Matrix De Rham complex and quantum A-infinity algebras

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We represent the equation defining the quantum $A_\infty-$algebras introduced in ([B1],[B2]) via GL-invariant tensors on matrix spaces $gl(A)$. This allows in particular to show that the cohomology of the Batalin-Vilkovisky differential from ([B1],[B2]) are zero.

*Notations.*We work in the tensor category of super vector spaces, over an algebraically closed field k, $char(k) = 0$. Let $V = V^{1|0} \oplus V^{0|1}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. We denote by $\overline{\alpha}$ the parity of an element α and by ΠV the super vector space with inversed parity.For a finite group G acting on a vector space U, we denote via U^G the space of invariants with respect to the action of G and by U_G the space of coinvariants $U_G = U/\{gv - v|v \in V, g \in G\}$. If G is finite then the averaging $(v) \to 1/|G| \sum_{g\in G} gv$ give a canonical isomorphism $U^G \simeq U_G$. Element $(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ of $A^{\otimes n}$ is denoted by $(a_1, a_2, \ldots, a_n)$. Cyclic words, i.e. elements of the subspace $(V^{\otimes n})^{\mathbb{Z}/n\mathbb{Z}}$ are denoted via $(a_1 \ldots a_n)^c$. The symbol δ_α^β denotes the Kronecker delta tensor: $\delta_\alpha^\beta = 1$ for $\alpha = \beta$ and zero otherwise. We denote by tr the super trace linear functional on $End(U)$, $tr(U) = \sum_a (-1)^{\overline{a}} U_a^a$.

1 The vector space F.

Let $F = \oplus_{n=0}^\infty F_n$ where

$$F_n = (V^{\otimes n} \otimes k[\mathbb{S}_n])^{\mathbb{S}_n}.$$

Here $k[\mathbb{S}_n]$ is the group algebra of the symmetric group $\mathbb{S}_n$, and $\mathbb{S}_n$ acts on $k[\mathbb{S}_n]$ by conjugation.

We defined in ([B1]) the differential on F which together with the odd symplectic bracket, coming from the cyclic operad structure, form the Batalin-Vilkovisky algebra . Here we shall use the invariant theory approach to cyclic homology ([FT],[L] and references therein) in order to represent our differential acting on F via $GL-$invariant geometry on the affine space $gl(\infty|\infty) \otimes V$. Let U be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. There is the natural left group $\mathbb{S}_n$-action on $U^{\otimes n}$ via

$$\sigma \in \mathbb{S}_n,\ \ \sigma : (u_1, \ldots, u_n) \to (-1)^\epsilon (u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(n)})$$

where ϵ is the standrad Koszul sign. Recall that we work in the tensor category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces, where the isomorphism $X \otimes Y \simeq Y \otimes X$ is realized via $(x, y) \to (-1)^{\overline{xy}}(y, x)$. This gives $k-$algebra morphism $\mu : k[\mathbb{S}_n] \to End_k(U^{\otimes n})$. The group $GL(U)$ of automorphisms of U acts diagonnally on $U^{\otimes n}$ and the image of $k[\mathbb{S}_n]$ is obviously in the invariant subspace of $End_k(U^{\otimes n})$. If U is a vector space with $\dim_k U^{1|0} \geq n$ then the $k-$algebra morphism μ is an isomorphism:

$$\mu : k[\mathbb{S}_n] \simeq (End_k(U^{\otimes n}))^{GL(U)} \tag{1}$$

according to the invariant theory ([GW]).

Proposition 1 *The vector space F_n is canonically identified via the map μ with $GL(U)$-invariant subspace of $n-$symmetric powers of the vector space $End_k(U) \otimes V$:*

$$F_n \simeq (S^n(End_k(U) \otimes V))^{GL(U)}, \tag{2}$$

where U is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with $\dim_k U^{1|0} \geq n$.

