Fractional Poisson Fields

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Abstract. This paper considers random balls in a D-dimensional Euclidean space whose centers are prescribed by a homogeneous Poisson point process and whose radii are prescribed by a specific power law. A random field is constructed by counting the number of covering balls at each point. Even though it is not Gaussian, this field shares the same covariance function as the fractional Brownian field (fBf). By analogy it is called fractional Poisson field (fPf). In this paper, we are mainly interested in the simulation of fPfs with index H in $(0, 1/2)$ and in the estimation of the H index. Our method is based on the analysis of structure functions. The fPf exhibits a multifractal behavior, contrary to that of the fBf which is monofractal.

Introduction

The random fields under consideration in this paper have been introduced in [15] and [2] as a limit of a rescaled shot-noise. More precisely, random balls in a D-dimensional Euclidean space are considered: the centers are prescribed by a homogeneous Poisson point process in $\mathbb{R}^D$ and the radii are prescribed by a power law. A shot-noise field is constructed by counting the number of covering balls (see [7, 12] for details on shot-noise processes) at each point. When there are enough balls with arbitrary small volumes, the associated field exhibits fractal properties at small scale and some global self-similar properties. In particular, its covariance function is a homogeneous function whose degree depends on the power exponent. Hence it shares the same covariance as a fractional Brownian field and it is called fractional Poisson field (fPf). For the fPf, the described procedure yields indices that range in $(0, 1/2)$.

A pioneer work in this area is due to Cioczek-Georges and Mandelbrot [5] where a sum of random micropulses in dimension one, or generalizations in higher dimensions, are properly rescaled and normalized in order to get a fractional Brownian field of index $H < 1/2$ (antipersistent fBf). In that paper, it is emphasized that the power law distribution prescribed for the length of the micropulses makes it impossible to get $H > 1/2$. Using similar models in dimension one, recent works ([6, 10, 16]) have examined the internet traffic modeling. The resulting signal is proved to exhibit a long range dependence ($H > 1/2$), in accordance with observations. Such a range for index H is made possible either by prescribing the connection lengths with heavy tails, or by forcing the number of long connections.

In the present paper, we are mainly interested in the simulation of an fPf and the estimation of its index. Let us note that simulating an fPf appears as very tractable since the basic objects are balls and the basic operation consists in counting. Moreover, the possibility of changing balls into other templates or changing the homogeneous Poisson process of centers into another point process yields a very large choice of patterns (see [4] for instance). In order to get simulations which are rapidly accurate at all scales, we force random balls with prescribed radii in each given slice $(\alpha^{j+1}, \alpha^j]$ ($\alpha \in (0, 1)$ is fixed and j ranges in $\mathbb{Z}$). We first

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simulate each slice and then proceed to the "piling" of the slices. This procedure is similar to the construction of the "general multitype Boolean model" in [11].

Concerning the index estimation, we use the structure functions as introduced in [14] or [19]. Roughly speaking, the q-structure function $S_q(f,\varepsilon)$ of a given function f is equal to the L^q-norm of the ε-increments of f (see [9] for this approach). When $q=2$ and $f=F_H$, the fPf of index H, the expectation of $S_2(F_H,\varepsilon)^2$ is proportional to ε^{2H} for any positive ε. We use this relation to provide two different estimators of H: The first takes into account the small scales and the second the intermediate ones. We also prove that the expectation of $S_{2q}(F_H,\varepsilon)^{2q}$ behaves like ε^{2H} for small ε and any nonnegative integer q. This should be interpreted as a non-monofractal behavior, in opposition with the fractional Brownian motion for which the expectation of $S_{2q}(B_H,\varepsilon)^{2q}$ behaves like ε^{2qH}.

Outline of the paper

The random fields we deal with are introduced in Section 2. First the piling field is obtained by summing the elementary slices between a lower and an upper scale. Then, letting the small scales go to 0 and the large scales go to infinity, we get the fPf. Due to the Central Limit Theorem, it is easy to obtain a fractional Brownian field from a family of independent fPfs. In Section 3, we introduce a notion of D-dimensional q-structure functions and compute them for an fPf. This reveals a multifractal behavior. Section 4 is devoted to the simulation procedure of an fPf on a cube as well as the numerical computation of the structure functions. In order to keep the irregularity property and to make sense from a numerical point of view, the smallest scale involved in both simulation and computation is taken to be equal to the grid resolution. In the last section, we estimate the H index of the previously simulated fPf. Two cases are presented: working at small scales (i.e. working with small balls) in dimension $D=1$ and working at intermediate scales for any D.

1. Notations and preliminaries

In this section some notations and formulas used throughout the paper are given.

Let $\{e_1,\ldots,e_D\}$ be the canonical basis of $\mathbb{R}^D$ ($D\geqslant 1$) and $\|\cdot\|$ be the Euclidean norm. We write $B(x,\varepsilon)$ for the closed ball of center x and radius $\varepsilon>0$ with respect to $\|\cdot\|$ and denote V_D the Lebesgue measure of $B(0,1)$. We recall that $V_D=\frac{\pi^{D/2}}{\Gamma(1+D/2)}$ where Γ is the usual Euler's function.

If A and B are two subsets of $\mathbb{R}^D$, we write $A\sqcup B$ for $A\cup B$ with $A\cap B=\emptyset$ and $A\triangle B$ for $(A\setminus B)\sqcup(B\setminus A)$. If A is a measurable subset of $\mathbb{R}^D$, we denote $|A|$ the D-dimensional Lebesgue measure of A.

Finally, for $y,x\in\mathbb{R}^D$ and $r\in[0,\infty)$, we write

$$\psi(y,x,r)=\mathbb{1}_{B(x,r)}(y)-\mathbb{1}_{B(x,r)}(0)=\mathbb{1}_{B(y,r)}(x)-\mathbb{1}_{B(0,r)}(x)\,. \tag{1}$$

Let us state some basic facts. First we clearly have

$$\forall\,\lambda>0\quad \psi(\lambda y,\lambda x,\lambda r)=\psi(y,x,r)\quad\text{and}\quad \forall\,p\in\mathbb{N}\setminus\{0\}\quad |\psi(y,x,r)|^p=\psi(y,x,r)^2\,.$$

Moreover, for any $y\in\mathbb{R}^D$ and $r\in\mathbb{R}^+$, one can find a constant $C(y)\in(0,+\infty)$ such that

$$\int_{\mathbb{R}^D}\psi(y,x,r)^2\,dx=|B(y,r)\triangle B(0,r)|\leqslant C(y)\min\left(r^D,r^{D-1}\right)\,. \tag{2}$$

Hence, for any $H \in (0,1/2)$ and $y \in \mathbb{R}^D$, the integral $\displaystyle\int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(y,x,r)^2\, r^{-D-1+2H}\, dx\, dr$ is finite. Using the rotation invariance of the Lebesgue measure, we obtain

$$\int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(y,x,r)^2\, r^{-D-1+2H}\, dx\, dr = c_H(D)\ \|y\|^{2H} \tag{3}$$

with

$$c_H(D) = \int_{\mathbb{R}^+} |B(e_1,r)\triangle B(0,r)|\ r^{-D-1+2H}\, dr\,. \tag{4}$$

In dimension $D=1$ one can check that

$$[1-r,1+r]\triangle\,[-r,r] = \begin{cases} [-r,r]\sqcup[1-r,1+r] & \text{if } r\leqslant 1/2\,,\\ [-r,1-r)\ \sqcup\ (r,1+r] & \text{if } r>1/2\,,\end{cases}$$

so that the constant $c_H(1)$ is explicitly computed as $c_H(1) = \frac{2^{1-2H}}{H(1-2H)}$.

In dimension $D=2$ or 3, explicit formulas for $|B(e_1,r)\triangle B(0,r)|$ can be found for instance in [18]. For higher dimensions, explicit formulas are not tractable.

2. Some random models

The purpose of this section is to give an iterative construction of fractional Poisson fields.

