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LIMITING ABSORPTION PRINCIPLE FOR THE DISSIPATIVE
HELMHOLTZ EQUATION

JULIEN ROYER

Abstract. Adapting Mourre’s commutator method to the dissipative setting, we
prove a limiting absorption principle for a class of abstract dissipative operators. A
consequence is the resolvent estimates for the high frequency Helmholtz equation when
trapped trajectories meet the set where the imaginary part of the potential is non-zero.
We also give the resolvent estimates in Besov spaces.

1. Introduction

We consider the following Helmholtz equation:
\[ \Delta A(x) + k_0^2(1 - N(x))A(x) + ik_0a(x)A(x) = A_0 \]  
(1.1)
This equation modelizes accurately the propagation of the electromagnetic field of a
laser in material medium. Here \( k_0 \) is the wave number of the laser in the vacuum, \( N \)
and \( a \) are smooth nonnegative functions representing the electronic density of material
medium and the absorption coefficient of the laser energy by material medium, and
\( A_0 \) is an incident known excitation (see [BLSS03]). Note that the laser wave cannot
propagate in regions where \( N(x) \geq 1 \), so it is assumed that \( 0 \leq N(x) < 1 \). An
important application of this problem is the design of very high power laser device such
as the Laser Mégajoule in France or the National Ignition Facility in the USA.

The oscillatory behaviour of the solution makes numerical resolution very expensive.
Fortunately, the wave length of the laser in the vacuum \( 2\pi k_0^{-1} \) is much smaller than the
scale of variation of \( N \). It is therefore relevant to consider the high frequency limit \( k_0 \to \infty \).
The simplified model with a constant absorption coefficient has been studied in many
papers. This coefficient is either assumed to be positive (see [BCKP02, BLSS03, WZ06])
in order to be regularizing, or only nonnegative ([Wan07]), in which case the outgoing
(or incoming) solution has to be chosen for \( A \), but in both cases the non-symmetric
part of the equation is in the spectral parameter, and what remains is a selfadjoint
Schrödinger operator. More precisely we are led to study an equation of the form:
\[ (-h^2 \Delta + V(x) - (E + i\mu_h))u_h = S_h \]
where \( h \sim k_0^{-1} \) is a small parameter.

When the absorption is not assumed to be constant, it has to be in the operator itself
and the selfadjoint theory no longer applies. However, the anti-adjoint part is small
compared to the selfadjoint Schrödinger operator, so by perturbation theory we can see
that some results concerning the selfadjoint part still apply to the perturbed operator.

In this paper we study the limiting absorption principle for the following dissipative
Schrödinger operator:
\[ H_h = -h^2 \Delta + V_1(x) - iv(h)V_2(x) \]
on \( L^2(\mathbb{R}^n) \), where \( V_1 \) is a real function, \( V_2 \) is nonnegative and \( \nu : [0, 1] \to [0, 1] \). Note that
\( \nu(h) = h \) for the Helmholtz equation.

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In the first section, we prove a uniform and dissipative version of the abstract commutator method introduced by E. Mourre in [Mou81] and developed in many papers (see for instance [PSS85], [MP84], [GGM88], [DJ01], [CGM04] and references therein). In particular we see that the anti-adjoint part with fixed sign allows us to weaken the Mourre condition we need to prove uniform estimates and limiting absorption principle on the upper half-plane. On the contrary, the result is not valid on the other side of the real axis.

In section 2 we apply this abstract result to the dissipative Schrödinger operator in the semi-classical setting, following the idea of C. Gérard and A. Martinez ([GM88], see also [FT87] for the semi-classical limiting absorption principle). In particular we get uniform estimates of the resolvent $(H_h - z)^{-1}$ for $h$ small enough, $\Im z > 0$ and $\Re z$ close to $E > 0$. In the selfadjoint case, the result is true if and only if $E$ is a non-trapping energy, that is if there is no bounded classical trajectory for the hamiltonian flow associated to the symbol $p(x,\xi) = \xi^2 + V_1(x)$ of $H_h$. In the dissipative case, the weakened Mourre condition gives a weaker condition on the classical behaviour: we only have to assume that any bounded trajectory of energy $E$ meets the set where $V_2 > 0$. Note that it is consistent with the selfadjoint result. Section 3 is devoted to prove that this condition is necessary (when $\nu(h) = h$, which is the case we are mainly interested in). To this purpose we use a selfadjoint dilation of the Schrödinger operator and we prove a dissipative Egorov theorem.

Finally, we show that the estimates we have proved in weighted spaces can be extended to estimates in Besov spaces, first for the abstract setting of section 2 and then for the Schrödinger operator.

2. Commutator method for a family of dissipative operators

We first recall that an operator $H$ of domain $D(H)$ in the Hilbert space $\mathcal{H}$ is said to be dissipative if:

$$\forall \varphi \in D(H), \quad \Im \langle H\varphi, \varphi \rangle \leq 0$$

2.1. Uniform conjugate operators. Let $(H_h)_{h \in [0,1]}$ be a family of dissipative operators on $\mathcal{H}$. We assume that $H_h$ is of the form $H_h = H_0^h - iV_h$ where $H_0^h$ is selfadjoint on a domain $D_H$ independant of $h$ and $V_h$ is selfadjoint, nonnegative and uniformly $H_0^h$-bounded with relative bound less than 1:

$$\exists a \in [0,1], \exists b \in \mathbb{R}, \forall h \in [0,1], \forall \varphi \in D_H, \quad \|V_h \varphi\| \leq a \left\|H_0^h \varphi\right\| + b \|\varphi\| \quad (2.1)$$

For any $h \in [0,1]$ and $\varphi \in D_H$, write: $\|\varphi\|_{\Gamma_h} = \left\|H_0^h \varphi\right\| + \|\varphi\|$. Consider now a family $(A_h)_{h \in [0,1]}$ of selfadjoint operators on a domain $D_A$ independant of $h$, $J \subset \mathbb{R}$ and $(\alpha_h)_{h \in [0,1]}$ with $\alpha_h \in [0,1]$.

**Definition 2.1.** The family $(A_h)$ is said to be uniformly conjugate to $(H_h)$ on $J$ with bounds $(\alpha_h)$ if the following conditions are satisfied:

(a) For every $h \in [0,1]$, $D_H \cap D_A$ is a core for $H_0^h$.

(b) $e^{itA_h}$ maps $D_H$ into itself for any $t \in \mathbb{R}$, $h \in [0,1]$, and:

$$\forall \varphi \in D_H, \quad \sup_{h \in [0,1], \|\cdot\| \leq 1} \left\|e^{itA_h} \varphi\right\|_{\Gamma_h} < \infty \quad (2.2)$$

(c) For every $h \in [0,1]$, the quadratic forms $i[H_0^h, A_h]$ and $i[V_h, A_h]$ defined on $D_H \cap D_A$ are bounded from below and closable. Moreover, if we denote by $i[H_0^h, A_h]$ and $i[V_h, A_h]$ their closures, then $D_H \subset D(i[H_0^h, A_h]) \cap D(i[V_h, A_h])$ and there exists $c > 0$ such that for $h \in [0,1]$ and $\varphi \in D_H$ we have:

$$\left\|i[H_0^h, A_h] \varphi\right\| + \left\|i[V_h, A_h] \varphi\right\| \leq c \sqrt{\alpha_h} \|\varphi\|_{\Gamma_h} \quad (2.3)$$
(d) There exists $c \geq 0$ such that for all $\varphi, \psi \in \mathcal{D}_H \cap \mathcal{D}_A$:

$$\left| \left\langle i[H_0^b, A_0^b] \varphi, A_0^b \psi \right\rangle - \left\langle A_0^b i[H_0^b, A_0^b] \varphi, \psi \right\rangle \right| \leq c \alpha_h \| \varphi \|_{\Gamma_h} \| \psi \|_{\Gamma_h} \tag{2.4}$$

and similar estimates hold for the forms $[i[V_h, A_h], A_h]$ and $[V_h, A_h]$.

(e) There exists $C_V \geq 0$ such that for all $h \in [0, 1]$:

$$\mathbb{1}_J(H_0^b) \left( i[H_0^b, A_0^b] + C_V V_h \right) \mathbb{1}_J(H_0^b) \supseteq \alpha_h \mathbb{1}_J(H_0^b) \tag{2.5}$$

where $\mathbb{1}_J$ denotes the characteristic function of $J$ and hence $(\mathbb{1}_J(H_0^b))_{J \subset \mathbb{R}}$ is the set of spectral projections for the selfadjoint operator $H_0^b$.

Let $\mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \}$ and for $J \subset \mathbb{R}$: $\mathbb{C}_{J,+} = \{ z \in \mathbb{C}_+ : \text{Re} z \in J \}$.

2.2. Abstract limiting absorption principle. We first prove a version of the quadratic estimates (see [Mou81, prop. II.5]) we need in our dissipative case:

**Proposition 2.2.** Let $T = T_R - iT_I$ be a dissipative operator on $\mathcal{H}$ with $T_R$ selfadjoint and $T_I$ nonnegative, selfadjoint and $T_R$-bounded. Then for all $z \in \mathbb{C}_+$ the operator $(T-z)$ has a bounded inverse. Moreover if $B$ is an operator such that $B^* B \leq T_I$ and $Q$ is a bounded selfadjoint operator, then we have:

$$\| B(T-z)^{-1} Q \| \leq \| Q(T-z)^{-1} Q \|^{\frac{1}{2}} \tag{2.6}$$

**Proof.** Since $T_R$ is closed and $T_I$ is $T_R$-bounded, $T$ is closed. For $z \in \mathbb{C}_+$ and $\varphi \in \mathcal{D}(H_0)$ we have:

$$\| (T-z) \varphi \| \| \varphi \| \geq | \text{Im} \langle (T-z) \varphi, \varphi \rangle | = \langle T_I \varphi, \varphi \rangle + \text{Im} z \| \varphi \|^2 \geq \text{Im} z \| \varphi \|^2$$

So $(T-z)$ is injective with closed range. We similarly prove that $\| (T^* - \overline{z}) \varphi \| \geq \text{Im} z \| \varphi \|$, so $\text{Ran}(T-z)$ is dense in $\mathcal{H}$ and hence equal to $\mathcal{H}$, which proves that $(T-z)$ has a bounded inverse. Let $\varphi \in \mathcal{H}$. We compute:

$$\| B(T-z)^{-1} Q \varphi \|^2 = \langle B^* B(T-z)^{-1} Q \varphi, (T-z)^{-1} Q \varphi \rangle$$

$$\leq \langle T_I (T-z)^{-1} Q \varphi, (T-z)^{-1} Q \varphi \rangle + \text{Im} z \langle (T-z)^{-1} Q \varphi, (T-z)^{-1} Q \varphi \rangle$$

$$\leq \frac{1}{2i} \langle Q(T^* - \overline{z})^{-1} |(T^* - \overline{z}) - (T-z)|| (T-z)^{-1} Q \varphi, \varphi \rangle$$

$$\leq \| Q(T-z)^{-1} Q \| \| \varphi \|^2$$

Let $\langle \cdot \rangle$ denote the function $x \mapsto \sqrt{1 + |x|^2}$. We can now state and prove the main theorem of this section:

**Theorem 2.3.** Let $(H_h)_{h \in [0,1]}$ be a family of dissipative operators of the form $H_h = H_0^b - iV_h$ as in section 2.2 and $(A_h)_{h \in [0,1]}$ a conjugate family to $(H_h)$ on the open interval $J \subset \mathbb{R}$ with bounds $(\alpha_h)_{h \in [0,1]}$. Then for any closed subinterval $I \subset J$ and all $s > \frac{1}{2}$, there exists a constant $c \geq 0$ such that:

$$\forall h \in [0,1], \forall z \in \mathbb{C}_{I,+}, \quad \| \langle A_h \rangle^{-s} (H_h - z)^{-1} \langle A_h \rangle^{-s} \| \leq \frac{c}{\alpha_h} \tag{2.7}$$

Moreover, we have for all $z, z' \in \mathbb{C}_{I,+}$:

$$\| \langle A_h \rangle^{-s} ((H_h - z)^{-1} - (H_h - z')^{-1}) \langle A_h \rangle^{-s} \| \leq c \alpha_h^{\frac{s}{1+s}} |z - z'|^{\frac{2s}{1+s}} \tag{2.8}$$

and for $E \in J$ the limit:

$$\langle A_h \rangle^{-s} (H_h - (E + i\mu))^{-1} - (H_h - (E + i\mu))^{-1} (A_h)^{-s} \quad \lim_{\mu \to 0^+}$$

exists in $\mathcal{L}(\mathcal{H})$ and is a $\frac{2s-1}{2s+1}$ Hölder continuous function of $E$. 

