}_\partial) \approx K^0(\mathcal{B}_\partial) \approx K^0(\mathbb{R}^N)$. Since the top line gives $\text{ind}_t^X$, we obtain the following result:

**Theorem 3.1.** For any manifold with boundary $X$, we have the equality of morphisms

$$\text{ind}_t^X = \text{ind}_t^X.$$
4. Perspectives

As discussed in [3, 4, 5], the index map \( \text{ind}^X_f \) computes the Fredholm index of a fully elliptic operator in the \( b \)-calculus of Melrose. We shall use the result proven here to give a formula in relation to that of Atiyah-Patodi-Singer (6).

References