Proof. The proof is essentually the application of the invariant theory as in the classical definition of cyclic homology via homology of general linear group (see [FT],[L]).We have the following sequence of isomorphisms of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces:

$$\begin{aligned}(V^{\otimes n} \otimes k[\mathbb{S}_n])^{\mathbb{S}_n} &\simeq (V^{\otimes n} \otimes (End_k(U^{\otimes n}))^{GL(U)})^{\mathbb{S}_n} \simeq \\ &\simeq (V^{\otimes n} \otimes (End_k(U)^{\otimes n})^{GL(U)})^{\mathbb{S}_n} \simeq ((V^{\otimes n} \otimes End_k(U)^{\otimes n})^{GL(U)})^{\mathbb{S}_n} \simeq \\ &\simeq (((V \otimes End_k(U))^{\otimes n})^{\mathbb{S}_n})^{GL(U)} \simeq (S^n(End_k(U) \otimes V))^{GL(U)}.\end{aligned}$$

Here we used the canonical isomorphism $End_k(U)^{\otimes n} \simeq End_k(U^{\otimes n})$, under which the permuting of $n-$tuples of endomorphisms by σ corresponds to the conjugation by $\mu(\sigma)$ .We also used the fact that $GL(U)-$action and $\mathbb{S}_n-$action mutually commute. ■

We shall denote the isomorphism (2) by $\widetilde{\mu}$. Let us denote by $\{E^{\beta}_{\alpha}\}$ the basis of elementary matrices in $End_k(U)$ corresponding to some basis $\{e_{\alpha}\}$ in U. Then the map (1) is written as

$$\mu : [\sigma] \to \sum_{\alpha_1,\ldots\alpha_n} (-1)^{\epsilon} E^{\alpha_{\sigma^{-1}(1)}}_{\alpha_1} \otimes \ldots \otimes E^{\alpha_{\sigma^{-1}(n)}}_{\alpha_n}$$

For an element $a_i \in V$ let us denote via

$$A^{\beta}_{i,\alpha} = a_i \otimes E^{\beta}_{\alpha}, \;\; A^{\beta}_{i,\alpha} \in End_k(U) \otimes V$$

the $V-$valued matrix correspoding to it. Then for

$$(a_1, \ldots, a_n) \otimes_{\mathbb{S}_n} \sigma \in (V^{\otimes n} \otimes k[\mathbb{S}_n])_{\mathbb{S}_n}$$

representing an element from F_n, the isomorphism (2) gives the following $GL(U)$-invariant symmetric tensor

$$\widetilde{\mu} : (a_1, \dots, a_n) \otimes_{\mathbb{S}_n} \sigma \to \sum_{\alpha_1, \dots \alpha_n} (-1)^{\widetilde{\epsilon}} A_{1,\alpha_1}^{\alpha_{\sigma^{-1}(1)}} \cdot \dots \cdot A_{n,\alpha_n}^{\alpha_{\sigma^{-1}(n)}}. \qquad (3)$$

where $\widetilde{\epsilon}$ is ϵ plus the Koszul sign arising from the isomorphism

$$End_k(U^{\otimes n}) \bigotimes V^{\otimes n} \simeq (End_k(U) \bigotimes V)^{\otimes n}$$

Let $\sigma = (\rho_1 \dots \rho_r) \dots (\tau_1 \dots \tau_t)$ be the cycle decomposition of σ.