Remark 2.4. As in [Mou81], if we only need resolvent estimates for an operator \( H = H_0 - iV \) where \( H_0 \) is selfadjoint and \( V \) is selfadjoint, nonnegative and \( H_0 \)-bounded, we look for a conjugate operator which satisfies the same assumptions as in definition 2.1 with a weaker Mourre condition:

\[
\mathbb{1}_J(H_0) (i[H_0, A]^0 + C_VV) \mathbb{1}_J(H_0) \geq \alpha \mathbb{1}_J(H_0) + \frac{1}{2} \mathbb{1}_J(H_0) K \mathbb{1}_J(H_0)
\]

where \( K \) is a compact operator on \( \mathcal{H} \). Indeed, for any \( E \in J \cap \sigma_c(H_0) \) (the continuous spectrum of \( H_0 \)) we can find \( \delta > 0 \) such that:

\[
\mathbb{1}_{[E-\delta,E+\delta]}(H_0)K\mathbb{1}_{[E-\delta,E+\delta]}(H_0) \geq -\frac{\alpha}{2} \mathbb{1}_{[E-\delta,E+\delta]}(H_0)
\]

hence \( A \) is conjugate to \( H \) on \([E-\delta,E+\delta]\) with bound \( \frac{\alpha}{2} \) in the sense of definition 2.1.

The proof of theorem 2.3 follows that of the selfadjoint analog:

Proof. Let \( I \subset J \) be a closed interval and \( s \in \left[\frac{1}{2}, 1\right] \) (the conclusions are weaker for \( s > 1 \)). Throughout the proof, \( c \) stands for a constant which may change but does not depend on \( z \in \mathcal{C}_I, \epsilon \in [0,1] \) and \( h \in [0,1] \).

1. Let \( \phi \in C_0^\infty(J, [0,1]) \) with \( \phi = 1 \) in a neighborhood of \( I \). We set \( P_h = \phi(h_0) \) and \( P_h^c = (1 - \phi)(h_0) \). We also define: \( \Theta_R, h = i[H_0^0, A]^0, \Theta_I, h = i[V_h, A]^0, \Theta_h = \Theta_R, h - i\Theta_I, h \) and \( \Theta_h^V = C_VV_h + \Theta_h, \Theta_V \) being given by assumption (e). Then by assumptions (c) and (2.3), \( \Theta_h^V \) is \( H_0^0 \)-bounded and:

\[
\|\Theta_h P_h\| + \|P_h \Theta_h\| \leq c\sqrt{\alpha_h} \tag{2.10}
\]

The operator \( V_h \) is \( H_0^0 \)-bounded and \( P_h \Theta_h^V P_h \) is bounded, so for all \( h, \epsilon \in [0,1] \) we can apply proposition 2.2 with \( T_R = H_0^0 - \epsilon P_h \Theta_h^V P_h \) and \( T_I = V_h + \epsilon P_h (C_VV_h + \Theta_h) P_h \).

Indeed by assumption (e) we have:

\[
0 \leq (\sqrt{\alpha_h} P_h)^2 = \alpha_h P_h (J_0^0)^2 P_h \leq P_h (C_VV_h + \Theta_h) P_h \tag{2.11}
\]

and hence \( T_I \) is nonnegative so \( G_{z,h}(\epsilon) = (H_0 - i\epsilon P_h \Theta_h^V P_h - z)^{-1} \) is well-defined for any \( z \in \mathcal{C}_+ \).

Then we write \( Q_h = Q_h = (A_h)^{-1} Q_h (A_h)^{-1} \) and finally: \( F_{z,h}(\epsilon) = Q_h(z)G_{z,h}(\epsilon)Q_h(\epsilon) \).

By functional calculus we have:

\[
\|Q_h(\epsilon)\| \leq 1 \quad \text{and} \quad \|A_h Q_h(\epsilon)\| = \|Q_h(\epsilon) A_h\| = \epsilon \tag{2.12}
\]

and the second part of proposition 2.2 with \( B = \sqrt{\alpha_h} \) and \( Q = Q_h \) for all \( h, \epsilon \in [0,1] \) gives:

\[
\left\|\sqrt{\alpha_h} G_{z,h}(\epsilon)Q_h(\epsilon)\right\| \leq \|F_{z,h}(\epsilon)\|^{\frac{1}{2}} \tag{2.13}
\]

2. By (2.11) and proposition 2.2 now applied with \( B = \sqrt{\alpha_h} \), we also have:

\[
\|P_h G_{z,h}(\epsilon)Q_h(\epsilon)\| \leq \frac{1}{\sqrt{\alpha_h} \epsilon} \|F_{z,h}(\epsilon)\|^{\frac{1}{2}} \tag{2.14}
\]

On the other hand:

\[
(1 + \sqrt{\alpha_h}) P_h G_{z,h}(\epsilon)Q_h(\epsilon) = (1 + \sqrt{\alpha_h}) P_h (H_0^0 - z)^{-1} (1 + i(V_h + \epsilon P_h \Theta_h^V P_h) G_{z,h}(\epsilon))Q_h(\epsilon)
\]

\[
= (1 + \sqrt{\alpha_h}) P_h (H_0^0 - z)^{-1} Q_h(\epsilon)
\]

\[
+i(1 + \sqrt{\alpha_h}) P_h (H_0^0 - z)^{-1} V_h G_{z,h}(\epsilon)Q_h(\epsilon)
\]

\[
+i\epsilon(1 + \sqrt{\alpha_h}) P_h (H_0^0 - z)^{-1} P_h \Theta_h P_h G_{z,h}(\epsilon)Q_h(\epsilon)
\]

\[
+i\epsilon C_V (1 + \sqrt{\alpha_h}) P_h (H_0^0 - z)^{-1} P_h V_h P_h G_{z,h}(\epsilon)Q_h(\epsilon)
\]

By functional calculus and (2.3) we have:

\[
\left\|(1 + \sqrt{\alpha_h}) P_h (H_0^0 - z)^{-1} (1 + \sqrt{\alpha_h})\right\| \leq c
\]
With (2.13), (2.10) and (2.14), this proves that the first three terms of (2.11) are bounded by \(c(1 + \|F_{z,h}(\varepsilon)\|^2)\). For the last term, since \(P_h\sqrt{V_h}\) is uniformly bounded, it only remains to estimate:

\[
\varepsilon \|\sqrt{V_h}P_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\| \leq \varepsilon \|\sqrt{V_h}G_{z,h}(\varepsilon)Q_h(\varepsilon)\| + \varepsilon \|\sqrt{V_h}P_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\|
\]

\[
\leq \varepsilon \|F_{z,h}(\varepsilon)\|^2 + \varepsilon \left(1 + \sqrt{\|V_h\|}P_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\right)
\]

For \(\varepsilon \in [0, \varepsilon_0]\), \(\varepsilon_0 > 0\) small enough, we finally obtain:

\[
\|P_h'G_{z,h}(\varepsilon)Q_h(\varepsilon)\| + \|\sqrt{V_h}P_h'G_{z,h}(\varepsilon)Q_h(\varepsilon)\| \leq c \left(1 + \|F_{z,h}(\varepsilon)\|^2\right)
\]  

(2.16)

Together with (2.14) this gives:

\[
\|F_{z,h}(\varepsilon)\| \leq \|G_{z,h}(\varepsilon)Q_h(\varepsilon)\| \leq c \left(1 + \frac{\|F_{z,h}(\varepsilon)\|^2}{\sqrt{\alpha_h\sqrt{\varepsilon}}}\right)
\]

(2.17)

and hence:

\[
\|F_{z,h}(\varepsilon)\| \leq \frac{c}{\alpha_h\varepsilon}
\]

(2.18)

Note that by (2.1) we also have:

\[
\bigg\|H_0^1G_{z,h}(\varepsilon)Q_h(\varepsilon)\bigg\| \leq \frac{1}{1-a} \|H_0G_{z,h}(\varepsilon)Q_h(\varepsilon)\| + \frac{b}{1-a} \|G_{z,h}(\varepsilon)Q_h(\varepsilon)\|
\]

\[
\leq c \left(1 + \frac{\|F_{z,h}(\varepsilon)\|^2}{\sqrt{\alpha_h\sqrt{\varepsilon}}}\right)
\]

(2.19)

while (2.13) and (2.16) give:

\[
\bigg\|\sqrt{V_h}PG_{z,h}(\varepsilon)Q_h(\varepsilon)\bigg\| \leq c \left(1 + \|F_{z,h}(\varepsilon)\|^2\right)
\]

(2.20)

3. We now estimate the derivative of \(F_{z,h}\) with report to \(\varepsilon\):

\[
\frac{d}{d\varepsilon}F_{z,h}(\varepsilon) = iC_VQ_h(\varepsilon)G_{z,h}(\varepsilon)P_hV_hP_hG_{z,h}(\varepsilon)Q_h(\varepsilon)
\]

\[
= iQ_h(\varepsilon)G_{z,h}(\varepsilon)P_h\Theta_hP_hG_{z,h}(\varepsilon)Q_h(\varepsilon)
\]

\[
+ \frac{dQ_h(\varepsilon)}{d\varepsilon}G_{z,h}(\varepsilon)Q_h(\varepsilon) + Q_h(\varepsilon)G_{z,h}(\varepsilon)\frac{dQ_h(\varepsilon)}{d\varepsilon}
\]

Functional calculus gives:

\[
\left\|\frac{dQ_h(\varepsilon)}{d\varepsilon}\right\| \leq c\varepsilon^{s-1}
\]

so the last two terms can be estimated by:

\[
\left\|\frac{dQ_h(\varepsilon)}{d\varepsilon}G_{z,h}(\varepsilon)Q_h(\varepsilon) + Q_h(\varepsilon)G_{z,h}(\varepsilon)\frac{dQ_h(\varepsilon)}{d\varepsilon}\right\| \leq c\varepsilon^{s-1} \left(1 + \frac{\|F_{z,h}(\varepsilon)\|^2}{\sqrt{\alpha_h\sqrt{\varepsilon}}}\right)
\]