2.1. **Elementary slices.**

Let $H\in\mathbb{R}$. For $\alpha\in(0,1)$ and $j\in\mathbb{Z}$ we consider $\Phi_j = \left\{(X_j^n,R_j^n)_n\right\}$ a Poisson point process in $\mathbb{R}^D\times\mathbb{R}^+$ of intensity

$$\nu_j(dx,dr) = dx\otimes\ r^{-D-1-2H}\,\mathbb{1}_{(\alpha^{j+1},\,\alpha^j]}(r)dr\,. \tag{5}$$

Note that Φ_j is well-defined since ν_j is a nonnegative finite measure on $\mathbb{R}^D\times\mathbb{R}^+$. We consider the associated so called "random balls field" T_j as defined in [3] that provides, at each point $y\in\mathbb{R}^D$, the number of balls $B(X_j^n,R_j^n)$ that contain the point y, namely

$$T_j(y) = \sum_{(X_j^n,R_j^n)\,\in\,\Phi_j} \mathbb{1}_{B(X_j^n,R_j^n)}(y)\,. \tag{6}$$

Equivalently, we can represent the field T_j through a stochastic integral

$$T_j(y) = \int_{\mathbb{R}^D\times\mathbb{R}^+} \mathbb{1}_{B(x,r)}(y) N_j(dx,dr), \tag{7}$$

where N_j is a Poisson random measure on $\mathbb{R}^D\times\mathbb{R}^+$ of intensity ν_j. We recall the basic facts (see [**?**] chapter 10 for instance) that for measurable sets $A\subset\mathbb{R}^D\times\mathbb{R}^+$ the random variable $N_j(A)$ is Poisson distributed with mean $\nu_j(A)$ and if $A_1,\dots,A_n$ are disjoint then $N_j(A_1),\dots,N_j(A_n)$ are independent. We also recall that for measurable functions $k:\mathbb{R}^D\times\mathbb{R}^+\to\mathbb{R}$, the stochastic integral $\int k(x,r)\, N_j(dx,dr)$ of k with respect to N_j exists a.s. if and only if

$$\int_{\mathbb{R}^D\times\mathbb{R}^+} \min\left(\,|k(x,r)|\,,1\right)\nu_j(dx,dr) < \infty. \tag{8}$$

Furthermore the characteristic function of $\int k(x,r)N_j(dx,dr)$ is given by (see [?] Lemma 10.2)

$$\mathbb{E}(e^{it\int k(x,r)N_j(dx,dr)}) = \exp\left(\int_{\mathbb{R}^D\times\mathbb{R}^+} (e^{itk(x,r)}-1)\,\nu_j(dx,dr)\right)\,,\ t\in\mathbb{R}\,. \tag{9}$$

In our setting, in order to ensure that the right hand side of (7) is well defined, it is sufficient to remark that, by Fubini's theorem,

$$\int_{\mathbb{R}^D\times\mathbb{R}^+} \mathbb{1}_{B(x,r)}(y)\,\nu_j(dx,dr) = V_D\int_{\alpha^{j+1}}^{\alpha^j} r^{-1+2H}dr = c_{\alpha,H}(D)\,\alpha^{2Hj} < +\infty\,,$$

where

$$c_{\alpha,H}(D) = V_D\,\frac{1-\alpha^{2H}}{2H}\,. \tag{10}$$

Let us remark that due to the translation invariance of the Lebesgue measure, the random field T_j is stationary. Moreover T_j admits moments of any order and according to [1], for $n\geqslant 1$, the n-th moment of $T_j(y)$ is given by

$$\begin{aligned}\mathbb{E}(T_j(y)^n) &= \sum_{(r_1,\ldots,r_n)\in I(n)} K_n(r_1,\ldots,r_n)\prod_{k=1}^n\left(\int_{\mathbb{R}^D\times\mathbb{R}^+}\mathbb{1}_{B(x,r)}(y)^k\,\nu_j(dx,dr)\right)^{r_k}\\ &= \sum_{(r_1,\ldots,r_n)\in I(n)} K_n(r_1,\ldots,r_n)\left(c_{\alpha,H}(D)\,\alpha^{2Hj}\right)^{\sum_{k=1}^n r_k}\,,\end{aligned} \tag{11}$$

where $I(n)=\left\{(r_1,\ldots,r_n)\in\mathbb{N}^n\,;\ \sum_{k=1}^n kr_k = n\right\}$ and $K_n(r_1,\ldots,r_n) = n!\left(\prod_{k=1}^n r_k!(k!)^{r_k}\right)^{-1}$.

Finally let us mention that the covariance of T_j is simply given by

$$\mathrm{Cov}(T_j(y),T_j(y')) = \int_{\alpha^{j+1}}^{\alpha^j}\left|B(y,r)\cap B(y',r)\right| r^{-D-1+2H}dr.$$

In particular, when we zoom in (out) on the random field T_j with a scale factor given by α^l for some $l>0$ ($l<0$) the covariance function of the zoom-in (out) field $\left(T_j(\alpha^l y)\right)_{y\in\mathbb{R}^D}$ is obtained by considering the field $(T_{j-l}(y))_{y\in\mathbb{R}^D}$ according to

$$\mathrm{Cov}(T_j(\alpha^l y),T_j(\alpha^l y')) = \alpha^{2Hl}\,\mathrm{Cov}(T_{j-l}(y),T_{j-l}(y')).$$

This identity can be stated in distribution. It yields a kind of aggregate similarity property as defined in [2].

Proposition 2.1. *Let $\alpha\in(0,1)$ and $j\in\mathbb{Z}$. Then, for any $l\in\mathbb{Z}$ such that $m=\alpha^{2Hl}\in\mathbb{N}$,*

$$\left\{T_{j+l}(\alpha^l y)\,;\ y\in\mathbb{R}^D\right\} \overset{fdd}{=} \left\{\sum_{k=1}^m T_j^{(k)}(y)\,;\ y\in\mathbb{R}^D\right\},$$

where $\left(T_j^{(k)}\right)_{k\geqslant 1}$ are iid copies of T_j.

Remark: In the case $H<0$ the exponent l must be nonnegative (zoom-in) while in the case $H>0$ it has to be strictly negative $l<0$ (zoom-out).

Proof. Let us assume that there exists $l \in \mathbb{Z}$ such that $m = \alpha^{2Hl} \in \mathbb{N}$. Let $n \geqslant 1$ and $y_1, \ldots, y_n \in \mathbb{R}^D$, $u_1, \ldots, u_n \in \mathbb{R}$.

$$\begin{aligned}
\log \mathbb{E} \exp\left(i \sum_{p=1}^{n} u_p T_{j+l}(\alpha^l y_p) \right) &= \int_{\mathbb{R}^D \times \mathbb{R}^+} \left(e^{i \sum\limits_{p=1}^{n} \mathbb{1}_{B(x,r)}(\alpha^l y_p)} - 1 \right) \nu_{j+l}(dx, dr) \\
&= \alpha^{2Hl} \int_{\mathbb{R}^D \times \mathbb{R}^+} \left(e^{i \sum\limits_{p=1}^{n} u_p \mathbb{1}_{B(x,r)}(y_p)} - 1 \right) \nu_j(dx, dr) \\
&= \sum_{k=1}^{m} \log \mathbb{E} \exp\left(i \sum_{p=1}^{n} u_p T_j(y_p) \right),
\end{aligned}$$

where the second line is obtained by a change of variables. Hence the result follows. □

Now, let us consider the associated piling random field. Let $(\Phi_j)_{j\in\mathbb{Z}}$ be independent Poisson point processes in $\mathbb{R}^D \times \mathbb{R}^+$, with each Φ_j of intensity $\nu_j(dx, dr)$ given by (5). For $j_{\min}, j_{\max} \in \mathbb{Z}$ with $j_{\min} \leqslant j_{\max}$ the piling random field $\sum\limits_{j=j_{\min}}^{j_{\max}} T_j$ can be considered as the random balls field associated with a Poisson random measure $N_{j_{\min},j_{\max}}$ of intensity

$$\nu_{j_{\min},j_{\max}}(dx, dr) = dx \otimes r^{-D-1+2H} \mathbb{1}_{(\alpha^{j_{\max}+1},\, \alpha^{j_{\min}}]}(r) dr\,.$$

In order to let $j_{\min} \to -\infty$ and $j_{\max} \to +\infty$ we introduce the centered random field

$$F_{j_{\min},j_{\max}}(y) = \sum_{j=j_{\min}}^{j_{\max}} \left(T_j(y) - T_j(0)\right)\ ,\ y \in \mathbb{R}^D\,. \tag{12}$$

2.2. **Fractional Poisson fields.**

Theorem 2.2. *Let $H \in (0, 1/2)$. Then, for all $n \geqslant 1$, $y_1, \ldots, y_n \in \mathbb{R}^D$, the sequence*

$$(F_{j_{\min},j_{\max}}(y_1), \ldots, F_{j_{\min},j_{\max}}(y_n))$$

has a limit in $L^2(\Omega, \mathbb{R}^n)$ as $j_{\min} \to -\infty$ and $j_{\max} \to +\infty$. Moreover, the limit defines a random field F_H that may be expressed as

$$F_H(y) = \sum_{j\in\mathbb{Z}} \left(T_j(y) - T_j(0)\right) \overset{fdd}{=} \int_{\mathbb{R}^D \times \mathbb{R}^+} \psi(y, x, r) \widetilde{N}(dx, dr) \tag{13}$$

with ψ given by (1) *and $\widetilde{N}$ a compensated Poisson random measure on $\mathbb{R}^D \times \mathbb{R}^+$ of intensity ν given by*

$$\nu(dx, dr) = dx \otimes r^{-D-1+2H}\, dr\,. \tag{14}$$

Proof. Let $y \in \mathbb{R}^D$. Remark that according to (1),

$$F_{j_{\min},j_{\max}}(y) = \int_{\mathbb{R}^D \times \mathbb{R}^+} \psi(y, x, r) N_{j_{\min},j_{\max}}(dx, dr) = \int_{\mathbb{R}^D \times \mathbb{R}^+} \psi(y, x, r) \widetilde{N}_{j_{\min},j_{\max}}(dx, dr),$$

where $\widetilde{N}_{j_{\min},j_{\max}}$ is the compensated Poisson random measure $N_{j_{\min},j_{\max}}(dx, dr) - \nu_{j_{\min},j_{\max}}(dx, dr)$.