Lemma 2 *The symmetric tensor* $\widetilde{\mu}((a_1, \dots, a_n) \otimes_{\mathbb{S}_n} \sigma)$ *is written in terms of the cycle decomposition of* σ *as*

$$(\sum_{\alpha_1, \dots \alpha_r} (-1)^{\epsilon_\rho} A_{\rho_1,\alpha_1}^{\alpha_r} \cdot A_{\rho_2,\alpha_2}^{\alpha_1} \dots \cdot A_{\rho_r,\alpha_r}^{\alpha_{r-1}}) \cdot \dots \cdot (\sum_{\gamma_1, \dots \gamma_t} (-1)^{\epsilon_\tau} A_{\tau_1,\gamma_1}^{\gamma_t} \cdot A_{\tau_2,\gamma_2}^{\gamma_1} \dots \cdot A_{\tau_t,\gamma_t}^{\gamma_{t-1}})$$

Proof. *It is sufficient to rearrange (3), so that the pairs of terms with the same repeating upper and lower indexes are placed one after the other. We leave to the interested reader to work out the signs arising from the standard Koszul rule.* ■

If we denote

$$Tr^\top(A_{\rho_1} \dots A_{\rho_r}) = \sum_{\alpha_1, \dots \alpha_r} (-1)^{\epsilon_\rho} A_{\rho_1,\alpha_1}^{\alpha_r} \cdot A_{\rho_2,\alpha_2}^{\alpha_1} \dots \cdot A_{\rho_r,\alpha_r}^{\alpha_{r-1}} \qquad (4)$$

then we can write $\widetilde{\mu}((a_1, \dots, a_n) \otimes_{\mathbb{S}_n} (\rho_1 \dots \rho_r) \dots (\tau_1 \dots \tau_t))$ as

$$Tr^\top(A_{\rho_1} \dots A_{\rho_r}) \cdot \dots \cdot Tr^\top(A_{\tau_1} \dots A_{\tau_t})$$

Remark 3 *The notation (4) can be justified by the fact that if* $V = k$ *and* $A_i \in End_k(U)$ *then it is the super trace of the action of* $A_1 \dots A_r$ *on the dual space* $Hom(U, k)$.

2 The differential and the bracket.

Let us assume that V has an *odd* symmetric inner product

$$l : V^{\otimes 2} \to \Pi k, l(x \otimes y) = (-1)^{\overline{x}\,\overline{y}} l(y \otimes x)$$

It follows in particular that the even and odd components of V are of the same dimension, $\dim_k V = (r|r)$.

We assume from now on that U is also the $\mathbb{Z}/2\mathbb{Z}$-graded vector space which has even and odd components of the same dimension:

$$\dim_k U = (N|N).$$

The super-trace functional
$$tr(E_\alpha^\alpha) = (-1)^{\overline{\alpha}}$$
defines the natural even inner product on the vector space $End_k(U)$:
$$tr(E_\alpha^\beta, E_{\widetilde{\beta}}^{\widetilde{\alpha}'}) = (-1)^{\overline{\beta}}\delta_\alpha^{\widetilde{\alpha}'}\delta_{\widetilde{\beta}}^\beta \tag{5}$$

It allows to extend the odd inner product l on V to the odd inner product $\widehat{l}$ defined on $End_k(U)\otimes V$. The latter space is therefore an affine space with constant odd symplectic structure. Its algebra of symmetric tensors $\oplus_{n=0}^\infty S^n(End_k(U)\otimes V)$ has natural structure of Batalin-Vilkovisky algebra. The odd inner product on $End_k(U)\otimes V$ can be written as
$$\widehat{l}(A_\alpha^\beta, \widetilde{A}_{\widetilde{\beta}}^{\widetilde{\alpha}'}) = (-1)^{\overline{\beta}+\overline{a}(\overline{\alpha}+\overline{\beta})}\delta_\alpha^{\widetilde{\alpha}'}\delta_{\widetilde{\beta}}^\beta l(a,\widetilde{a}).$$