By (2.21) we have:

\[
\|Q_h(\varepsilon)G_{z,h}(\varepsilon)P_hV_hP_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\| \leq c(1 + \|F_{z,h}(\varepsilon)\|)
\]

and for the second term we replace \(P_h\Theta_hP_h\) by \(\Theta_h - P_h\Theta_hP_h' - P_h'\Theta_hP_h'\), which gives:

\[
iQ_h(\varepsilon)G_{z,h}(\varepsilon)P_h\Theta_hP_hG_{z,h}(\varepsilon)Q_h(\varepsilon) = D_1 + D_2 + D_3 + D_4
\]
with:
\[
\| D_2 \| = \left\| Q_h(\varepsilon) G_{z,h}(\varepsilon) P_h \Theta_h P'_h G_{z,h}(\varepsilon) Q_h(\varepsilon) \right\|
\leq \left\| Q_h(\varepsilon) G_{z,h}(\varepsilon) \right\| \left\| P_h \Theta_h \right\| \left\| P'_h G_{z,h}(\varepsilon) Q_h(\varepsilon) \right\|
\leq c \left( 1 + \frac{1}{\sqrt{\alpha_h \varepsilon}} \left\| F_{z,h}(\varepsilon) \right\| \right)^{\frac{3}{2}} \times c \sqrt{\alpha_h} \times \left( 1 + \left\| F_{z,h}(\varepsilon) \right\| \right)^{\frac{1}{2}}
\leq c \left( 1 + \frac{1}{\sqrt{\varepsilon}} \left\| F_{z,h}(\varepsilon) \right\| \right)
\]

\( D_3 \) is estimated similarly, while we use (2.19) for \( D_4 \):
\[
\| D_4 \| = \left\| Q_h(\varepsilon) G_{z,h}(\varepsilon) P'_h \Theta_h P'_h G_{z,h}(\varepsilon) Q_h(\varepsilon) \right\|
\leq \left\| Q_h(\varepsilon) G_{z,h}(\varepsilon) \right\| \left\| \Theta_h(H^0_0 - i)^{-1} P'_h \right\| \left\| (H^0_0 + i) G_{z,h}(\varepsilon) Q_h(\varepsilon) \right\|
\leq c \left( 1 + \frac{1}{\sqrt{\varepsilon}} \right) \left\| F_{z,h}(\varepsilon) \right\|
\]

To estimate \( D_1 \), we are going to use the choice of \( \Theta_{R,h} \) and \( \Theta_{I,h} \) as commutators with \( H_h \). By proposition II.6 in [Mou81], \( G_{z,h}(\varepsilon) \) maps \( D_A \) into \( D_H \cap D_A \), so we can compute, in the sense of quadratic forms on \( D_H \cap D_A \):
\[
Q_h(\varepsilon) G_{z,h}(\varepsilon) \Theta_h G_{z,h}(\varepsilon) Q_h(\varepsilon)
= iQ_h(\varepsilon) G_{z,h}(\varepsilon) [H_h, A_h] G_{z,h}(\varepsilon) Q_h(\varepsilon)
= iQ_h(\varepsilon) G_{z,h}(\varepsilon) [H_h - z - i \varepsilon P_h \Theta_h^V P_h, A_h] G_{z,h}(\varepsilon) Q_h(\varepsilon)
- \varepsilon Q_h(\varepsilon) G_{z,h}(\varepsilon) [P_h \Theta_h^V P_h, A_h] G_{z,h}(\varepsilon) Q_h(\varepsilon)
= iQ_h(\varepsilon) [A_h, G_{z,h}(\varepsilon)] Q_h(\varepsilon)
- \varepsilon Q_h(\varepsilon) G_{z,h}(\varepsilon) [P_h \Theta_h^V P_h, A_h] G_{z,h}(\varepsilon) Q_h(\varepsilon)
\tag{2.21}
\]

For \( \varphi, \psi \in D_H \cap D_A \), we have:
\[
\langle G_{z,h}(\varepsilon) Q_h(\varepsilon) \varphi, A_h Q_h(\varepsilon) \psi \rangle \leq c \alpha_h^{-\frac{1}{2}} e^{s^{\frac{1}{2}}} \left\| F_{z,h}(\varepsilon) \right\| \varphi \| \psi \|
\]
according to (2.17) and (2.12).

By proposition II.6 in [Mou81], the quadratic form \( [P_h \Theta_h^V P_h, A_h] \) has the properties of \( [\Theta_h^V, A_h] \) given by assumption (d). With (2.19) this proves:
\[
\varepsilon \left| \langle [P_h \Theta_h^V P_h, A_h] G_{z,h}(\varepsilon) Q_h(\varepsilon) \varphi, G_{z,h}(\varepsilon)^* Q_h(\varepsilon) \psi \rangle \right|
\leq c \alpha_h \varepsilon \left\| G_{z,h}(\varepsilon) Q_h(\varepsilon) \varphi \right\|_{\Gamma_h} \left\| G_{z,h}(\varepsilon) Q_h(\varepsilon) \psi \right\|_{\Gamma_h}
\leq c (1 + \left\| F_{z,h}(\varepsilon) \right\|) \left\| \varphi \right\| \left\| \psi \right\|
\]

So both terms in (2.21) extend to bounded operators and:
\[
\| D_1 \| \leq c \alpha_h^{-\frac{1}{2}} e^{s^{\frac{1}{2}}} \left( 1 + \left\| F_{z,h}(\varepsilon) \right\| \right) + c (1 + \left\| F_{z,h}(\varepsilon) \right\|)
\]
and hence we have proved:
\[
\left\| \frac{d}{d\varepsilon} F_{z,h}(\varepsilon) \right\| \leq c + \frac{c}{\sqrt{\varepsilon}} \left\| F_{z,h}(\varepsilon) \right\| + \frac{c e^{s^{\frac{1}{2}}}}{\sqrt{\alpha_h}} \left\| F_{z,h}(\varepsilon) \right\|^{\frac{1}{2}}
\]
which can also be written:
\[
\left\| \frac{d}{d\varepsilon} \alpha_h F_{z,h}(\varepsilon) \right\| \leq c + \frac{c}{\sqrt{\varepsilon}} \left\| \alpha_h F_{z,h}(\varepsilon) \right\| + c e^{s^{\frac{1}{2}}} \left\| \alpha_h F_{z,h}(\varepsilon) \right\|^{\frac{1}{2}}
\tag{2.22}
\]

4. Using lemma 3.3 in [MPS84] with (2.18) and (2.22), we get that \( F_{z,h}(\varepsilon) \) can be continuously continued for \( \varepsilon = 0 \). Furthermore, the constants in this lemma do not depend on the function but only on the estimates. Since \( (\alpha_h F_{z,h}(\varepsilon)) \) and its derivative with report to \( \varepsilon \) are estimated uniformly in \( h \), we can conclude that \( \alpha_h F_{z,h}(0) \) is uniformly
We also write $\text{Op}$

But this assumption is useless if we can take $C$ Hölder-continuous with respect to $E$.

Let $V_{\nu}$ for some $\rho > (up to any order) and:

$\nu$ had to put the limit $\mu$ for $\pi h$

Since $5$. Application to the dissipative Helmholtz equation

$\text{Jecko}$ for pointing out that this can be avoided.

Moreover with (2.17) we get:

and gives:

$\nu = \min(1, \nu)$

Remark 2.5. We added the uniform estimate on $[V_h, A_h]$ in assumptions (d) because we had to put $V_h$ in the $\varepsilon$-term of $G_{z,h}(\varepsilon)$ in order to use the weak Mourre estimate (2.7).

But this assumption is useless if we can take $\mathcal{C}_V = 0$ in (2.4).\

3. Application to the dissipative Helmholtz equation

In this section we apply the abstract Mourre theory to the dissipative Schrödinger operator. Let $V_1 \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with:

\[ \forall \alpha \in \mathbb{N}^n, \forall x \in \mathbb{R}^n, \quad |\partial^\alpha V_1(x)| \leq C_\alpha \langle x \rangle^{-\rho - |\alpha|} \quad (3.1) \]

for some $\rho > 0$ and $C_\alpha > 0$. Let $V_2 \in C^\infty(\mathbb{R}^n, \mathbb{R})$ nonnegative, with bounded derivatives (up to any order) and:

\[ V_2(x) \rightarrow 0 \quad (3.2) \]

We consider on $L^2(\mathbb{R}^n)$ the operator:

\[ H_h = -h^2 \Delta + V_1 - i\nu(h)V_2 \]

where $\nu(h) \in [0, 1]$. We denote by $H_1^h = -h^2 \Delta + V_1(x)$ the selfadjoint part of $H_h$, $\bar{\nu}(h) = \min(1, \nu(h)/h)$ and:

\[ \mathcal{O} = \{(x, \xi) \in \mathbb{R}^{2n} : V_2(x) > 0\} \]

We also write $\text{Op}_h^w(a)$ for the Weyl-quantization of a symbol $a$ (see [Rob87], [Mar02], [EZ]):

\[ \text{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\frac{x-y}{h}\xi} a \left( \frac{x+y}{2}, \xi \right) u(y) dy d\xi \]

Initially, estimate $\|V_h(H_0^h + i)^{-1}\| = O(\sqrt{\mu})$ was also required in assumption (c). I thank Th. Jecko for pointing out that this can be avoided.
3.1. Hamiltonian flow. Let \( p : (x, \xi) \mapsto \xi^2 + V_1(x) \) be the symbol of \( H^1 \), and \( (x_0, \xi_0) \mapsto \phi^t(x_0, \xi_0) = (\mathcal{F}(t, x_0, \xi_0), \mathcal{F}(t, x_0, \xi_0)) \in \mathbb{R}^{2n} \) the corresponding Hamiltonian flow:

\[
\begin{align*}
\partial_t \mathcal{F}(t, x_0, \xi_0) &= 2\mathcal{F}(t, x_0, \xi_0) \\
\partial_t \mathcal{F}(t, x_0, \xi_0) &= -\nabla V_1(\mathcal{F}(t, x_0, \xi_0)) \\
\phi^0(x_0, \xi_0) &= (x_0, \xi_0)
\end{align*}
\]

For \( I \subset \mathbb{R} \) we introduce:

\[
\begin{align*}
\Omega_b(I) &= \{ w \in p^{-1}(I) : (\mathcal{F}(t, w))_{t \in \mathbb{R}} \text{ is bounded} \} \\
\Omega^\infty_b(I) &= \{ w \in p^{-1}(I) : |(\mathcal{F}(t, w)|_{t \to \pm \infty} \to \infty \}
\end{align*}
\]

We recall a few basic facts about this flow:

**Proposition 3.1.** (i) For \( a \in C^\infty(\mathbb{R}^{2n}) \) we have \( \partial_t (a \circ \phi^t) = \{ p, a \circ \phi^t \} \) where \( \{ \cdot, \cdot \} \) is the Poisson bracket.

(ii) If \( I \subset \mathbb{R}^+ \) is closed, there exists \( R_0(I) > 0 \) such that for any \( R \geq R_0(I) \), a trajectory of energy in \( I \) which leaves \( B_2(R) \) (in the future or in the past) cannot come back.

(iii) If \( I \subset \mathbb{R}^+ \), \( p^{-1}(I) = \Omega_b(I) \cup \Omega^\infty_b(I) \cup \Omega^\infty_b(I).

(iv) If \( I \subset \mathbb{R}^+ \) is open, then \( \Omega_b(I) \) is compact in \( \mathbb{R}^{2n} \).