Let $j'_{\min} < j_{\min}$ and $j'_{\max} > j_{\max}$, then

$$\left\|F_{j'_{\min},j'_{\max}}(y) - F_{j_{\min},j_{\max}}(y)\right\|_2^2 = \mathrm{Var}\left(F_{j'_{\min},j'_{\max}}(y) - F_{j_{\min},j_{\max}}(y)\right)$$

$$= \int_{\mathbb{R}^D \times (\alpha^{j'_{\max}+1}, \alpha^{j_{\max}+1}]} \psi(y,x,r)^2 r^{-D-1+2H} dxdr + \int_{\mathbb{R}^D \times (\alpha^{j_{\min}}, \alpha^{j'_{\min}}]} \psi(y,x,r)^2 r^{-D-1+2H} dxdr\,.$$

Using (2), one can find a constant $C(y) > 0$ such that

$$\left\|F_{j'_{\min},j'_{\max}}(y) - F_{j_{\min},j_{\max}}(y)\right\|_2^2 \leqslant C(y)\left(\alpha^{2Hj_{\max}} + \alpha^{(-1+2H)j_{\min}}\right),$$

which proves that $(F_{j_{\min},j_{\max}}(y))_{j_{\min},j_{\max}}$ is a Cauchy sequence in $L^2(\Omega)$ since $H \in (0,1/2)$. This concludes the proof for the convergence in $L^2(\Omega, \mathbb{R}^n)$. We call F_H the limit field.

Now let us consider $\widetilde{N}$ a compensated Poisson random measure of intensity ν given by (14). Let us remark that for $H \in (0,1/2)$ and any $y \in \mathbb{R}^D$, $\psi(y,\cdot,\cdot) \in L^2(\mathbb{R}^D \times \mathbb{R}^+, \nu(dx,dr))$ such that one can define the random field $X = \left\{\int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(y,x,r)\widetilde{N}(dx,dr)\,;\, y \in \mathbb{R}^D\right\}$. Considering the limit of the characteristic function of $(F_{j_{\min},j_{\max}}(y_1), \ldots, F_{j_{\min},j_{\max}}(y_n))$ as $j_{\min} \to -\infty$ and $j_{\max} \to +\infty$ leads to $F_H \overset{fdd}{=} X$. □

Proposition 2.3. *Let $H \in (0,1/2)$. The random field F_H given by* (13) *admits moments of all orders. It is centered, with stationary increments and its covariance function is given by*

$$\forall\, y, y' \in \mathbb{R}^D \quad \mathrm{Cov}(F_H(y), F_H(y')) = \frac{c_H(D)}{2}\left(\|y\|^{2H} + \|y'\|^{2H} - \|y-y'\|^{2H}\right), \tag{15}$$

where the constant $c_H(D)$ is defined in (4).

Since F_H shares the same covariance function as the fractional Brownian field with the Hurst parameter H (see the seminal paper [17]), we give the following definition:

Definition 2.1. F_H is called fractional Poisson field of index H with $H \in (0,1/2)$.

Proof of Proposition 2.3. Let us observe that F_H is clearly centered by (13). Moreover, since $\psi(\cdot + y_0, \cdot, \cdot) - \psi(y_0, \cdot, \cdot) = \psi(\cdot, \cdot - y_0, \cdot)$ for any $y_0 \in \mathbb{R}^D$, the random field F_H has stationary increments by the translation invariance of the Lebesgue measure. Note also that (3) implies that $F_H(y)$ admits moments of all orders in view of [1]. Moreover, using the increments stationarity, for all $y, y' \in \mathbb{R}^D$,

$$\mathrm{Cov}(F_H(y), F_H(y')) = \frac{1}{2}\left(\mathrm{Var}(F_H(y)) + \mathrm{Var}(F_H(y')) - \mathrm{Var}(F_H(y-y'))\right),$$

since $F_H(0) = 0$ almost surely. Using (3) again yields

$$\mathrm{Var}(F_H(y)) = \int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(y,x,r)^2 \nu(dx,dr) = c_H(D)\, \|y\|^{2H}\,.$$

Henceforth the constant $c_H(D)$ introduced in (4) is convenient. □

Finally, let us recall that F_H is not Gaussian. Let us compute its marginal distributions. Using (13), we observe that for $t \in \mathbb{R}$,

$$\log \mathbb{E}\left(\exp\left(itF_H(y)\right)\right) = \int_{\mathbb{R}^D\times\mathbb{R}^+} \left(e^{it\psi(y,x,r)} - 1 - it\psi(y,x,r)\right)\nu(dx,dr)\,. \tag{16}$$

Hence, using the fact that $\int_{\mathbb{R}^D\times\mathbb{R}^+}\psi(y,x,r)\nu(dx,dr) = 0$ and $|\psi(y,x,r)| = \psi(y,x,r)^2$ we may recast (16) into

$$\begin{aligned}\log\mathbb{E}\left(\exp\left(itF_H(y)\right)\right) &= \frac{1}{2}\left(e^{it}-1-it\right)\int_{\mathbb{R}^D\times\mathbb{R}^+}\psi(y,x,r)^2\nu(dx,dr)\\ &\qquad\qquad + \frac{1}{2}\left(e^{-it}-1+it\right)\int_{\mathbb{R}^D\times\mathbb{R}^+}\psi(y,x,r)^2\nu(dx,dr)\\ &= \frac{c_H(D)}{2}\,\|y\|^{2H}\,\left(\left(e^{it}-1-it\right)+\left(e^{-it}-1+it\right)\right)\,.\end{aligned}$$

This is the logarithmic characteristic functional of the difference of two independent random variables that both have a Poisson distribution with intensity $\frac{c_H(D)}{2}\,\|y\|^{2H}$.

2.3. **Aggregate similarity.**

Note that F_H does not share the same sample paths regularity as the fractional Brownian field, nor it self-similarity. However it exhibits what is called an aggregate similarity property (see Definition 3.5 of [2]).

Proposition 2.4. *Let $H\in(0,1/2)$. Then, the random field F_H given by* (13) *is aggregate similar: for all $m\in\mathbb{N}$ with $m\geqslant 1$ and $a_m = m^{1/2H}$*

$$\left\{F_H(a_m y)\,;\,y\in\mathbb{R}^D\right\}\overset{fdd}{=}\left\{\sum_{k=1}^m F_H^{(k)}(y)\,;\,y\in\mathbb{R}^D\right\},$$

where $\left(F_H^{(k)}\right)_{k\geqslant 1}$ are iid copies of F_H.

Proof. Let $m\in\mathbb{N}$ with $m\geqslant 1$ and set $a_m = m^{1/2H}$. Let $y_1,\dots,y_n\in\mathbb{R}^D$, $u_1,\dots,u_n\in\mathbb{R}$.

$$\begin{aligned}\log\mathbb{E}\exp\Bigg(i\sum_{p=1}^n u_pF_H(a_my_p)\Bigg) &= \int_{\mathbb{R}^D\times\mathbb{R}^+}\Bigg(e^{i\sum\limits_{p=1}^n u_p\psi(a_my_p,x,r)}-1-i\sum_{p=1}^n u_p\psi(a_my_p,x,r)\Bigg)\nu(dx,dr)\\ &= a_m^{2H}\int_{\mathbb{R}^D\times\mathbb{R}^+}\Bigg(e^{i\sum\limits_{p=1}^n u_p\psi(y_p,x,r)}-1-i\sum_{p=1}^n u_p\psi(y_p,x,r)\Bigg)\nu(dx,dr)\\ &= \sum_{k=1}^m\log\mathbb{E}\exp\Bigg(i\sum_{p=1}^n u_pF_H(y_p)\Bigg)\,.\end{aligned}$$

Hence the result. □

Let us remark that a fractional Brownian field B_H of the Hurst parameter H is also aggregate similar. Actually, for $m\geqslant 1$ and $a_m = m^{1/2H}$ we have

$$B_H(a_m\cdot)\overset{fdd}{=}a_m^H B_H(\cdot)\overset{fdd}{=}\sum_{k=1}^m B_H^{(k)}(\cdot)\,,$$

with $\left(B_H^{(k)}\right)_{k\geqslant 1}$ iid copies of B_H. Here, the first equality is obtained with the so-called self-similarity property of B_H and the second one by using the Gaussian law.