If we choose a basis $\{a_\nu\}$ in V, then the Batalin-Vilkovisky operator acting on the symmetric algebra $\oplus_{n=0}^\infty S^n(End_k(U)\otimes V)$ is written, using the generators $A_{\nu,\alpha}^\beta$, as
$$\Delta = \sum_{\nu\kappa,\alpha\beta}(-1)^\varepsilon\frac{l_{\nu\kappa}}{2}\frac{\partial^2}{\partial A_{\nu,\alpha}^\beta\partial A_{\kappa,\beta}^\alpha} \tag{6}$$
where $l_{\nu\kappa} = l(a_\nu, a_\kappa)$, $\varepsilon = \overline{\beta} + \overline{a}_\nu(\overline{\alpha}+\overline{\beta})$. Similarly we have the standard odd Poisson bracket corresponding to the affine space with constant odd symplectic structure:
$$\{A_{\nu,\alpha}^\beta A_{\kappa,\widetilde{\beta}}^{\widetilde{\alpha}}\} = (-1)^\varepsilon\delta_\alpha^{\widetilde{\alpha}'}\delta_{\widetilde{\beta}}^\beta l_{\nu\kappa}$$

Since the inner product $\widehat{l}$ is $GL(U)$-invariant , therefore both the second-order

odd operator Δ and the bracket $\{\bullet,\bullet\}$ are $GL(U)$-invariant. It defines the differential and the bracket on the $GL(U)-$invariant subspace
$$\oplus_{n=0}^{n=s}S^n(End_k(U)\otimes V)^{GL(U)}, \tag{7}$$
which coincides with $\oplus_{n=0}^{n=s}F$ by (2) if $\dim_k U$ is sufficiently big ($N \geq s$).

We defined in ([B1],[B2]) the Batalin-Vilkovisky operator acting on F. It is the combination of "dissection-gluing" operator acting on cycles with contracting by the tensor of the inner product. The space F is naturally isomorphic to the symmetric algebra of the space of cyclic words:
$$F = Symm(\oplus_{j=0}^\infty(V^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})$$

It is easy to see by considering the cycle decomposition of permutations. The second order Batalin-Vilkovisky operator from ([B1],[B2]) is completely determined by its action on the second symmetric power and it sends a product of two cyclic words $(a_{\rho_1}\dots a_{\rho_r})^c(a_{\tau_1}\dots a_{\tau_t})^c$ to

$$\sum_{p,q}(-1)^{\varepsilon_1}l_{\rho_p\tau_q}(a_{\rho_1}\dots a_{\rho_{p-1}}a_{\tau_{q+1}}\dots a_{\tau_{q-1}}a_{\rho_{p+1}}\dots a_{\rho_r})^c + \tag{8}$$
$$+\sum_{p+1<q}(-1)^{\varepsilon_2}l_{\rho_p\rho_q}(a_{\rho_1}\dots a_{\rho_{p-1}}a_{\rho_{q+1}}\dots a_{\rho_r})^c(a_{\rho_{p+1}}\dots a_{\rho_{q-1}})^c(a_{\tau_1}\dots a_{\tau_t})^c$$
$$+\sum_{p+1<q}(-1)^{\varepsilon_3}l_{\tau_p\tau_q}(a_{\rho_1}\dots a_{\rho_r})^c(a_{\tau_1}\dots a_{\tau_{p-1}}a_{\tau_{q+1}}\dots a_{\tau_t})^c(a_{\tau_{p+1}}\dots a_{\tau_{q-1}})^c$$

where ε_i are the signs from [B1] in the case of the modular operad based on the spaces $k[S_n]$.

Theorem 4 *The operator Δ defined on the $GL(U)-$invariant subspace (7), where $\dim_k U = (N|N)$ is sufficiently big ($N \geq s$), coincides with the differential defined on $\oplus_{n=0}^{n=s}F_n$ in ([B1],[B2]).*