(v) If \( I \subset \mathbb{R}^+ \) is open, then \( \Omega^\infty_b(I) \) and \( \Omega^\infty_b(I) \) are open.

3.2. Limiting absorption principle for the dissipative Schrödinger operator.

**Theorem 3.2.** Let \( E > 0 \) and \( s \geq \frac{1}{2} \). If all bounded trajectories of energy \( E \) meet \( \mathcal{O} \), then there exists \( c \geq 0, h_0 > 0 \) and \( I = [E - \delta, E + \delta], \delta > 0 \), such that:

(i) For all \( h \in ]0, h_0[ \):

\[
\sup_{z \in C_I^+} \left\| \langle x \rangle^{-s} (H_h - \gamma)^{-1} \langle x \rangle^{-s} \right\| \leq \frac{c}{h \nu_h(h)} \tag{3.3}
\]

(ii) For all \( h \in [0, h_0] \) and \( z, z' \in C_{I^+} \):

\[
\left\| \langle x \rangle^{-s} \left( (H_h - \gamma)^{-1} - (H_h - \gamma')^{-1} \right) \langle x \rangle^{-s} \right\| \leq c (h \nu_h(h))^{-\frac{1}{2s+1}} |z - z'|^{\frac{2s+1}{2s+1}} \tag{3.4}
\]

(iii) For \( \lambda \in I \) and \( h \in [0, h_0] \) the limit:

\[
\langle x \rangle^{-s} (H_h - \lambda)^{-1} \langle x \rangle^{-s} = \lim_{\mu \to 0^+} \langle x \rangle^{-s} (H_h - \lambda + \mu i)^{-1} \langle x \rangle^{-s} \tag{3.5}
\]

exists in \( L(L^2(\mathbb{R}^n)) \) and is a \( \frac{2s+1}{2s+1} \)-Hölder continuous function of \( \lambda \).

**Remark 3.3.** This condition that a damping perturbation of the Schrödinger operator allows to weaken a non-trapping condition already appears in [AK07] where dispersive estimates are obtained for the Schrödinger operator on an exterior domain.

**Remark 3.4.** We are mainly interested in the cases \( \nu(h) = h (\nu(h) = 1) \), as mentioned in the introduction, and \( \nu(h) = h^2 (\nu(h) = h) \) which appears in the study of the high energy limit for the Schrödinger operator \( -\Delta - iV_2 - z, \Re z \geq 1 \) (see [AK07, §1.2]).

**Remark 3.5.** If \( E \) is a non-trapping energy, we have the usual estimate in \( O(h^{-1}) \), no matter how small the anti-adjoint part is.

The proof of theorem 3.2 follows that of the selfadjoint case given in [GM88]: we construct a conjugate family of operators using the quantization of an escape function and then we check that this operators can be replaced by \( \langle x \rangle \) in the results of theorem 2.3. The only difference is that we need to prove a weaker Mourre estimate so we are allowed to consider a function which is not an escape function where \( V_2 \) is not zero. Let us denote:

\[
A_h = \frac{1}{2} (x.hD + hD.x) \tag{3.6}
\]

the generator of dilations.
Proposition 3.6. For any \( r \in C^\infty_0(\mathbb{R}^{2n}, \mathbb{R}) \) the operators \( \tilde{v}(h)F_h = \tilde{v}(h)(A_h + \text{Op}_h^w(r)) \) are selfadjoint and satisfy assumptions (a) to (d) for a conjugate operator to \( H_h \).

The proof of this proposition is not really changed by the imaginary part of \( V \), so we omit it. The important assumption is the Mourre estimate (e), for which we need to choose \( r \) more carefully:

Proposition 3.7. Assume that every bounded trajectory of energy \( E \) goes through \( \mathcal{O} \), then there exists \( \varepsilon > 0 \) and \( r \in C^\infty_0(\mathbb{R}^{2n}, \mathbb{R}) \) such that \( \tilde{v}(h)F_h = \tilde{v}(h)(A_h + \text{Op}_h^w(r)) \) is conjugate to \( H_h \) on \( J = [E - \varepsilon, E + \varepsilon] \) with bounds \( c_0 h \tilde{v}(h) \), where \( c_0 > 0 \).

Proof. 1. We first remark that the assumption on bounded trajectories can be extended to a neighborhood of \( E \) : there exists \( \varepsilon \in [0, E/12] \) such that any bounded trajectory of energy in \( [E - 3\varepsilon, E + 3\varepsilon] \) meets \( \mathcal{O} \). Indeed, assume that for any \( n \in \mathbb{N} \) we can find \( w_n \in \Omega_b([E/2, 2E]) \) such that \( p(w_n) \to E \) and \( w_n \notin \mathcal{O} \). Let

\[
\mathcal{O}^\phi = \bigcup_{t \in \mathbb{R}} \phi^{-t}(\mathcal{O})
\]

Maybe after extracting a subsequence we can assume that \( w_n \to w \in \Omega_b([E/2, 2E]) \). As \( p \) is continuous, we have \( p(w) = E \), and hence \( w \in \mathcal{O}^\phi \) which is open. This gives a contradiction. We set \( J = [E - \varepsilon, E + \varepsilon] \), \( J_2 = [E - 2\varepsilon, E + 2\varepsilon] \) and \( J_3 = [E - 3\varepsilon, E + 3\varepsilon] \).

2. Let \( R \geq R_0(\mathcal{J}_3) \) (given in proposition 3.4) so large that \( \Omega_b(\mathcal{J}_3) \subset B_x(R) \), where \( B_x(R) = \{(x, \xi) \in \mathbb{R}^{2n} : |x| < R \} \), and:

\[
|2V_1(x) + x \cdot \nabla V_1(x)| \leq \frac{E}{2} \quad \text{when} \quad |x| \geq R
\]

Let \( b \in C^\infty(\mathbb{R}^n) \) equal to \( x, \xi \) outside \( B_x(R + 1) \) and zero in a neighborhood of \( \mathcal{B}_x(R) \). Then, if \( p(x, \xi) \in J_3 \) and \( |x| \geq R + 1 \) we have:

\[
\{p, b\}(x, \xi) = 2p(x, \xi) - 2V_1(x) - x \cdot \nabla V_1(x) \geq E
\]

and \( \{p, b\} = 0 \) in \( B_x(R) \).

3. Let \( w \in \Omega_b(\mathcal{J}_3) \) and \( T_w \in \mathbb{R} \) such that \( \phi^{T_w}(w) \in \mathcal{O} \). As \( \phi^{T_w} \) is continuous, we can find \( \gamma_w > 0 \) and an open neighborhood \( \mathcal{V}_w \) of \( w \) in \( \mathbb{R}^{2n} \) such that for any \( z \in \mathcal{V}_w \) we have \( \phi^{T_w}(z) \in \mathcal{O}_w \) where \( \mathcal{O}_w \) stands for \( \{(x, \xi) \in \mathbb{R}^{2n} : V_2(x) > \gamma \} \). Let \( \mathcal{U}_w \) be another neighborhood of \( w \) with \( \mathcal{U}_w \subset \mathcal{V}_w \), \( g_w \in C^\infty_0(\mathbb{R}^{2n}, [0, 1]) \) be supported in \( \mathcal{V}_w \) and equal to 1 on \( \mathcal{U}_w \), and \( f \in C^\infty(\mathbb{R}^{2n}) \) defined for \( z \in \mathbb{R}^{2n} \) by:

\[
f_w(z) = \int_0^{T_w} g_w(\phi^{-t}(z)) \, dt
\]

The first term is supported in \( \mathcal{V}_w \), nonnegative and equal to 1 on \( \mathcal{U}_w \), while the support of the second is in \( \phi^{T_w}(\mathcal{V}_w) \subset \mathcal{O}_w \). In particular \( \{p, f_w\} \) is compactly supported, nonnegative outside \( \mathcal{O}_w \), and equal to 1 in \( \mathcal{U}_w \). As \( \Omega_b(\mathcal{J}_3) \) is compact, we can find \( w_1, \ldots, w_N \in \Omega_b(\mathcal{J}_3) \) for some \( N \in \mathbb{N} \) such that \( \Omega_b(\mathcal{J}_3) \subset \mathcal{U} := \bigcup_{j=1}^N \mathcal{U}_{w_j} \). Let \( \gamma = \min_{1 \leq j \leq N} \gamma_{w_j} \), and \( f = \sum_{j=1}^N f_{w_j} \). Then \( \{p, f\} \) is compactly supported, nonnegative outside \( \mathcal{O}_\gamma \) and greater than or equal to 1 in \( \mathcal{U} \).
4. We can find a constant $C_V \geq 0$ such that $\{p, f\} + C_V V_2 \geq 1$ on $\mathcal{O}_\gamma$, so that $\{p, f\} + C_V V_2$ is nonnegative on $\mathbb{R}^{2n}$ and at least 1 on $\mathcal{U}$.

5. Let:

$$\mathcal{U}_\pm = \Omega_\pm^\pm(J_3) \cap B_x(R + 2)$$

We have:

$$\mathcal{U}_+ \cup \mathcal{U}_- \cup \mathcal{U} \cup p^{-1}(\mathbb{R} \setminus \mathcal{J}_2) \cup (\mathbb{R}^{2n} \setminus \overline{B}_x(R + 1)) = \mathbb{R}^{2n}$$

Considering a partition of unity for this open cover of $\mathbb{R}^{2n}$ provides two functions $g_\pm \in C_0^\infty(\mathbb{R}^{2n}, [0, 1])$ supported in $\mathcal{U}_\pm$ such that $g_\infty = g_+ + g_-$ is equal to 1 in a neighborhood of the compact set:

$$K_\infty = p^{-1}(\mathcal{J}_2) \cap \overline{B}_x(R + 1) \setminus \mathcal{U}$$

There exists $T \geq 0$ such that for any $w \in \mathbb{R}^{2n}$ we can find a neighborhood $\mathcal{V}$ of $w$ and $\tau_\pm \geq 0$ such that for any $v \in \mathcal{V}$ and $t \geq 0$ we have:

$$0 \leq g_\pm(\phi^{\pm t}(v)) \leq 1_{[\tau_\pm, T + \tau_\pm]}(t)$$

As a consequence the functions:

$$f_\pm = \mp \int_0^{+\infty} (g_\pm \circ \phi^{\pm t}) \, dt$$

are well-defined, bounded (by $T$) and $C^\infty$ on $\mathbb{R}^{2n}$. The same calculation as for $f$ shows that $\{p, f_\pm\} = g_\pm \geq 0$. Hence we can find a constant $C_{\infty} \geq 0$ such that for $f_\infty = f_+ + f_-$ we have:

$$\{p, b + C_{\infty} f_\infty\} \geq E \quad \text{on } K_\infty$$

and we already know that $\{p, b + C_{\infty} f_\infty\} \geq \{p, b\}$ is nonnegative on $p^{-1}(\mathcal{J}_2) \setminus K_\infty$.

6. Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $B(R + 2)$. Since we can replace $\zeta$ by $x \mapsto (\mu x)$ with $\mu$ small enough, we can assume that:

$$|C_\infty f\{p, \zeta\}|_{L^\infty(p^{-1}(\mathcal{J}_2))} \leq 2C_{\infty} T \sup_{p^{-1}(\mathcal{J}_2)} |\zeta, \nabla \zeta(x)| \leq \frac{E}{2}$$

With (3.7) and (3.8) this gives:

$$\{p, b + \zeta f_\infty\} \geq \frac{E}{2} \quad \text{on } p^{-1}(\mathcal{J}_2) \setminus \mathcal{U}$$

and $\{p, b + \zeta f_\infty\}$ is still nonnegative on $p^{-1}(\mathcal{J}_2)$ since $\nabla \zeta = 0$ on $\mathcal{U}$. Taking $r(x, \xi) = x, \xi - b(x, \xi) + C_{\infty} f_\infty + f$ then $r \in C_0^\infty(\mathbb{R}^{2n})$ and:

$$\{p, x, \xi + r\} + C_V V_2 \geq 2c_0 \quad \text{on } p^{-1}(\mathcal{J}_2) \text{ with } 2c_0 = \min \left(1, \frac{E}{2}\right)$$

7. Let $F_h = A_h + \text{Op}_h^x(r) = \text{Op}_h^x(x, \xi + r)$. The principal symbol of the operator $ih^{-1}[H_{1, h}, F_h]$ is $\{p, x, \xi + r\}$. Let $\chi \in C_0^\infty(\mathbb{R})$ supported in $\mathcal{J}_2$ and equal to 1 on $J$. By [Rob87] or [HRS3] the operator $\chi(H_1^h)$ is a pseudo-differential operator of principal symbol $\chi \circ p$. As a consequence the principal symbol of the operator:

$$i_h \chi(H_1^h)[H_1^h, F_h] \chi(H_1^h) + C_V V_2 - 2c_0 \chi(H_1^h)^2$$

is nonnegative, so by Gårding inequality (see theorem 4.27 in [EZ]) there is $C \geq 0$ such that, after multiplication by $\tilde{\nu}(h)$:

$$\chi(H_1^h)i \left[H_1^h, \tilde{\nu}(h)F_h\right] \chi(H_1^h) + h\tilde{\nu}(h)C_V V_2 \geq 2h\tilde{\nu}(h)c_0 \chi(H_1^h)^2 - C h^2 \tilde{\nu}(h)$$

Taking $h$ small enough and multiplying by $1_J(H_1^h)$ on both sides give:

$$1_J(H_1^h) \left(i \left[H_1^h, \tilde{\nu}(h)F_h\right] + h\tilde{\nu}(h)C_V V_2\right) 1_J(H_1^h) \geq h\tilde{\nu}(h)c_0 1_J(H_1^h)^2$$
Then \((\nu(h) - h\tilde{\nu}(h))1_J(H_1^h)C_VV_21_J(H_1^h) \geq 0\) so we have:

\[
1_J(H_1^h) \left( i \left[ H_1^h, \tilde{\nu}(h)F_h \right] + C_V\nu(h)V_2 \right) 1_J(H_1^h) \geq h\tilde{\nu}(h)c_0 1_J(H_1^h)^2
\]

which is the Mourre estimate we need. Note that if \(E\) is non-trapping we can take \(f = 0\), \(C_V = 0\), and use the estimate:

\[
1_J(H_1^h)1[H_1^h,F_h]1_J(H_1^h) \geq h\tilde{\nu}(h)c_0 1_J(H_1^h)^2
\]
even if \(h \geq \nu(h)\), which justifies remark \([3.3]\). \(\square\)

This proposition shows that for any closed subinterval \(I\) of \(J\) theorem \([3.2]\) is true with \(\langle \tilde{\nu}(h)F_h \rangle^{-s}\) instead of \(\langle x \rangle^{-s}\). The operator \(\langle \tilde{\nu}(h)F_h \rangle^s \langle \tilde{\nu}(h)A_h \rangle^{-s}\) is bounded uniformly in \(h\) (this is true for \(s = 0\) and \(s = 1\) hence for any \(s \in [0,1]\) by complex interpolation), so conclusions of theorem \([3.2]\) are valid with \(\langle \tilde{\nu}(h)A_h \rangle^{-s}\). Now write:

\[
(H_h - z)^{-1} = (H_h - i)^{-1} - (z - i)(H_h - i)^{-2} + (z - i)^2(H_h - i)^{-1}(H_h - z)^{-1}(H_h - i)^{-1}
\]

Since \(\langle \tilde{\nu}(h)A_h \rangle^s (H_h - i)^{-1} \langle x \rangle^{-s}\) is uniformly bounded (see \([PSS81]\) lemma 8.2)), this gives:

\[
\begin{align*}
\| \langle x \rangle^{-s} (H_h - z)^{-1} \langle x \rangle^{-s} \| & \leq c + c \| \langle x \rangle^{-s} (H_h - i)^{-1} (H_h - z)^{-1} (H_h - i)^{-1} \langle x \rangle^{-s} \| \\
& \leq c + c \| \langle x \rangle^{-s} (H_h - i)^{-1} (\tilde{\nu}(h)A_h)^s \| \| \langle \tilde{\nu}(h)A_h \rangle^{-s} (H_h - z)^{-1} (\tilde{\nu}(h)A_h)^{-s} \langle x \rangle^{-s} \| \\
& \leq \frac{c}{h\tilde{\nu}(h)}
\end{align*}
\]

where \(c\) does not depend on \(z \in \mathbb{C}_{I,+}\) with \(\text{Im } z \leq 1\). This is \([3.3]\). Then \([3.4]\) and hence existence of the limit \([3.3]\) follow similarly.

4. Necessity of the condition on trapped trajectories

We consider in this section the operator \(H_h = -h^2\Delta + V_1 - ihV_2\) we introduced to study the Helmholtz equation. We prove that our assumption that every bounded trajectory of energy \(E\) should meet the open set \(\mathcal{O}\) is actually necessary in order to have the uniform estimates and the limiting absorption principle as in theorem \([3.2]\). When \(V_2 = 0\), this is proved in \([Wan87]\).

**Theorem 4.1.** Assume that for some \(s \in \left[ \frac{1}{2}, \frac{1+s}{2} \right]\) \((\rho > 0\) given by \([3.1]\)), there exists \(\varepsilon, h_0 > 0\) such that the limit:

\[
\langle x \rangle^{-s} (H_h - (\lambda + i0))^{-1} \langle x \rangle^{-s}
\]

exists for all \(\lambda \in J = \{E - \varepsilon, E + \varepsilon[ \) and \(h \in [0, h_0]\) with the estimates:

\[
\| \langle x \rangle^{-s} (H_h - z)^{-1} \langle x \rangle^{-s} \| \leq \frac{C}{h}
\]

uniformly in \(z \in \mathbb{C}_{I,+}\) and \(h \in [0, h_0]\), then every bounded trajectory of energy \(E\) goes through \(\mathcal{O}\).

To prove this theorem we use the contraction semigroup generated by \(H_h\) (given by Hille-Yosida theorem, see for instance theorem 3.5 in \([EN06]\)):

\[
U_h(t) = e^{-\frac{t}{h}H_h}, \quad t \geq 0
\]

We first need a dissipative version of the Egorov theorem. Let \(q \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^{2n})\) be defined by:

\[
q(t, w) = \exp \left( -2 \int_0^t V_2(\phi^s(w)) \, ds \right)
\]

(where \(V_2(x, \xi)\) means \(V_2(x)\) for \((x, \xi) \in \mathbb{R}^{2n}\)).
Theorem 4.2. Let $a \in C^\infty(\mathbb{R}^{2n})$ be a symbol whose derivatives are bounded (in $L^\infty(\mathbb{R}^n)$). Then for all $t \geq 0$ we have:

$$U_h(t)^* \text{Op}_h^w(a)U_h(t) = \text{Op}_h^w((a \circ \phi^s)q(t)) + hR(t, h) \tag{4.1}$$

where $R$ is bounded in $\mathcal{L}(L^2(\mathbb{R}^n))$ uniformly in $h \in [0, 1]$ and $t$ in a compact subset of $\mathbb{R}_+$.

Remark 4.3. More precisely, we prove that there exists a family $(b(\tau, h))_{\tau \geq 0}$ of classical symbols with bounded derivatives (uniformly for $\tau$ in a compact subset of $\mathbb{R}_+$) such that for all $t \geq 0$:

$$R(t, h) = \int_0^t U_h(\tau)^* \text{Op}_h^w(b(\tau, h))U_h(\tau) \, d\tau \tag{4.2}$$

Remark 4.4. If we replace one of the $U_h(t)$ by $U_1^h(t) = e^{-\frac{it}{h}H_0^h}$ in the left-hand side of (4.1) then we have to replace $q$ by

$$q_1 : (x, \xi) \mapsto \exp \left(-\int_0^t V_2(\phi^s(w)) \, ds \right) \tag{4.3}$$

in the right-hand side (with the two occurrences of $U_h(t)$ replaced by $U_1^h(t)$ and $q$ replaced by 1, theorem 4.2 is just the usual Egorov theorem).

Proof. We follow the proof of the usual Egorov theorem (see for instance [Rob87, § IV.4]). Let $t \geq 0$. For $\tau \in [0, t]$ and $w \in \mathbb{R}^{2n}$ write:

$$\tilde{a}(\tau, w) = a(\phi^{t-\tau}(w)) \exp(S(\tau, w)) \quad \text{where} \quad S(\tau, w) = -2\int_\tau^t V_2(\phi^{s-\tau}(w)) \, ds$$

and:

$$B_h(\tau) = U_h(\tau)^* \text{Op}_h^w(\tilde{a}(\tau))U_h(\tau)$$

so that the estimate we have to prove is: $B_h(t) - B_h(\tau) = O(h)$ in $\mathcal{L}(L^2(\mathbb{R}^n))$. We have:

$$\partial_\tau \tilde{a}(\tau) = -\{p, a \circ \phi^{t-\tau}\} \exp(S(\tau)) + 2 \left( V_2 + \int_\tau^t \{p, V_2 \circ \phi^{s-\tau}\} \, ds \right) \tilde{a}(\tau)$$

$$= -\{p, a \circ \phi^{t-\tau}\} \exp(S(\tau)) + 2V_2\tilde{a}(\tau) - \{p, S(\tau)\} \tilde{a}(\tau)$$

$$= 2V_2\tilde{a}(\tau) - \{p, \tilde{a}(\tau)\}$$

The function $\tau \mapsto B_h(\tau)$ is of class $C^1$ in the weak sense and:

$$B'_h(\tau) = U_h(\tau)^* \tilde{B}_h(\tau)U_h(\tau)$$

with:

$$\tilde{B}_h(\tau) = \frac{i}{h} [H^h_1, \text{Op}_h^w(\tilde{a}(\tau))] - V_2 \text{Op}_h^w(\tilde{a}(\tau)) - \text{Op}_h^w(\tilde{a}(\tau))V_2 + \text{Op}_h^w(\partial_\tau \tilde{a}(\tau))$$

$$= \text{Op}_h^w(c(\tau, h))$$

for some classical symbol $c(\tau, h) = \sum_{j \in \mathbb{N}} h^j c_j(\tau)$, and in particular:

$$c_0(\tau) = \{p, \tilde{a}(\tau)\} - V_2\tilde{a}(\tau) - \tilde{a}(\tau)V_2 + \partial_\tau \tilde{a}(\tau) = 0$$

Setting $b = h^{-1}c$ we get (4.1)-(4.2), in the weak sense and hence in $\mathcal{L}(L^2(\mathbb{R}^n))$. □

Proposition 4.5. Assume that the assumptions of theorem 4.4 are satisfied. Then for any $\chi \in C_0^\infty(\mathbb{R})$ supported in $J$ there exists $c \geq 0$ such that for all $h \in [0, h_0]$ and $z \in \mathbb{C}_+$ we have:

$$\left\| \langle x \rangle^{-s} \chi(H^h_1)(H^h - z)^{-1} \chi(H^h_1) \langle x \rangle^{-s} \right\| \leq \frac{c}{h} \tag{4.4}$$

Remark 4.6. We have similar estimates for $(H^h - z)^{-1}$. 