2.4. **Central Limit Theorem for the fractional Poisson field.**
Let $\left(F^{(k)}_{j_{\min},j_{\max}}\right)_{k\geqslant 1}$ be iid copies of $F_{j_{\min},j_{\max}}$. According to the Central Limit Theorem :

$$\left\{\frac{1}{\sqrt{K}}\sum_{k=1}^{K} F^{(k)}_{j_{\min},j_{\max}}(y)\,;\,y\in\mathbb{R}^D\right\} \xrightarrow[K\to+\infty]{fdd} \left\{W_{j_{\min},j_{\max}}(y)\,;\,y\in\mathbb{R}^D\right\}, \tag{17}$$

where $W_{j_{\min},j_{\max}}$ is a centered Gaussian process with stationary increments and covariance function given by :

$$\operatorname{Cov}\left(W_{j_{\min},j_{\max}}(y), W_{j_{\min},j_{\max}}(y')\right) = \frac{1}{2}\left(v_{j_{\min},j_{\max}}(y) + v_{j_{\min},j_{\max}}(y') - v_{j_{\min},j_{\max}}(y-y')\right)$$

with

$$v_{j_{\min},j_{\max}}(y) = \int_{(\alpha^{j_{\max}+1},\,\alpha^{j_{\min}}]} |B(y,r)\triangle B(0,r)|\, r^{-D-1+2H}\, dr\,.$$

Note that for any $K\geqslant 1$, the random field $\sum_{k=1}^{K} F^{(k)}_{j_{\min},j_{\max}}$ can also be considered as a centered random balls field

$$\sum_{k=1}^{K} F^{(k)}_{j_{\min},j_{\max}}(y) = \int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(y,x,r)\,\widetilde{N^K}(dx,dr)$$

with $\widetilde{N^K}$ a compensated Poisson random measure of intensity $K\,\nu_{j_{\min},j_{\max}}$. Hence the asymptotic result (17) can also be understood as the classical normal convergence obtained for shot noise fields when the number of shots tends to $+\infty$. Rates of convergence can also be derived in this context (see [12] for instance).

In the particular case $H\in(0,1/2)$, letting $(j_{\min},j_{\max})$ going to $(-\infty,+\infty)$ we get $v_{j_{\min},j_{\max}}(y)\to c_H(D)\,\|y\|^{2H}$ with the constant $c_H(D)$ prescribed by (4). Then

$$\frac{1}{\sqrt{c_H(D)}}\,W_{j_{\min},j_{\max}} \xrightarrow[(j_{\min},j_{\max})\to(-\infty,+\infty)]{fdd} B_H$$

where B_H is the fractional Brownian field. Similar ideas have been early developed in [5].

3. Structure functions

Let $f:\mathbb{R}^D\to\mathbb{R}$ be an integrable function. If $q>0$, we define the *q-structure function* of f by

$$\forall\,\varepsilon\geqslant 0 \qquad S_q(f,\varepsilon) = \left(\frac{1}{D}\sum_{i=1}^{D}\int_{[0,1]^D} |f(t+\varepsilon e_i) - f(t-\varepsilon e_i)|^q\, dt\right)^{1/q}. \tag{18}$$

These quantities have been used in one dimension to study the fractal behavior of f (see [14, 19, 9]). When f is regular enough one may expect that $\lim_{\varepsilon\to 0} S_q(f,\varepsilon)=0$ for all $q>0$. More precisely we are interested in a power-law behavior through a relation of the type $S_q(f,\varepsilon)\simeq_0 \varepsilon^{H(q)}$ for a certain constant $H(q)>0$, where $u(\varepsilon)\simeq_0 v(\varepsilon)$ means that $u(\varepsilon)/v(\varepsilon)$ is bounded from above and from below for small ε. We say that f is multifractal when $H(q)$ is not constant (see [14]).

Note that if one is interested in the isotropic properties of f then one may use anisotropic versions of $S_q(f,\varepsilon)$ with $\varepsilon \in \mathbb{R}_+^D$. Here we do not use it since our model is clearly isotropic.

Assume that f is a random function satisfying

$$\sup_{t\in[-a,1+a]^D} \mathbb{E}\,|f(t)|^q < \infty \quad \text{for some } a > 0\,.$$

Then, by Fubini's theorem one can consider

$$\mathbb{E}\left(S_q^q(f,\varepsilon)\right) = \frac{1}{D}\sum_{i=1}^{D}\left(\int_{[0,1]^D} \mathbb{E}\,|f(t+\varepsilon e_i) - f(t-\varepsilon e_i)|^q\,dt\right)\,. \tag{19}$$

3.1. **The 2-structure function of a fractional Poisson field.**

The value $q=2$ allows us to give exact formulas.

Proposition 3.1. *Let F_H be the fractional Poisson field of index $H \in (0,1/2)$. Then :*

(i) For all $\varepsilon \geqslant 0$, one has :

$$\mathbb{E}\left(S_2^2(F_H,\varepsilon)\right) = c_H(D)\,(2\,\varepsilon)^{2H} \quad \text{where } c_H(D) \text{ is defined by (4).} \tag{20}$$

(ii) For all $\varepsilon > \alpha^{j_{\min}}$, one has :

$$\mathbb{E}\left(S_2^2(F_{j_{\min},j_{\max}},\varepsilon)\right) = \frac{\pi^{D/2}}{\Gamma(1+D/2)\,H}\,\left(\alpha^{2Hj_{\min}} - \alpha^{2H(j_{\max}+1)}\right)\,. \tag{21}$$

(iii) When $D=1$ one has : $\mathbb{E}\left(S_2^2(F_{j_{\min},j_{\max}},\varepsilon)\right) =$

$$\begin{cases} \dfrac{4}{1-2H}\left(\alpha^{(2H-1)(j_{\max}+1)} - \alpha^{(2H-1)j_{\min}}\right)\varepsilon & \text{if } \varepsilon \in [0,\alpha^{j_{\max}+1})\,, \\ \dfrac{4}{1-2H}\,\varepsilon\left(\varepsilon^{2H-1} - \alpha^{(2H-1)j_{\min}}\right) + \dfrac{2}{H}\left(\varepsilon^{2H} - \alpha^{2H(j_{\max}+1)}\right) & \text{if } \varepsilon \in [\alpha^{j_{\max}+1},\alpha^{j_{\min}}]\,, \\ \dfrac{2}{H}\left(\alpha^{2Hj_{\min}} - \alpha^{2H(j_{\max}+1)}\right) & \text{if } \varepsilon > \alpha^{j_{\min}}\,. \end{cases} \tag{22}$$

Proof. First we observe that for all $\varepsilon \geqslant 0$, $t \in \mathbb{R}^D$ and $i \in \{1,\dots,D\}$:

$$\begin{aligned} F_H(t+\varepsilon e_i) - F_H(t-\varepsilon e_i) &= \int_{\mathbb{R}^D\times\mathbb{R}^+} (\psi(t+\varepsilon e_i,x,r) - \psi(t-\varepsilon e_i,x,r))N(dx,dr) \\ &= \int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(2\varepsilon e_i, x-t+\varepsilon e_i, r)N(dx,dr)\,. \end{aligned} \tag{23}$$

(i) We compute $\mathbb{E}\left(F_H(t+\varepsilon e_i) - F_H(t-\varepsilon e_i)\right)^2$ using (23) :

$$\begin{aligned} \mathbb{E}\left(F_H(t+\varepsilon e_i) - F_H(t-\varepsilon e_i)\right)^2 &= \int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(2\varepsilon e_i, x-t+\varepsilon e_i, r)^2\, r^{-D-1+2H}\,dx\,dr \\ &= \int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(2\varepsilon e_i, u, r)^2\, r^{-D-1+2H}\,du\,dr\,. \end{aligned}$$

Therefore, using (3) :

$$\mathbb{E}\left(F_H(t+\varepsilon e_i) - F_H(t-\varepsilon e_i)\right)^2 = c_H(D)\,\left\|2\varepsilon e_i\right\|^{2H} = c_H(D)\,(2\varepsilon)^{2H}\,.$$

Hence the result since the above quantity does not depend on index i nor on variable t.

(ii) Since (23) holds replacing F_H by $F_{j_{\min},j_{\max}}$ and N by $\widetilde{N}$, we obtain in the same way, for $\varepsilon \geqslant 0$ and $t \in \mathbb{R}^D$:

$$\mathbb{E}\left(F_{j_{\min},j_{\max}}(t+\varepsilon e_i) - F_{j_{\min},j_{\max}}(t-\varepsilon e_i)\right)^2 = \int_{\mathbb{R}^D \times (\alpha^{j_{\max}+1},\, \alpha^{j_{\min}}]} \psi(2\varepsilon e_i, u, r)^2\, r^{-D-1+2H}\, du\, dr\,.$$

Notice that if $\varepsilon > r$ then :

$$\int_{\mathbb{R}^D} \psi(2\varepsilon e_i, u, r)^2 du = \int_{\mathbb{R}^D} \mathbb{I}_{B(2\varepsilon e_i, r)\sqcup B(0,r)}(u) du = 2V_D\, r^D\,.$$

Thus for all $\varepsilon > \alpha^{j_{\min}}$:

$$\begin{aligned}\mathbb{E}(F_{j_{\min},j_{\max}}(t+\varepsilon e_i) - F_{j_{\min},j_{\max}}(t-\varepsilon e_i))^2 &= 2V_D \int_{\alpha^{j_{\max}+1}}^{\alpha^{j_{\min}}} r^{-1+2H} dr \\ &= \frac{V_D}{H}\left(\alpha^{2Hj_{\min}} - \alpha^{2H(j_{\max}+1)}\right)\,.\end{aligned}$$

Hence the result, since the latter quantity does not depend on t nor on i.