Proof. As Δ is of second order with respect to the multiplication it is sufficient to consider the case when the cycle decomposition of σ has two cycles $\sigma = (\rho_1\dots\rho_r)(\tau_1\dots\tau_t)$, so that

$$\widetilde{\mu}((a_1,\dots,a_n)\otimes\sigma) = Tr^{\top}(A_{\rho_1}\dots A_{\rho_r})Tr^{\top}(A_{\tau_1}\dots A_{\tau_t})$$

is the product of two generators of the algebra. Then, applying

$$\Delta = \sum_{\nu,\kappa;\theta,\beta}(-1)^{\varepsilon}\frac{l_{\nu\kappa}}{2}\frac{\partial^2}{\partial A^{\beta}_{\nu,\theta}\partial A^{\theta}_{\kappa,\beta}}$$

we get three terms. First, there is the term

$$\sum_{p,q;\theta,\beta}(-1)^{\varepsilon_1}l_{\rho_p\tau_q}\Big(\sum_{\substack{\alpha_1,\dots\alpha_r;\\ \alpha_{p-1}=\beta,\alpha_p=\theta}}(-1)^{\epsilon_p}A^{\alpha_r}_{\rho_1,\alpha_1}\dots\widehat{A^{\beta}_{\rho_p,\theta}}\dots A^{\alpha_{r-1}}_{\rho_r,\alpha_r}\Big)\cdot$$
$$\cdot\Big(\sum_{\substack{\gamma_1,\dots\gamma_t\\ \gamma_{q-1}=\theta,\gamma_q=\beta}}(-1)^{\widetilde{\epsilon}_q}A^{\gamma_t}_{\tau_1,\gamma_1}\dots\widehat{A^{\theta}_{\tau_q,\beta}}\dots A^{\gamma_{t-1}}_{\tau_t,\gamma_t}\Big)$$

Here and further in this proof we leave to the reader to verify that the signs match correctly. Alternatively the matching follows from the identification of the underlying modular operads, see below, for where the matching of signs is trivial. Rewriting this term so that the pairs of terms with repeating lower and upper indexes follow one after the other we get

$$\sum_{p,q;\theta,\beta}(-1)^{\varepsilon_1}l_{\rho_p\tau_q}\cdot$$
$$\cdot\Big(\sum_{\{\alpha\},\{\gamma\}}(-1)^{\widetilde{\epsilon}}A^{\alpha_r}_{\rho_1,\alpha_1}\dots A^{\alpha_{p-2}}_{\rho_{p-1},\beta}A^{\beta}_{\tau_{q+1},\gamma_{q+1}}\dots A^{\gamma_{q-2}}_{\tau_{q-1},\theta}A^{\theta}_{\rho_{p+1},\alpha_{p+1}}\dots A^{\alpha_{r-1}}_{\rho_r,\alpha_r}\Big)$$

where $\{\alpha\}=\{\alpha_1,\dots\widehat{\alpha}_{p-1},\widehat{\alpha}_{p\dots}\alpha_r\}$, $\{\gamma\}=\{\gamma_1,\dots\widehat{\gamma}_{q-1},\widehat{\gamma}_q\dots\gamma_t\}$. This can be written as

$$\sum_{p,q}(-1)^{\varepsilon_1}l_{\rho_p\tau_q}Tr^{\top}(A_{\rho_1}\dots A_{\rho_{p-1}}A_{\tau_{q+1}}\dots A_{\tau_{q-1}}A_{\rho_{p+1}}\dots A_{\rho_r})$$

which is exactly the term corresponding to the first term in (8). The second term:

$$\sum_{p<q;\theta,\beta}(-1)^{\varepsilon}l_{\rho_p\rho_q}(\sum_{\substack{\alpha_1,\dots\alpha_r;\\ \alpha_{p-1}=\beta,\alpha_p=\theta\\ \alpha_{q-1}=\theta,\alpha_q=\beta}}(-1)^{\epsilon_p}A^{\alpha_r}_{\rho_1,\alpha_1}\dots\widehat{A^{\beta}_{\rho_p,\theta}}\dots\widehat{A^{\theta}_{\rho_q,\beta}}\dots A^{\alpha_{r-1}}_{\rho_r,\alpha_r})\cdot$$
$$\cdot(\sum_{\gamma_1,\dots\gamma_t}(-1)^{\epsilon_\tau}A^{\gamma_t}_{\tau_1,\gamma_1}\dots A^{\gamma_{t-1}}_{\tau_t,\gamma_t}).$$