Proof. First, we can find $c \geq 0$ such that estimate (4.4) holds for $z \in \mathbb{C}_{J,+}$ by assumption and uniform boundedness of $\langle x \rangle^{-s} \chi(H^h_0) (x) \langle x \rangle^s$ with respect to $h$ (note that this last statement holds for $s = 0$ by functional calculus and for $s = 1$, we use the fact that $\chi(H^h_0)$ is a pseudo-differential operator whose symbol has bounded derivatives and $[x, \text{Op}^w_h(b)] = -ih \text{Op}^w_h(\partial_x b)$; then the claim follows for any $s \in [0, 1]$ by complex interpolation).

Then, there exists $\delta > 0$ such that for all $z \in \mathbb{C}_{\mathbb{R} \setminus J,+}$ we have $d(z, \text{supp} \chi) \geq \delta$. As a consequence, the operator $\chi(H^h_0)(H^h_0 - z)^{-1}$ is bounded uniformly in $z \in \mathbb{C}_{\mathbb{R} \setminus J,+}$ and $h \in [0, h_0]$. Hence, using twice the resolvent equation, we can write:

$$\left\| \chi(H^h_0)(H^h_0 - z)^{-1} \chi(H^h_0) \right\| \leq c + h^2 \left\| \chi(H^h_0)(H^h_0 - z)^{-1} V_2(H^h_0 - z)^{-1} V_2(H^h_0 - z)^{-1} \chi(H^h_0) \right\| \leq c \left( 1 + h \left\| \sqrt{h} V_2(H^h_0 - z)^{-1} \sqrt{h} V_2 \right\| \right) \leq c$$

where the last step is given by proposition 2.2 applied with $T = H_h = H^h_0 - ihV_2$ and $B = Q = \sqrt{h} V_2$.

Proposition 4.7. Assume that the assumptions of theorem 4.1 are satisfied. Then for any $\chi \in C_0^\infty(\mathbb{R})$ supported in $J$ there exists $C_\chi \geq 0$ such that for all $\psi \in L^2(\mathbb{R}^n)$ and $h \in [0, h_0]$ we have:

$$\int_0^{+\infty} \left\| \langle x \rangle^{-s} \chi(H^h_0) U_h(t) \psi \right\|^2 \, dt \leq C_\chi \|\psi\|^2 \tag{4.5}$$

Proof. Let $K_h$ be the selfadjoint dilation of $H_h$ on the Hilbert space $\mathcal{K} \supset L^2(\mathbb{R}^n)$ given in appendix A. Let $P$ be the orthogonal projection of $\mathcal{K}$ on $L^2(\mathbb{R}^n)$ and $A_h = \langle x \rangle^{-s} \chi(H^h_0) \in \mathcal{L}(\mathcal{K})$, where operators on $L^2(\mathbb{R}^n)$ are extended by $0$ on $L^2(\mathbb{R}^n)^\perp \subset \mathcal{K}$. Let $\varphi = (\varphi_0, \varphi_\perp) \in \mathcal{K} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)^\perp$. For $z \in \mathbb{C}_+$ we have:

$$\left| \langle A_h \varphi, ((K_h - z)^{-1} - (K_h - \mathbf{1})^{-1}) A_h \varphi \rangle_{\mathcal{K}} \right| = \left| \left\langle \varphi_0, \langle x \rangle^{-s} \chi(H^h_0) \left( (H^h_0 - z)^{-1} - (H^h_0 - \mathbf{1})^{-1} \right) \chi(H^h_0) \langle x \rangle^{-s} \varphi_0 \right\rangle_{L^2(\mathbb{R}^n)} \right| \leq \frac{2c}{h} \|\varphi_0\|^2_{L^2(\mathbb{R}^n)} \leq \frac{2c}{h} \|\varphi\|^2_{\mathcal{K}}$$

where $c$ is given by proposition 1.5. The same applies if $\text{Im} z < 0$, so by theorem XIII.25 in [RS79], where $h$-dependence has to be treated for our semiclassical setting, this proves that $A_h$ is $K_h$-smooth and:

$$\sup_{h \in [0, h_0]} \sup_{\|\psi\| = 1} \int_{\mathbb{R}} \left\| A_h e^{-\frac{i}{h} K_h} \varphi \right\|^2_{\mathcal{L}(\mathcal{K})} \, dt < \infty \tag{4.6}$$

But for $\psi \in L^2(\mathbb{R}^n)$ (which we identify with $(\psi, 0) \in \mathcal{K}$), $h \in [0, h_0]$ and $t \geq 0$ we have:

$$\left\| \langle x \rangle^{-s} \chi(H^h_0) U_h(t) \psi \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \left\| \langle x \rangle^{-s} \chi(H^h_0) Pe^{-\frac{i}{h} K_h} P \psi \right\|_{\mathcal{L}(\mathcal{K})} = \left\| A_h e^{-\frac{i}{h} K_h} \psi \right\|_{\mathcal{L}(\mathcal{K})}$$

so (4.6) gives (4.5).

Proposition 4.8. Let $T \geq 0$ and $\chi \in C_0^\infty$ as in proposition 1.3. There exists $h_T > 0$ and $C_\chi' \geq 0$ such that for any $\psi \in L^2(\mathbb{R}^n)$ and $h \in [0, h_T]$ we have:

$$\int_0^T \left\| \langle x \rangle^{-s} \chi(H^h_0) U_h^h(t) Q_h(T) \psi \right\|^2 \, dt \leq C_\chi' \|\psi\|^2 \tag{4.7}$$

where $Q_h(T) = \text{Op}^w_h(q_1(T))$, $q_1$ being defined in (1.3).
Proof. According to Egorov theorem applied with the symbol \( a(x, \xi) = 1 \) we have:

\[
U_h^t(-t)U_h(t) = Q_h(t) + hR(t, h)
\]

where \( R \) is bounded in \( L(L^2(\mathbb{R}^n)) \) uniformly for \( h \in [0, 1] \) and \( t \in [0, T] \). On the other hand, writing \( Q_h(t, T) = \text{Op}_h^\ast(q_1(t, T)) \) with \( q_1(t, T) = \left( e^{-\int_t^T V_{200t}^r dr} \right) \) for \( t \in [0, T] \) we have by theorem 5.1 in [EZ]:

\[
\|Q_h(t, T)\| \leq C + O(\sqrt{h}) \quad \text{and} \quad Q_h(T) = Q_h(t, T)Q_h(t) + O(h)
\]

where \( C \) does not depend on \( t, T \) and \( h \), and the sizes of the remainders in \( L(L^2(\mathbb{R}^n)) \) depend on \( T \) but can be estimated uniformly on \( t \in [0, T] \). Then if \( \|\psi\| = 1 \) we have:

\[
\int_0^T \left\| \langle x \rangle^{-s} \chi(H_1^h)U_h^t(t)Q_h(T)\psi \right\|^2 dt
\]

\[
\leq \int_0^T \left\| \langle x \rangle^{-s} \chi(H_1^h)U_h^t(t)Q_h(t)\psi \right\|^2 dt + O(h)
\]

\[
\leq \int_0^T \left\| Q(2t, T + t) \langle x \rangle^{-s} \chi(H_1^h)U_h(t)\psi \right\|^2 dt + O(h)
\]

\[
\leq \left( C + O(\sqrt{h}) \right) \int_0^T \left\| \langle x \rangle^{-s} \chi(H_1^h)U_h(t)\psi \right\|^2 dt + O(h)
\]

\[
\leq CC_\chi + O(h)
\]

where \( C_\chi \) is given by proposition 4.4. The remainder is uniformly bounded in \( \psi \) so we can choose \( h_T > 0 \) small enough to make it less than 1 and the result follows with \( C_\chi = CC_\chi + 1 \). \( \square \)

We can now prove theorem 4.1 as in [Van87]:

Proof of theorem 4.1. Let \( A_h \) be the generator of dilations defined in (3.6) and \( \chi, \varphi, \psi \in C_0^\infty(\mathbb{R}) \) supported in \( J \) such that \( \chi(E) = 1 \) and \( \chi(\lambda) = \lambda \varphi(\lambda) \psi(\lambda) \) for all \( \lambda \in \mathbb{R} \). We have:

\[
H_1^hU_1^h(T) = \frac{1}{2T} \left( [A_h, U_1^h(T)] + \int_0^T U_1^h(T - t)W(x)U_1^h(t) dt \right)
\]

where \( W(x) = -2V_1(x) - x.\nabla V_1(x) \) and hence there exists \( c \geq 0 \) such that for all \( T \geq 0 \) and \( h \in [0, h_T] \) \( (h_T > 0 \) depends on \( T)\):

\[
\left\| \langle x \rangle^{-s} Q_h(T)\chi(H_1^h)U_1^h(T)Q_h(T)\langle x \rangle^{-s} \right\|
\]

\[
= \left\| \langle x \rangle^{-s} Q_h(T)\varphi(H_1^h)H_1^hU_1^h(T)\psi(H_1^h)Q_h(T)\langle x \rangle^{-s} \right\|
\]

\[
\leq \frac{c}{T} (1 + \|F_h(T)\|)
\]

where:

\[
F_h(T) = \int_0^T \langle x \rangle^{-s} Q_h(T)\varphi(H_1^h)U_1^h(T - t)W(x)U_1^h(t)\psi(H_1^h)Q_h(T)\langle x \rangle^{-s} dt
\]

Indeed, we have \( |Q_h(T)| \leq C + O(\sqrt{h}) \), hence for \( h \in [0, h_T] \) with \( h_T > 0 \) small enough we have \( \|Q_h(T)\| \leq 2C \). Furthermore \( A_h \) is uniformly \( H_1^h \)-bounded, so we have:

\[
\left\| \langle x \rangle^{-s} Q_h(T)\varphi(H_1^h)[A_h, U_1^h(T)]\psi(H_1^h)Q_h(T)\langle x \rangle^{-s} \right\| \leq c
\]

uniformly in \( T \geq 0 \) and \( h \in [0, h_T] \).
Let us now chose $\theta \in C_0^\infty(\mathbb{R}^n)$ with support in $B(0, 2)$ and equal to 1 on $B(0, 1)$, and define $W_1(T, x) = W(x)\theta(x/T)$, $W_2(T, x) = W(x) - W_1(T, x)$ and $F_h(T)$ with the same expression as $F_h(T)$ with $W$ replaced by $W_j$ ($j = 1, 2$). As $W$ decays like $V_1$ (see (3.1)), there exists $c > 0$ such that for all $T \geq 0$ and $h \in [0, h_T]$ we have $\|F_h(T)\| \leq cT^{1-\rho}$. To estimate $F_h$ we compute, for $\|f\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} = 1$:

$$\begin{align*}
\left| \left\langle F_h(T) f, g \right\rangle \right| &\leq \int_0^T \left\| (x)^{-s} U_h(t) \psi(H_1) Q_h(T) (x)^{-s} f \right\| \left\| (x)^{-s} W_1(t, x) \right\| dt \\
&\quad \times \left\| (x)^{-s} U_h(T-t) \varphi(H_1) Q_h(T) (x)^{-s} g \right\| dt \\
&\leq cT^{2s-\rho} \int_0^T \left\| (x)^{-s} \psi(H_1) U_h(t) Q_h(T) (x)^{-s} f \right\|^2 dt \\
&\quad \times \int_0^T \left\| (x)^{-s} \varphi(H_1) U_h(T-t) Q_h(T) (x)^{-s} g \right\|^2 dt \\
&\leq c T^{2s-\rho}
\end{align*}$$

where $c$ is independent of $T \geq 0$ and $h \in [0, h_T]$. Finally we have:

$$\|F_h(T)\| \leq cT^{1-\delta} \quad (4.9)$$

with $\delta = \min(1 + \rho - 2s, \rho) > 0$ and $c \geq 0$ independent of $T \geq 0$ and $h \in [0, h_T]$.