(iii) When $D = 1$ we have :

$$\begin{aligned}\mathbb{E}\left(F_{j_{\min},j_{\max}}(t+\varepsilon) - F_{j_{\min},j_{\max}}(t-\varepsilon)\right)^2 &= \int_{\mathbb{R}\times(\alpha^{j_{\max}+1},\, \alpha^{j_{\min}}]} \mathbb{I}_{[2\varepsilon-r,2\varepsilon+r]\triangle[-r,r]}(u)\, r^{2H-2}\, du\, dr \\ &= 4\int_{\alpha^{j_{\max}+1}}^{\alpha^{j_{\min}}} \min\{r,\varepsilon\}\, r^{2H-2}\, dr\,,\end{aligned}$$

the latter equality following from

$$[2\varepsilon - r, 2\varepsilon + r]\triangle[-r,r] = \begin{cases}[2\varepsilon - r, 2\varepsilon + r] \sqcup [-r,r] & \text{if } r \leqslant \varepsilon\,, \\ [-r, 2\varepsilon - r] \sqcup [r, 2\varepsilon + r] & \text{if } r > \varepsilon\,.\end{cases}$$

To conclude we just have to consider the position of ε and the fact that again the latter quantity does not depend on t. □

Let us have a look at the case $D = 1$. According to (22), $\mathbb{E}\left(S_2^2(F_{j_{\min},j_{\max}}, \varepsilon)\right)$ has the same order as ε for ε small enough. This shows that $F_{j_{\min},j_{\max}}$ has regular increments on average. So, in order to keep the irregularity of the original Poisson field F_H (see (20)), we cannot consider ε in $(0, \alpha^{j_{\max}+1})$. The smallest value for ε ensuring the correct approximation $\mathbb{E}\left(S_2^2(F_{j_{\min},j_{\max}}, \varepsilon)\right) \simeq \mathbb{E}\left(S_2^2(F,\varepsilon)\right) \simeq \varepsilon^{2H}$ is $\alpha^{j_{\max}+1}$. Moreover, in practice, ε is chosen as small as possible, that is as δ, the resolution of the grid. This remark justifies again the assumption $\alpha^{j_{\max}+1} = \delta$.

3.2. **The q-structure functions of a fractional Poisson field.**

For some well-known functions f, one can give the exact behavior of $S_q(f,\varepsilon)$. For example if f is a random Weierstrass function W_H with the Hurst parameter H, then (see [9]) :

$$\forall\, q > 0 \quad S_q(W_H, \varepsilon) \simeq_0 \varepsilon^H\, a.s. \qquad \text{and} \qquad \mathbb{E}(S_q^q(W_H,\varepsilon)) \simeq_0 \varepsilon^{qH}\,.$$

For a fractional Brownian field B_H, we have, for all $t \in \mathbb{R}^D$, $i = 1, \ldots, D$ and $\varepsilon > 0$,

$$B_H(t+\varepsilon e_i) - B_H(t-\varepsilon e_i) \stackrel{d}{=} B_H(2\varepsilon) \stackrel{d}{=} (2\varepsilon)^H B_H(1)\,,$$

and so

$$\forall\, q > 0 \quad \mathbb{E}(S_q^q(B_H,\varepsilon)) \simeq_0 \varepsilon^{qH}\,. \tag{24}$$

Since the fPf shares some properties with B_H, a natural question is to wonder whether F_H satisfies (24). One can observe that the assertion (i) of Proposition 3.1 implies that it does for $q = 2$. However the answer is negative, so F_H is multifractal in the sense of [14]. To state this result, we will study $\mathbb{E}(S_q^q(F_H,\varepsilon))$ for even integer values of q.

Theorem 3.2. *Let F_H be the fractional Poisson field of index $H \in \left(0, \frac{1}{2}\right)$. Then, for all $p \in \mathbb{N} \setminus \{0\}$,*

$$\mathbb{E}\left(S_{2p}^{2p}(F_H,\varepsilon)\right) \simeq_0 \varepsilon^{2H}\,.$$

Proof. According to (19) we have to compute $\mathbb{E}\left(F_H(t+\varepsilon e_i) - F_H(t-\varepsilon e_i)\right)^{2p}$ for $\varepsilon \geqslant 0$, $t \in [0,1]^D$ and $p \geqslant 1$. Let us recall that $\psi \in L^{2p}\left(\mathbb{R}^D \times \mathbb{R}^+, \nu(dx,dr)\right)$ with for any $k \geqslant 1$:

$$\begin{cases} \displaystyle\int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(2\varepsilon e_i, x-t+\varepsilon e_i, r)^{2k-1}\,\nu(dx,dr) = 0 \\[2ex] \displaystyle\int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(2\varepsilon e_i, x-t+\varepsilon e_i, r)^{2k}\,\nu(dx,dr) = c_H(D)(2\varepsilon)^{2H}\,. \end{cases}$$

Then, according to [1] (with the convention that $0^0 = 1$), using (23), we have

$$\begin{aligned} &\mathbb{E}\left((F_H(t+\varepsilon e_i) - F_H(t-\varepsilon e_i))^{2p}\right) \\ &\quad = \sum_{(r_1,\ldots,r_{2p})\in I(2p)} K_{2p}(r_1,\ldots,r_{2p}) \prod_{k=1}^{2p} \left(\int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(2\varepsilon e_i, x-t+\varepsilon e_i, r)^k\,\nu(dx,dr)\right)^{r_k} \\ &\quad = \sum_{(0,r_2,0,\ldots,r_{2p})\in I(2p)} K_{2p}(0,r_2,0,\ldots,r_{2p}) \prod_{k=1}^{p} \left(\int_{\mathbb{R}^D\times\mathbb{R}^+} \psi(2\varepsilon e_i, x-t+\varepsilon e_i, r)^{2k}\,\nu(dx,dr)\right)^{r_{2k}} \\ &\quad = \sum_{(r_1,\ldots,r_p)\in I(p)} \widetilde{K}_p(r_1,\ldots,r_p)\left(c_H(D)\varepsilon^{2H}\right)^{\sum_{k=1}^{p} r_k}\,, \end{aligned}$$

where for $p \geqslant 1$, $\widetilde{K}_p(r_1,\ldots,r_p) = (2p)!\left(\prod_{k=1}^{p} r_k!((2k)!)^{r_k}\right)^{-1}$. Integrating with respect to $t \in [0,1]^D$ and averaging according to the directions $i \in \{1,\ldots,D\}$ we get

$$\mathbb{E}\left(S_{2p}^{2p}(F_H,\varepsilon)\right) = \sum_{(r_1,\ldots,r_p)\in I(p)} \widetilde{K}_p(r_1,\ldots,r_p)\left(c_H(D)\varepsilon^{2H}\right)^{\sum_{k=1}^{p} r_k}\,.$$

Note that $e_p = (0,\ldots,0,1) \in I(p)$ and for any $(r_1,\ldots,r_p) \in I(p) \setminus \{e_p\}$ we have $\sum_{k=1}^{p} r_k \geqslant 2$ such that

$$\mathbb{E}\left(S_{2p}^{2p}(F_H,\varepsilon)\right) = c_H(D)(2\varepsilon)^{2H} + \varepsilon^{4H}\,u(\varepsilon)\,,$$

where $u(\varepsilon)$ is bounded for small ε. This concludes the proof. $\square$

4. Simulations

All the Matlab codes are available at http://www.math-info.univ-paris5.fr/∼demichel

4.1. Simulation of a fractional Poisson field on a cube.

We focus here on the simulation of the fractional Poisson field F_H for $H \in (0,1/2)$. According to Theorem 2.2 we consider $F_{j_{\min},j_{\max}}$ as an approximation for F_H. We generate exact simulations of the fields T_j and $F_{j_{\min},j_{\max}}$ (see (6) and (12)) on the cube $[c,c+d]^D$ with $c \in \mathbb{R}$ and $d \geqslant 1$, containing 0. Let us recall that T_j may be written as the sum

$$T_j = \sum_{(X_j^n,R_j^n)\in\Phi_j} \mathbb{1}_{B(X_j^n,R_j^n)}\,.$$

If X_j^n is at a distance of the cube larger than R_j^n then no point of the cube is covered by $B(X_j^n,R_j^n)$. Since $R_j^n \leqslant \alpha^j$, when simulating the slice number j, it is enough to pick up centers of balls randomly in the enlarged cube $[c-\alpha^j, c+d+\alpha^j]^D$.