Assume first that the erased terms $\widehat{A^{\beta}_{\rho_p,\theta}}$ and $\widehat{A^{\theta}_{\rho_q,\beta}}$ are not sitting next to each other , i.e. $p+1<q$. Then, rewriting this expression so that the pairs of terms with the same repeating lower and upper indexes follow one after the other, gives

$$\sum_{p+1<q}(-1)^{\varepsilon_2}l_{\rho_p\rho_q}\cdot$$
$$\cdot Tr^{\top}(A_{\rho_1}\dots A_{\rho_{p-1}}A_{\rho_{q+1}}\dots A_{\rho_r})Tr^{\top}(A_{\rho_{p+1}}\dots A_{\rho_{q-1}})Tr^{\top}(A_{\tau_1}\dots A_{\tau_t})$$

However if $p+1=q$ then we get instead

$$\sum_{p}(-1)^{\widetilde{\varepsilon}}l_{\rho_p\rho_{p+1}}Tr^{\top}(A_{\rho_1}\dots A_{\rho_{p-1}}A_{\rho_{p+2}}\dots A_{\rho_r})Tr^{\top}(Id)Tr^{\top}(A_{\tau_1}\dots A_{\tau_t})$$

where $Tr^{\top}(Id)=\sum_{\alpha}(-1)^{\overline{\alpha}}$, which is equal to zero because even and odd parts of U are of the same dimension

$$\dim_k U^{even}=\dim_k U^{odd}. \tag{9}$$

The third term is similar to the second and it gives

$$\sum_{p+1<q}(-1)^{\varepsilon_3}l_{\tau_p\tau_q}\cdot$$
$$\cdot Tr^{\top}(A_{\rho_1}\dots A_{\rho_r})\cdot Tr^{\top}(A_{\tau_1}\dots A_{\tau_{p-1}}A_{\tau_{q+1}}\dots A_{\tau_t})Tr^{\top}(A_{\tau_{p+1}}\dots A_{\tau_{q-1}})$$

So we get the three terms corresponding exactly to the Batalin -Vilkovisky operator defined in ([B1],[B2]) ■

Since the map $\widetilde{\mu}$ respects the multiplicative structure, we have the similar result concerning the odd symplectic bracket.

Proposition 5 *The odd symplectic bracket on the $GL(U)-$invariant subspaces*

$$S^n(End(U)\otimes V)^{GL(U)}\otimes S^{n'}(End(U)\otimes V)^{GL(U)}\to S^{n+n'-2}(End(U)\otimes V)^{GL(U)}$$

coincides with the odd symplectic bracket

$$F_n\otimes F_{n'}\to F_{n+n'-2}$$

described in ([B1],[B2]).

The important consequence of the theorem 4 is that the cohomology of the differential Δ acting on F are zero, except for the constants $F_0 = k$.

Theorem 6 *The cohomology of the Batalin-Vilkovisky differential acting on F are trivial: $H^*(\oplus_{n=1}^{\infty}F_n,\Delta)=0$.*

Proof. The cohomology of the Batalin Vilkovisky operator Δ acting on the symmetric algebra of the vector space $End_k(U)\otimes V$ are trivial, because this complex is isomorphic to the De Rham complex of the vector space $End_k(U)\otimes V^{1|0}$. The reductivity of $GL(U)$ implies that the cohomology of Δ acting on the $GL(U)$-invariant subspace are also trivial. It follows that $\ker\Delta|_{F_n} = \operatorname{im}\Delta|_{F_{n++2}}$. ■

3 Modular operad structure on $k[S_n]$.

We defined in ([B1],[B2]) the modular operad $\mathcal{S}$ with components $\mathcal{S}((n)) = k[S_n]$. The subspace of cyclic permutations corresponds to the cyclic operad of associative algebras with scalar product. The relation with $GL(U)-$invariant tensors on the matrix spaces allows to give a straightforward definition for this modular operad structure.