Let $(z, \zeta) \in \Omega_h(E)$ (if $\Omega_h(E)$ is empty then there is nothing to prove) and $T \geq 0$. Let $W_h(z, \zeta) = \exp(ih^{-\frac{1}{2}}(\zeta.x - z.D))$ (see [Wan87, § 3.1]) and:

$$G_h(T) = W_h(z, \zeta)^* \left\langle \frac{h^2}{2} x \right\rangle^{-s} R_h(T) \chi(P_1^h) V_h(T) R_h(T) \left\langle \frac{h^2}{2} x \right\rangle^{-s} V_h(-T) W_h(z, \zeta)$$

where $P_1^h = -h\Delta + V_1(h^2 x)$, $V_h(T) = \exp(-\frac{T}{h} P_1^h)$ and $R_h(T) = q_1(T)^w(h^2 x, h^\frac{1}{2} \zeta)$. These three operators are conjugate to $H_1^h$, $U_h(T)$ and $Q_h(T)$ by the unitary transformation $f \mapsto \left( x \mapsto h^\frac{1}{2} f(h^2 x) \right)$, so for $T \geq 0$ and $h \in [0, h_T]$ we have by (4.8) and (4.9):

$$\|G_h(T)\| = \left\| \left\langle \frac{h^2}{2} x \right\rangle^{-s} R_h(T) \chi(P_1^h) V_h(T) R_h(T) \left\langle \frac{h^2}{2} x \right\rangle^{-s} \right\|$$

$$= \left\| (x)^{-s} Q_h(T) \chi(H_1^h) U_1^h(T) Q_h(T) (x)^{-s} \right\|$$

$$\leq cT^{-\delta}$$

where $c$ does not depend on $T$ and $h \in [0, h_T]$. On the other hand, using [Wan87, lemma 3.1] and [Wan87, theorem 4.2] we have:

$$G(T) = \left\langle \frac{h^2}{2} x + z \right\rangle^{-s} q_1(T)^w(h^2 x + z, h^\frac{1}{2} D + \zeta) \left( \chi \circ p \right)^w(h^2 x + z, h^\frac{1}{2} D + \zeta)$$

$$\times W_h(z, \zeta)^* V_h(q_1(T)^w(h^2 x, h^\frac{1}{2} D) \left\langle \frac{h^2}{2} x \right\rangle^{-s} V_h(-T) W_h(z, \zeta) + O(h)$$

$$\xrightarrow[h \to 0]{} \langle z \rangle^{-s} q_1(T, z, \zeta) \chi(p(z, \zeta)) q_1(T, \phi^T(z, \zeta)) (x(T, z, \zeta)) \langle x(T, z, \zeta) \rangle^{-s}$$

This proves:

$$\langle z \rangle^{-s} q_1(T, z, \zeta) q_1(T, \phi^T(z, \zeta)) \langle x(T, z, \zeta) \rangle^{-s} \leq cT^{-\delta}$$

where $c$ does not depend on $T$, but $x(T, z, \zeta)$ stays in a bounded subset of $\mathbb{R}^n$, so we must have:

$$q_1(T, z, \zeta) q_1(T, \phi^T(z, \zeta)) \xrightarrow[T \to +\infty]{} 0$$

which, by definition of $q_1$, cannot be true unless the classical trajectory starting from $(z, \zeta)$ goes through $\mathcal{O}$ (see (13)).

\[ \square \]
In order to obtain in Besov spaces the resolvent estimates we proved in weighted spaces, we need another resolvent estimate (see proposition 5.2). We begin with a lemma which turns properties on $G_{z,h}(\varepsilon) = (H_h - i\varepsilon P_h \Theta_h^V P_h - z)^{-1}$ (see section 2) into properties on $(H_h - i\varepsilon \Theta_h^V - z)^{-1}$.

**Lemma 5.1.** With assumptions and notations of theorem 2.3, for all $h, \varepsilon \in ]0,1]$ and $z \in \mathbb{C}_{I,+}$ the operator $(H_h - i\varepsilon \Theta_h^V - z)$ has a bounded inverse (denoted by $G_{z,h}^1(\varepsilon)$) which satisfies the following estimates:

\[
\left\| G_{z,h}^1(\varepsilon) \right\| + \left\| H_h^0 G_{z,h}^1(\varepsilon) \right\| \leq \frac{c}{\alpha_h \varepsilon} \quad (5.1)
\]

\[
\left\| G_{z,h}^1(\varepsilon) \langle A_h \rangle^{-1} \right\| + \left\| H_h^0 G_{z,h}^1(\varepsilon) \langle A_h \rangle^{-1} \right\| \leq \frac{c}{\alpha_h \varepsilon} \quad (5.2)
\]

\[
\left\| \sqrt{V_h} G_{z,h}^1(\varepsilon) \right\| \leq \frac{c}{\sqrt{\alpha_h \varepsilon}} \quad (5.3)
\]

\[
\left\| \sqrt{V_h} G_{z,h}^1(\varepsilon) \langle A_h \rangle^{-1} \right\| \leq c \quad (5.4)
\]

where $c$ is independent of $\varepsilon, h \in ]0,1]$ and $z \in \mathbb{C}_{I,+}$ for some closed subinterval $I$ of $J$.

**Proof.** We keep all the notations of the proof of theorem 2.3, in particular $P_h = \phi(H_h^0)$, $P_h' = 1 - P_h$, $G_{z,h}(\varepsilon) = (H_h - i\varepsilon P_h \Theta_h^V P_h)^{-1}, \ldots$. Applying proposition 2.2 with $B = \sqrt{\alpha_h \varepsilon} P_h$ and $Q = P_h$ gives:

\[
\left\| P_h G_{z,h}(\varepsilon) P_h \right\| \leq \frac{1}{\alpha_h \varepsilon}
\]

Calculations (2.13)–(2.16) with $Q_h(\varepsilon)$ replaced by $P_h$ and $P_h'$ show:

\[
\left\| P_h' G_{z,h}(\varepsilon) P_h \right\| \leq \frac{c}{\sqrt{\alpha_h \varepsilon}}, \quad \left\| P_h G_{z,h}(\varepsilon) P_h' \right\| \leq c \quad (5.5)
\]

We also have $\left\| P_h G_{z,h}(\varepsilon) P_h' \right\| \leq \frac{c}{\sqrt{\alpha_h \varepsilon}}$ and hence:

\[
\left\| G_{z,h}(\varepsilon) \right\| + \left\| H_h^0 G_{z,h}(\varepsilon) \right\| \leq \frac{c}{\alpha_h \varepsilon} \quad (5.6)
\]

Now three applications of proposition 2.2 with $B = \sqrt{V_h}$ give:

\[
\left\| \sqrt{V_h} G_{z,h}(\varepsilon) \langle A_h \rangle^{-1} \right\| + \left\| \sqrt{V_h} G_{z,h}(\varepsilon) P_h' \right\| \leq c, \quad \left\| \sqrt{V_h} G_{z,h}(\varepsilon) \right\| \leq \frac{c}{\sqrt{\alpha_h \varepsilon}} \quad (5.7)
\]

Then, as in [JMP84], we prove that:

\[
G_{z,h}(\varepsilon) = G_{z,h}(\varepsilon) + i\varepsilon G_{z,h}(\varepsilon) P_h' (1 - i\varepsilon \Theta_h^V P_h G_{z,h}(\varepsilon) P_h')^{-1} \Theta_h^V P_h G_{z,h}(\varepsilon)
\]

is well-defined for $\varepsilon$ small enough and is a bounded inverse of $(H_h - i\varepsilon \Theta_h^V P_h)$ which satisfies estimates (5.5)–(5.7) as $G_{z,h}(\varepsilon)$. Then it remains to define:

\[
G_{z,h}^1(\varepsilon) = G_{z,h}(\varepsilon) + i\varepsilon G_{z,h}(\varepsilon) \Theta_h^V (1 - i\varepsilon P_h' G_{z,h}(\varepsilon) \Theta_h^V P_h')^{-1} P_h' G_{z,h}(\varepsilon)
\]

for $\varepsilon$ small enough and check the conclusions of the lemma.

**Proposition 5.2.** Let $s > 1$ and $I$ a closed subinterval of $J$. Then there exists $c \geq 0$ such that for all $z \in \mathbb{C}_{I,+}$ and $h \in ]0,1]$:

\[
\left\| \mathbb{1}_{\mathbb{R}_-}(A_h) (H_h - z)^{-1} \langle A_h \rangle^{-s} \right\| \leq \frac{c}{\alpha_h} \quad (5.8)
\]

**Proof.** We follow the proof of theorem 2.3 in [JMP84]. Let

\[
F_{z,h}(\varepsilon) = \mathbb{1}_{\mathbb{R}_-}(A_h) \exp(\varepsilon A_h) G_{z,h}(\varepsilon) \langle A_h \rangle^{-s}
\]
By (5.2) we already know that:
\[
\left\| \tilde{F}_{z,h}(\varepsilon) \right\| \leq \frac{c}{\alpha_h \sqrt{\varepsilon}} \tag{5.9}
\]

Then we compute in the sense of quadratic forms on \( D_H \cap D_A \):
\[
\begin{align*}
\frac{d}{d\varepsilon} \tilde{F}_{z,h}(\varepsilon) &= \mathbb{1}_{\mathbb{R}_-}(A_h)e^{\varepsilon A_h}A_h G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \\
&\quad + i\mathbb{1}_{\mathbb{R}_-}(A_h)e^{\varepsilon A_h}(\tilde{H}_{V_h} + i[H_h, A_h])G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \\
&\quad + iC_V \mathbb{1}_{\mathbb{R}_-}(A_h)e^{\varepsilon A_h}G^1_{z,h}(\varepsilon)V_h G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \\
&\quad - \varepsilon \mathbb{1}_{\mathbb{R}_-}(A_h)e^{\varepsilon A_h}G^1_{z,h}(\varepsilon)\langle \Theta^\varepsilon_h, A \rangle G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s}
\end{align*}
\]

We use complex interpolation to estimate the first term:
\[
\left\| \mathbb{1}_{\mathbb{R}_-}(A_h)e^{\varepsilon A_h}G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\| \\
\leq \left\| \mathbb{1}_{\mathbb{R}_-}(A_h)e^{\varepsilon A_h}G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\|^{1-\frac{1}{2}} \left\| \mathbb{1}_{\mathbb{R}_-}(A_h)e^{\varepsilon A_h}G^1_{z,h}(\varepsilon) \right\|^\frac{1}{2}
\]