Let us denote $c_{\alpha,H} = (\alpha^{-D+2H}-1)/(D-2H)$ and consider the measure on $\mathbb{R}^D \times \mathbb{R}^+$

$$\widetilde{\nu_j}(dx,dr) = c_{\alpha,H}(d+2\alpha^j)^D\alpha^{-j(D-2H)}\,\mu_j(dx)\otimes\rho_j(dr) \tag{25}$$

where

$$\begin{cases} \mu_j(dx) = \dfrac{1}{(d+2\alpha^j)^D}\,\mathbb{1}_{[c-\alpha^j,\,c+d+\alpha^j]^D}(x)\,dx \\ \\ \rho_j(dr) = c_{\alpha,H}^{-1}\,\alpha^{j(D-2H)}r^{-D-1+2H}\,\mathbb{1}_{(\alpha^{j+1},\,\alpha^j]}(r)\,dr \end{cases} \tag{26}$$

are respectively two probability measures for centers and radii of random balls.

We simulate T_j considering

$$T_j(y) = \sum_{n=1}^{\Lambda_j} \mathbb{1}_{B(X_j^n,R_j^n)}(y)\,, \quad y \in [c,c+d]^D\,, \tag{27}$$

where

- $(X_j^n)_n$ is a family of iid random variables with law $\mu_j(dx)$
- $(R_j^n)_n$ is a family of iid random variables with law $\rho_j(dr)$
- Λ_j is a Poisson random variable with parameter $c_{\alpha,H}(d+2\alpha^j)^D\alpha^{-j(D-2H)}$.

Let us recall that a simple way to generate the sequence $(R_j^n)_n$ is to use the pseudo-inverse method : if V_j^n is with uniform law on $[0,1]$ then $R_j^n = \alpha^j(\alpha^{-D+2H} - (\alpha^{-D+2H}-1)V_j^n)^{-1/(D-2H)}$.

Consequently, considering independent realizations of $(T_j)_{j_{\min}\leqslant j\leqslant j_{\max}}$ one may simulate $F_{j_{\min},j_{\max}}$ using (12) and

$$\sum_{j=j_{\min}}^{j_{\max}} T_j(y) = \sum_{j=j_{\min}}^{j_{\max}} \left(\sum_{n=1}^{\Lambda_j} \mathbb{1}_{B(X_j^n,R_j^n)}(y)\right),\ \ y \in [c,c+d]^D\,, \tag{28}$$

since $0 \in [c,c+d]^D$.

In practice $F_{j_{\min},j_{\max}}$ is only obtained on a discrete subset of $[c, c+d]^D$, say a grid with a step of size $\delta \in (0,1)$. Thus it no longer makes sense to consider balls with radii smaller than δ. Since the smallest radius is greater than $\alpha^{j_{\max}+1}$, one will assume that $\alpha^{j_{\max}+1} \geqslant \delta$. On the other hand, to get the most precise details, it seems natural to assume exactly that $\alpha^{j_{\max}+1} = \delta$. Consequently, $j_{\max}$ may be chosen freely. Given the resolution δ the slice factor α will be fixed by $\alpha^{j_{\max}+1} = \delta$.

Concerning the choice of $j_{\min}$, let us explain its effect. We can distinguish two different kinds of balls $B(X_j^n, R_j^n)$ according to their size :

- the large balls for $R_j^n > \alpha^{j+1} \geqslant d$: they give the general geometry of the graph of $F_{j_{\min},j_{\max}}$ since they are visible,
- the small balls for $R_j^n \leqslant \alpha^j \leqslant d$: they give the local irregularity of the graph.

When more interested in the irregularity of the field than in its look-like geometry, one may choose $j_{\min} \geqslant \log d / |\log \alpha|$.

4.2. **Simulation of the q-structure functions.**

We explain now how to use structure functions in a practical way. To simplify we only deal with the case $D = 1$ and $[c, c+d]^D = [0,1]$. Let us consider a function $f : \mathbb{R} \to \mathbb{R}$. We suppose that we can simulate f on a regular subdivision $\tau = \{\, \tau_i = iN^{-1} \,\}$ of $[-1, 2]$ with step $\delta = N^{-1}$ (where $N \geqslant 2$). We consider $S_q(f, \varepsilon)$ given by (18) for $\varepsilon \in [N^{-1}, 1] \cap \tau$. More precisely, we choose a finite sequence of $M \geqslant 2$ integers n_m (with $n_1 = 1$ and $n_M = N$) and put

$$\varepsilon_m = \tau_{n_m} = n_m N^{-1} \,. \tag{29}$$

Let us note that the smallest ε considered corresponds to the resolution of the grid N^{-1}. Then, for $1 \leqslant m \leqslant M$, we approximate the integral defining $S_q^q(f, \varepsilon_m)$ by its Riemann sum:

$$S_q^q(f, \varepsilon_m) = \int_0^1 |f(t+\varepsilon_m) - f(t-\varepsilon_m)|^q \, dt \approx \frac{1}{N} \sum_{i=1}^{N} |f(\tau_{i+n_m}) - f(\tau_{i-n_m})|^q \,. \tag{30}$$

Let us emphasize that Proposition 3.1 still holds when replacing $S_2^2(f, \varepsilon_m)$ by the discrete sum (30).

Now we focus on the behavior of $S_q^q(f, \varepsilon_m)$ with respect to ε_m. Formula (20) invites us to use log-log plots. For $1 \leqslant m \leqslant M$, let us write $\eta_m = \log n_m = \log(N\varepsilon_m)$ and

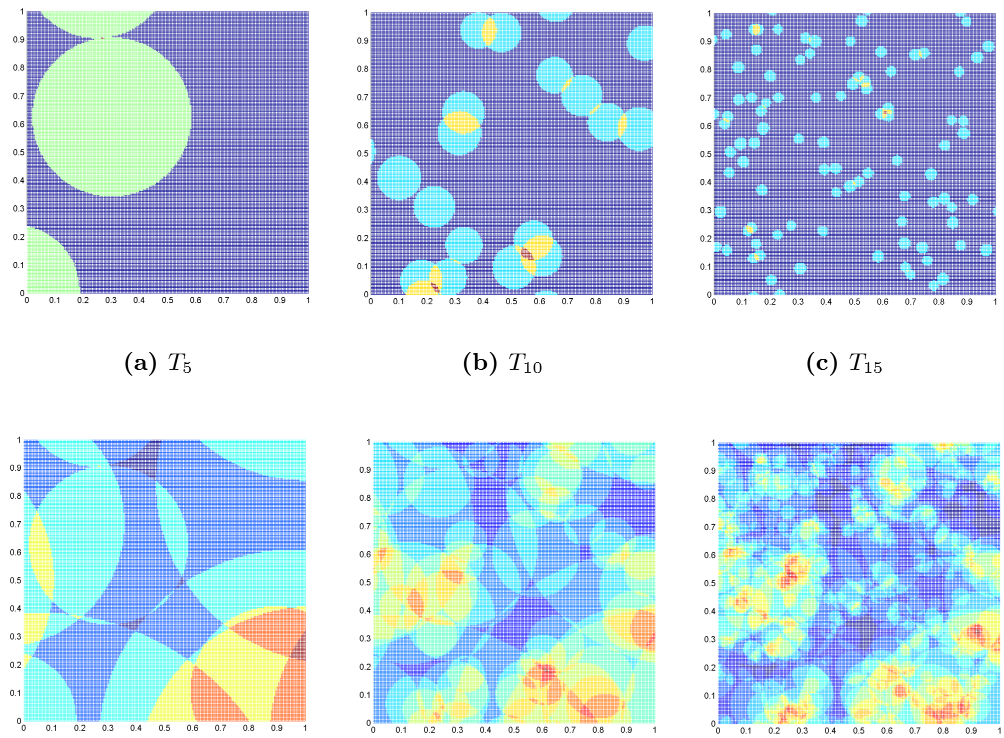
$$L_q(f, \eta_m) = \frac{1}{q} \log S_q^q(f, \varepsilon_m) = \frac{1}{q} \log S_q^q\left(f, N^{-1} 10^{\eta_m}\right) \tag{31}$$

and consider the log-log plot $\{(\eta_m, L_q(f, \eta_m)) \; ; \; 1 \leqslant m \leqslant M\}$.

In order to obtain the η_m approximately equally spaced, one usually assumes that the ε_ms have an arithmetic progression (but n_m should be an integer). However since $n_1 = 1$ and $n_2 \geqslant 2$, we always have $\eta_2 - \eta_1 \geqslant \log 2$. In order to get the minimal lag $\eta_2 - \eta_1$ we set $n_2 = 2$.

5. Estimation of index H

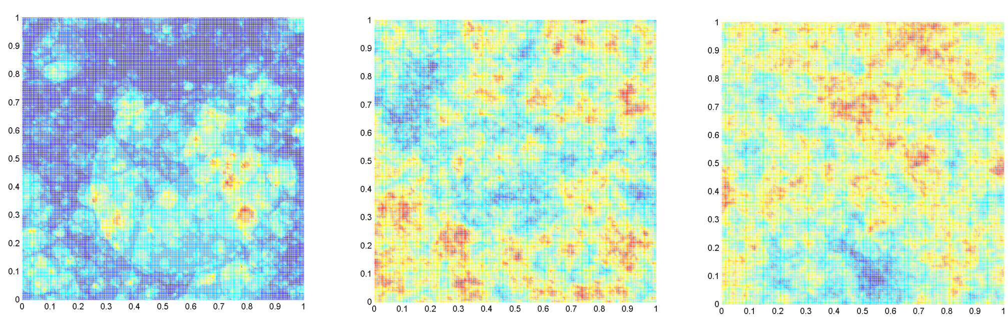
In this section we are interested in the estimation of index H from realizations of $F_{j_{\min},j_{\max}}$, an approximation of F_H.



(a) T_5 **(b)** T_{10} **(c)** T_{15}

(d) $F_{0,5}$ **(e)** $F_{0,10}$ **(f)** $F_{0,15}$

FIGURE 1. Step by step piling for the fractional Poisson field.



(a) $K = 1$ **(b)** $K = 5$ **(c)** $K = 10$

FIGURE 2. Normal convergence when $K \to +\infty$.

There are two ways to estimate H: looking at the small scales through the behavior of $S_q^q(\varepsilon, F_H)$ when ε goes to 0, and looking at the intermediate scales when $\varepsilon \simeq 1$. Each way carries a practical difficulty. The first one supposes that the field can be seen at scales arbitrarily smaller than the resolution of the grid whereas the second requires that all the contributing balls be visible.

We show that it is possible to overcome these difficulties. The main tools are the 2-structure functions given by (18). We build two estimators using Proposition 3.1. Such methods have been used for the "fractal sums of pulses" (see [8] chapter 7). The first method can be seen as a generalization of the quadratic variation method used for the estimation of the Hurst index of a Brownian motion (see [13]).

In the following $j_{\min}$ and $j_{\max}$ will be assumed to be known. Moreover we fix an integer $N \geqslant 1$ such that $\alpha^{j_{\max}+1} = N^{-1}$. In practice it will be linked to the resolution by $\delta = N^{-1}$.

5.1. **The $D = 1$ case.**
Here we assume that $D = 1$ and $j_{\min} \geqslant 0$. Since it is not always possible to see balls with radius larger than $\alpha^{j_{\min}}$, we focus on the small balls or equivalently on the small scales. We use the 2-structure functions for small ε.

Proposition 5.1. *Assume that $D = 1$. Let $0 \leqslant j_{\min} < j_{\max}$ and let $\left(F^{(k)}_{j_{\min},j_{\max}}\right)_{k\geqslant 1}$ be iid copies of $F_{j_{\min},j_{\max}}$ of index H. Finally, let us define for all $K \geqslant 1$:*

$$\widehat{\gamma_2}(K) = \frac{1}{2\log 2} \log \frac{\sum_{k=1}^{K} S_2^2\left(F^{(k)}_{j_{\min},j_{\max}}, 2N^{-1}\right)}{\sum_{k=1}^{K} S_2^2\left(F^{(k)}_{j_{\min},j_{\max}}, N^{-1}\right)} . \tag{32}$$

Then, when K goes to $+\infty$, $\widehat{\gamma_2}(K)$ converges almost surely to $h_1(H)$ where

$$h_1(H) = H + \frac{1}{2\log 2} \log\left(\frac{1-(2\alpha^{-j_{\min}}N^{-1})^{1-2H} + (1-2H)(1-2^{-2H})/(2H)}{1-(\alpha^{-j_{\min}}N^{-1})^{1-2H}}\right) . \tag{33}$$

Proof. Since $\alpha^{j_{\max}+1} = N^{-1}$, Proposition 3.1 implies that for all $\varepsilon \in [N^{-1}, \alpha^{j_{\min}}]$:

$$\mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}}, \varepsilon\right)\right) = \frac{4}{1-2H}\,\varepsilon^{2H} - \frac{4}{1-2H}\,\varepsilon\,\alpha^{(1-2H)j_{\min}} + \frac{2}{H}\left(\varepsilon^{2H} - N^{-2H}\right) .$$

Therefore we obtain :

$$\frac{\mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}}, 2N^{-1}\right)\right)}{\mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}}, N^{-1}\right)\right)} = 2^{2H}\left(\frac{1-(2\alpha^{-j_{\min}}N^{-1})^{1-2H} + (1-2H)(1-2^{-2H})/(2H)}{1-(\alpha^{-j_{\min}}N^{-1})^{1-2H}}\right) .$$

Thus :

$$\begin{aligned} &\frac{1}{2\log 2}\log\frac{\mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}}, 2N^{-1}\right)\right)}{\mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}}, N^{-1}\right)\right)} \\ &\quad = H + \frac{1}{2\log 2}\log\left(\frac{1-(2\alpha^{-j_{\min}}N^{-1})^{1-2H} + (1-2H)(1-2^{-2H})/(2H)}{1-(\alpha^{-j_{\min}}N^{-1})^{1-2H}}\right) . \end{aligned} \tag{34}$$

But one has, by the law of large numbers :

$$\mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}},\cdot\right)\right) = \lim_{K\to+\infty} \frac{1}{K}\sum_{k=1}^{K} S_2^2\big(F^{(k)}_{j_{\min},j_{\max}},\cdot\big)\,.$$

So $\widehat{\gamma_2}(K) \underset{K\to+\infty}{\longrightarrow} h_1(H)$ almost surely and the result follows from (34). □

From this result, we deduce a first estimator $\widehat{H}_1(K)$ for index H setting $\widehat{H}_1(K) = h_1^{-1}\left(\widehat{\gamma_2}(K)\right)$.

In practice the $F^{(k)}_{j_{\min},j_{\max}}$ are simulated on a regular grid with step $\delta = N^{-1}$ using the results of Section 4.1. We compute $S_2^2\big(F^{(k)}_{j_{\min},j_{\max}}, N^{-1}\big)$ and $S_2^2\big(F^{(k)}_{j_{\min},j_{\max}}, 2N^{-1}\big)$ for each $F^{(k)}_{j_{\min},j_{\max}}$ using the discrete sums (30). This gives $\widehat{\gamma_2}(K)$ and we find $\widehat{H}_1(K)$ by solving the equation $h_1(h) = \widehat{\gamma_2}(K)$ with a numerical approximation procedure (e.g. the standard Newton method).

Let us give an example. We consider $F_{j_{\min},j_{\max}}$ for different values of H. The processes are simulated on a regular grid of $[-1, 2]$ with step $\delta = 5.10^{-4}$ (so $N = 2000$). We chose $j_{\min} = 0$ and $j_{\max} = 15$. Table 1 shows the results for $K = 500$.

$h_1(H)$	$\widehat{\gamma_2}$	H	$\widehat{H}_1$
0.47399	0.47425	0.45	0.45159
0.46519	0.46553	0.40	0.40182
0.45531	0.45589	0.35	0.35282
0.44473	0.44452	0.30	0.29906
0.43375	0.43492	0.25	0.25528
0.42265	0.42196	0.20	0.19688
0.41161	0.41051	0.15	0.14496
0.40075	0.40111	0.10	0.10169
0.39013	0.39028	0.05	0.05074

TABLE 1. Estimation of H with $(j_{\min}, j_{\max}) = (0, 15)$ and $K = 500$.

We see that H is well approximated by $\widehat{H}_1(K)$ whatever H is.

Let us remark that we cannot provide such an estimator in higher dimensions looking at $S_2^2(F_{j_{\min},j_{\max}}, \varepsilon)$ only for small ε. When $D = 1$, assertion (iii) of Proposition 3.1 gives us an exact expression for $S_2^2(F_{j_{\min},j_{\max}}, \varepsilon)$ but we have none for $D > 1$. Thus, whatever D is, we have to use assertion (ii) of Proposition 3.1 to build an estimator .

5.2. **The $D \geqslant 1$ and $j_{\min} > 0$ case.**

Since $j_{\min} > 0$ it is possible to see all balls whatever their radii. This allows us to propose an estimator for H in the general multidimensional case $D \geqslant 1$. We may focus on the large scales and use the 2-structure functions for ε near 1.

Under the assumption $j_{\min} > 0$ the interval $(\alpha^{j_{\min}}, 1]$ is not empty. Let M^* be the smallest integer such that $\varepsilon_{M^*} = n_{M^*} N^{-1} > \alpha^{j_{\min}}$. Then for all $m \in \{M^*, \ldots, M\}$, $\varepsilon_m \in (\alpha^{j_{\min}}, 1]$ (note that the larger $j_{\min}$ is the more points we have).