For the second term we write:
\[
\left\| \chi(A_h)e^{\varepsilon A_h}G^1_{z,h}(\varepsilon)\langle A_h \rangle^{-s} \right\| \leq \left\| G^1_{z,h}(\varepsilon)\sqrt{V_h} \right\| \left\| \sqrt{V_h} G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\|
\]

and finally, by assumption (d) and (5.1)-(5.2):
\[
\varepsilon \left\| G^1_{z,h}(\varepsilon)\langle \Theta^\varepsilon_h, A \rangle G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\| \leq c\varepsilon \alpha_h \left\| G^1_{z,h}(\varepsilon) \right\|_{\Gamma_h} \left\| G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\|_{\Gamma_h}
\]

This gives:
\[
\left\| \frac{d}{d\varepsilon} \alpha_h \tilde{F}_{z,h}(\varepsilon) \right\| \leq c \left( \varepsilon^{-\frac{1}{2}} \left\| \alpha_h \tilde{F}_{z,h}(\varepsilon) \right\|^{1-\frac{1}{2}} + \varepsilon^{-\frac{1}{2}} \right)
\]

which, together with (5.9), gives the result. \(\square\)

Let \(\Omega_0 = ]-1,1[\) and \(\Omega_j = \{ \lambda \in \mathbb{R} : 2^{j-1} \leq |\lambda| < 2^j \}\) for \(j \in \mathbb{N}^*\). For a selfadjoint operator \(F\) on \(\mathcal{H}\) and \(s \geq 0\), the abstract Besov space \(B_s(F)\) is defined by:
\[
B_s(F) = \left\{ u \in \mathcal{H} : \left\| u \right\|_{B_s(F)} < \infty \right\}
\]

where:
\[
\left\| u \right\|_{B_s(F)} = \sum_{j \in \mathbb{N}} 2^{js} \left\| \mathbb{1}_{\Omega_j}(F)u \right\|_{\mathcal{H}}
\]

The norm of its dual space \(B^*_s(F)\) with respect to the scalar product on \(\mathcal{H}\) is:
\[
\left\| v \right\|_{B^*_s} = \sup_{j \in \mathbb{N}} 2^{-js} \left\| \mathbb{1}_{\Omega_j}(F)v \right\|_{\mathcal{H}}
\]

When \(F\) is the multiplication by \(x\) on \(L^2(\mathbb{R}^n)\) we recover the usual Besov spaces \(B_s\) and \(B^*_s\) and the norm we have just defined for \(B^*_s\) is equivalent to the usual one:
\[
\sup_{R \geq 1} R^{-s} \left( \int_{|x| < R} |v(x)|^2 \, dx \right)^{\frac{1}{2}}
\]
Then there exists $h \in \mathbb{C}$ such that for any $z \in \mathbb{C}_{l,h}$ and $h \in [0,1]$ we have:

$$\| (H_h - z)^{-1} \|_{B_s(A_h) \rightarrow B^*_s(A_h)} \leq \frac{c}{\alpha_h}$$

Now that we have theorem 2.3 and proposition 5.2, we can follow word by word the proof of the analog theorem for selfadjoint operators (see theorem 2.2 in [Van07]). Applied to our dissipative Schrödinger $H_h = -h^2\Delta + V_1(x) - iv(h)V_2(x)$, this gives:

**Theorem 5.4.** Let $E > 0$ and $s \geq \frac{1}{2}$. If all bounded trajectories of energy $E$ meet $\mathcal{O}$, then there exists $\varepsilon, h_0 > 0$ and $c \geq 0$ such that with $J = [E - \varepsilon, E + \varepsilon]$ we have for all $x \in \mathbb{C}_{l,h}$ and $h \in [0, h_0]$:

$$\| (H_h - z)^{-1} \|_{B_s \rightarrow B^*_s} \leq \frac{c}{h\hat{\nu}(h)}$$

Proof. We already have a conjugate family $(\hat{\nu}(h)F_h)$ for $(H_h)$. So we only have to apply the abstract theorem 5.3, (3.10), and the estimate:

$$\| (H_h - i)^{-1} \|_{B_s \rightarrow B_s} \leq c$$

with a similar estimate for dual spaces. To prove (5.10), we use the idea given in [Hor84, 14.1]. For any $u \in B_s(F)$ and $k \in \mathbb{N}$, since the $\mathbb{1}_{\Omega_j}(F)u$ for $j \in \mathbb{N}$ are pairwise orthogonal we have:

$$\| u \|_{B_s(F_h)} = \sum_{0 \leq j \leq k} 2^{js} \| \mathbb{1}_{\Omega_j}(F_h)u \| + \sum_{j > k} 2^{-js} \| 2^{2js}\mathbb{1}_{\Omega_j}(F_h)u \|

\leq \left( \sum_{j \leq k} 2^{2js} \right) \left( \sum_{j \leq k} \| \mathbb{1}_{\Omega_j}(F_h)u \|^2 \right)^{\frac{1}{2}} + \left( \sum_{j > k} 2^{-2js} \right) \left( \sum_{j > k} \| 2^{2js}\mathbb{1}_{\Omega_j}(F_h)u \|^2 \right)^{\frac{1}{2}}

\leq c_s 2^{ks} \| u \| + c_s 2^{-ks} \| \mathbb{1}_{\Omega_j}(F_h)u \|

\text{and hence, for } \varphi \in B_s, \text{ using the fact that the operator } \langle F_h \rangle^{2s}(H_h - i)^{-1}\langle x \rangle^{-2s} \text{ is bounded in } \mathcal{L}(L^2(\mathbb{R}^n)) \text{ uniformly in } h \text{ we have:}

$$\| (H_h - i)^{-1} \varphi \|_{B_s(F_h)} \leq \sum_{k \in \mathbb{N}} \| (H_h - i)^{-1}\mathbb{1}_{\Omega_k}(x)\varphi \|_{B_s(F_h)}

\leq c_s \sum_{k \in \mathbb{N}} 2^{ks} \| (H_h - i)^{-1}\mathbb{1}_{\Omega_k}(x)\varphi \| + c_s \sum_{k \in \mathbb{N}} 2^{-ks} \| (X)^{2s}\mathbb{1}_{\Omega_k}(x)\varphi \|

\leq c_s \sum_{k \in \mathbb{N}} 2^{ks} \| \mathbb{1}_{\Omega_k}(x)\varphi \| + c_s \sum_{k \in \math{N}} 2^{-ks} \| (X)^{2s}\mathbb{1}_{\Omega_k}(x)\varphi \|

\leq c_s \| \varphi \|_{B_s} + c_s \sum_{k \in \mathbb{N}} 2^{ks} \| \mathbb{1}_{\Omega_k}(x)\varphi \|

\leq c_s \| \varphi \|_{B_s} \quad \square$$

**APPENDIX A. UNITARY DILATIONS AND DISSIPATIVE SCHRODINGER OPERATORS**

In order to use the selfadjoint theory to study dissipative operators, we have mostly used the assumption that $H$ is a perturbation of its selfadjoint part $H_1$. However, by the theory of unitary dilations, there are other selfadjoint operators we can use:
Definition A.1. Let $T$ be a bounded operator of the Hilbert space $\mathcal{H}$. A bounded operator $U$ on a Hilbert space $\mathcal{K}$ is said to be a dilation of $T$ if $\mathcal{H} \subset \mathcal{K}$ and for all $\varphi, \psi \in \mathcal{H}$ and $n \in \mathbb{N}$ we have:

$$\langle U^n \varphi, \psi \rangle_{\mathcal{K}} = \langle T^n \varphi, \psi \rangle_{\mathcal{H}}$$

The theory of unitary dilations for a contraction is developed in the book of B.S.-Nagy and C. Foias (\cite{NF67}). In particular, we know that every contraction has a unitary operator which satisfies:

$$U \in L^2(\mathcal{H})$$

Definition A.1.

The unitary group $(U(t))_{t \in \mathbb{R}}$ on $\mathcal{K}$ is generated by a selfadjoint operator $K$ on $\mathcal{K}$, the properties of which we can use to study the dissipative operator $H$. Note that $K$ is usually said to be a selfadjoint dilation of $H$ but is not a dilation of $H$ in the sense of definition A.1.

Much is said on the abstract theory in \cite{NF67}, but there is a more detailed study of the dissipative Schrödinger operator case in \cite{Pav77}. In particular an example of dilation is given. Here we recall this example in the semiclassical setting:

Proposition A.2. Let $-h^2\Delta + V_1 - iV_2$ be a dissipative Schrödinger operator on $L^2(\mathbb{R}^n)$ as in section 3. $W_h = \sqrt{2h}V_2$, $\Omega = \text{supp} V_2$, $K = L^2(\mathbb{R}^n, L^2(\Omega)) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n, L^2(\Omega))$ and $P$ the orthogonal projection of $K$ on $L^2(\mathbb{R}^n)$. Then the operator:

$$K_h : \varphi = (\varphi_-, \varphi_0, \varphi_+) \mapsto \left(-i\varphi_-, H^h_1 \varphi_0 - \frac{W_h}{2}(\varphi_-(0) + \varphi_+(0)), -i\varphi_+^h\right)$$

with domain:

$$\mathcal{D}(K_h) = \{(\varphi_-, \varphi_0, \varphi_+) : \varphi_\pm \in H^1(\mathbb{R}^n, L^2(\Omega)) \text{ and } \varphi_+(0) - \varphi_-(0) = iW_h \varphi_0 \} \subset \mathcal{K}$$

(where $H^1$ is the Sobolev space of $L^2$-functions with first derivative in $L^2$) is a selfadjoint operator which satisfies:

$$\forall z \in \mathbb{C}_+, \quad P(K_h - z)^{-1} |_{L^2(\mathbb{R}^n)} = (H_h - z)^{-1}$$

$$\forall z \in \mathbb{C}_+, \quad P(K_h - \bar{z})^{-1} |_{L^2(\mathbb{R}^n)} = (H_h^* - \bar{z})^{-1}$$

$$\forall t \geq 0, \quad Pe^{-\frac{it}{h}K_h} |_{L^2(\mathbb{R}^n)} = e^{-\frac{it}{h}H_h}$$

$$\forall t \leq 0, \quad Pe^{-\frac{it}{h}K_h} |_{L^2(\mathbb{R}^n)} = e^{-\frac{it}{h}H_h^*}$$

Proof. The proof of the proposition is straightforward calculations. We first have to check that $K$ is symmetric, that $\mathcal{D}(K^*) \subset \mathcal{D}(K)$ and then that for $z \in \mathbb{C}_+$ we have $(\psi_-, \psi_0, \psi_+) = (K - z)^{-1}(\varphi_-, \varphi_0, \varphi_+)$ where:

$$\psi_-(r) = i \int_{-\infty}^r e^{iz(r-s)} \varphi_-(s) \, ds$$

$$\psi_0 = (H_h - z)^{-1}(\varphi_0 + W_h \varphi_- (0))$$

$$\psi_+(r) = (\varphi_-(0) + iW_h \psi_0) e^{izr} + i \int_0^r e^{iz(r-s)} \varphi_+(s) \, ds$$

and an analog for $(K - \bar{z})^{-1}$. To prove the last statement, we show that the operator of the semigroup $t \mapsto Pe^{-\frac{it}{h}K_h} |_{L^2(\mathbb{R}^n)}$ must be $H$ using the result on the resolvent. Details are given in \cite{Pav77}.

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References


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