Proposition 5.2. *Let $0 < j_{\min} < j_{\max}$ and let $\left(F^{(k)}_{j_{\min},j_{\max}}\right)_{k\geqslant 1}$ be iid copies of $F_{j_{\min},j_{\max}}$ of index H. Finally, let us define for all $K \geqslant 1$:*

$$\widehat{\pi_2}(K) = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{1}{M-M^*+1}\sum_{m=M^*}^{M} S_2^2\left(F^{(k)}_{j_{\min},j_{\max}}, \varepsilon_m\right)\right) . \tag{35}$$

Then, when K goes to $+\infty$, $\widehat{\pi_2}(K)$ converges almost surely to $h_2(H)$ where

$$h_2(H) = \frac{\pi^{D/2}}{\Gamma(1+D/2)H}\left(\alpha^{2Hj_{\min}} - N^{-2H}\right) . \tag{36}$$

Proof. Proposition 3.1 (ii) implies :

$$\forall\, m \in \{M^*, \ldots, M\} \qquad \mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}}, \varepsilon_m\right)\right) = \frac{\pi^{D/2}}{\Gamma(1+D/2)H}\left(\alpha^{2Hj_{\min}} - N^{-2H}\right) .$$

Thus:

$$\frac{1}{M-M^*+1}\sum_{m=M^*}^{M} \mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}}, \varepsilon_m\right)\right) = \frac{\pi^{D/2}}{\Gamma(1+D/2)H}\left(\alpha^{2Hj_{\min}} - N^{-2H}\right) . \tag{37}$$

Now, by the law of large numbers, one has:

$$\mathbb{E}\left(S_2^2\left(F_{j_{\min},j_{\max}}, \cdot\right)\right) = \lim_{K\to+\infty}\frac{1}{K}\sum_{k=1}^{K} S_2^2\left(F^{(k)}_{j_{\min},j_{\max}}, \cdot\right) . \tag{38}$$

So $\widehat{\pi_2}(K) \underset{K\to+\infty}{\longrightarrow} h_2(H)$ almost surely and the result follows from (37). □

As previously, we deduce a second estimator $\widehat{H}_2(K)$ for index H considering $\widehat{H}_2(K) = h_2^{-1}\left(\widehat{\pi_2}(K)\right)$. In practice we proceed in the same way as for $\widehat{H}_1(K)$.

Let us give another example. We look again at an fPf of index H, $F_{j_{\min},j_{\max}}$, for different values of H. We assume $D = 1$. The processes are again simulated on a regular grid of $[-1, 2]$ with step $\delta = 5.10^{-4}$ (so $N = 2000$). We chose $j_{\min} = 5$ and $j_{\max} = 15$. We use the two estimators $\widehat{H}_1(K)$ and $\widehat{H}_2(K)$ with $K = 500$. Table 2 reports the different results.

We see that in both cases $j_{\min} = 0$ and $j_{\min} = 5$, the estimates given by $\widehat{H}_1$ are with the same precision. When $j_{\min} = 5$, $\widehat{\pi_2}$ is far from $h_2(H)$ whereas $\widehat{\gamma_2}$ is close to $h_1(H)$. However the estimator $\widehat{H}_2$ is better than $\widehat{H}_1$. Actually the function h_2 is more convenient for a numerical inversion since its derivative is larger than the h_1 one (see Fig. 3).

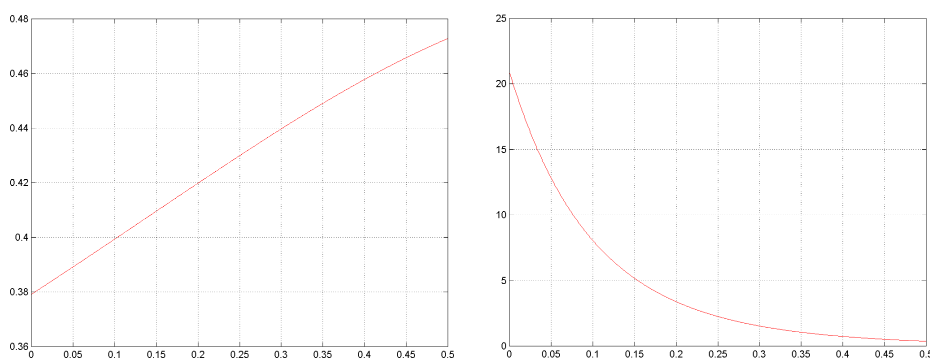
$h_1(H)$	$\widehat{\gamma_2}$	H	$\widehat{H}_1$
0.46579	0.46579	0.45	0.45001
0.45784	0.45853	0.40	0.40415
0.44910	0.44847	0.35	0.34656
0.43974	0.43863	0.30	0.29427
0.42993	0.42895	0.25	0.24513
0.41983	0.41942	0.20	0.19800
0.40959	0.41122	0.15	0.15795
0.39934	0.39921	0.10	0.09937
0.38917	0.38904	0.05	0.04938

(a) ***Using*** $\widehat{H}_1$

$h_2(H)$	$\widehat{\pi_2}$	H	$\widehat{H}_2$
0.51932	0.52374	0.45	0.44877
0.73624	0.74679	0.40	0.39800
1.05565	1.05701	0.35	0.34982
1.53341	1.52188	0.30	0.30099
2.26068	2.27908	0.25	0.24897
3.38875	3.41444	0.20	0.19909
5.17488	5.20930	0.15	0.14924
8.06343	8.05267	0.10	0.10015
12.83808	12.87241	0.05	0.04971

(b) ***Using*** $\widehat{H}_2$

TABLE 2. Estimation of H with $(j_{\min}, j_{\max}) = (5, 15)$ and $K = 500$.



(a) $H \mapsto h_1(H)$ **(b)** $H \mapsto h_2(H)$

FIGURE 3. Functions h_1 et h_2 with $(j_{\min}, j_{\max}) = (5, 15)$ and $N = 2000$.

Finally Figure 4 shows the 2-structure curves for $H = 0.25$ (see (31)) where $S_2(F_{j_{\min},j_{\max}}, \varepsilon_m)$ is replace by its empirical mean $\frac{1}{K}\sum_{k=1}^{K} S_2(F^{(k)}_{j_{\min},j_{\max}}, \varepsilon_m)$ with $K = 500$. Near 0 one cannot distinguish the curve from its tangent. For $\eta \geqslant \eta_{M^*} = 2.2695$ the curve is near a constant.

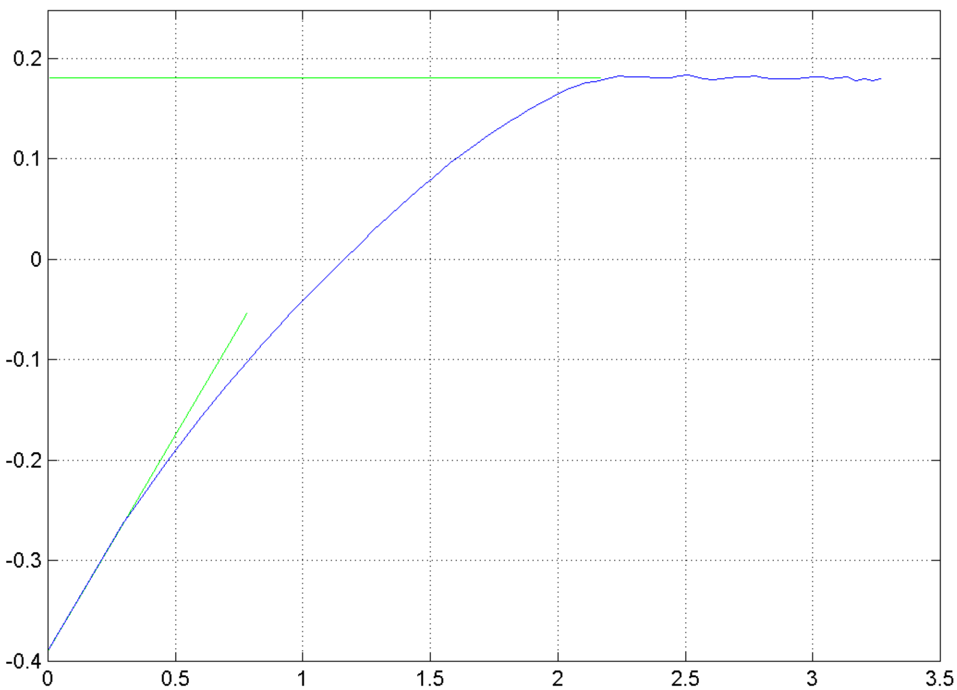


FIGURE 4. The 2-structure curve of $F_{j_{\min},j_{\max}}$ for $(j_{\min}, j_{\max}) = (5, 15)$, $H = 0.25$ and $K = 500$.

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