We work in the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces and the appropriate modification of the modular operad is defined as the algebra over triple, which is the functor on $\mathbb{S}-$modules given by

$$\mathbb{M}\mathcal{V}((n))=\bigoplus_{G\in\Gamma((n))}\mathcal{V}((G))_{Aut(G)}$$

Let us consider the endomorphism modular operad $\mathcal{E}[End_k(U)]$, associated with the vector space $End_k(U)$, $\dim_k U=(N|N)$, equipped with the even scalar product defined by the super trace (5). We have

$$\mathcal{E}[End_k(U)]((n))=End_k(U)^{\otimes n}$$

and contractions along graphs are defined via contractions with the two-tensor corresponding to the super trace. The structure maps of $\mathcal{E}[End_k(U)]$ are invariant under the $GL(U)$-action. Consider the $GL(U)-$invariant modular suboperad $\mathcal{E}[End_k(U)]^{GL(U)}$. Because of (1) its components for $n<N$ are the same as the components of the operad $\mathcal{S}$

$$\mathcal{E}[End_k(U)]((n))^{GL(U)}\simeq\mathcal{S}((n)).$$

For the space $U' = U \oplus k^{1|1}$, the natural maps $\mathcal{E}[End_k(U)]((n))^{GL(U)} \to \mathcal{E}[End_k(U')]((n))^{GL(U')}$ are isomorphisms for $n < N$. Let us consider the modular operad $\mathcal{E}[End_k]^{GL}$ which is the direct limit of $\mathcal{E}[End_k(U_i)]^{GL(U_i)}$, $\dim_k U_i = (N_i|N_i)$, $N_i \to \infty$:

$$\mathcal{E}[End_k]^{GL} = \lim_{\to} \mathcal{E}[End_k(U_i)]^{GL(U_i)}$$

Recall that the basic contraction operators

$$\mu^{\mathcal{S}}_{ff'} : \mathcal{S}((I \sqcup \{f, f'\})) \to \mathcal{S}((I))$$

are defined for the modular operad $\mathcal{S}$ as the linear maps

$$k[Aut(I \sqcup \{f, f'\})] \to k[Aut(I)]$$

defined by on permutations of the set $(I \sqcup \{f, f'\})$ via

$$\begin{aligned} (\rho_1 \dots \rho_{p-1} f \rho_{p+1} \dots \rho_r) \dots (\tau_1 \dots \tau_{q-1} f' \tau_{q+1} \dots \tau_t) \quad \to \\ \to (\rho_1 \dots \rho_{p-1} \rho_{p+1 \dots} \rho_r) \dots (\tau_1 \dots \tau_{q-1} \tau_{q+1} \dots \tau_t) \end{aligned}$$

if the elements f and f' are in the different cycles of the permutation, and via

$$\begin{aligned} (\rho_1 \dots \rho_{p-1} f \rho_{p+1} \dots \rho_{q-1} f' \rho_{q+1} \dots \rho_r) \dots (\tau_1 \dots \tau_t) \quad \to \\ \to (\rho_1 \dots \rho_{p-1} \rho_{p+1} \dots \rho_{q-1} \rho_{q+1} \dots \rho_r) \dots (\tau_1 \dots \tau_t) \end{aligned} \tag{10}$$

$$(\rho_1 \dots \rho_{p-1} f f' \rho_{p+1} \dots \rho_r) \dots (\tau_1 \dots \tau_t) \to 0 \tag{11}$$

if the elements f and f' are in the same cycle of the permutation.

Proposition 7 *The modular operad $\mathcal{S}$ is isomorphic to the modular operad $\mathcal{E}[End_k]^{GL}$*

Proof. The proof consists essentually of the same calculations as in the proof of the theorem 4. In particular the condition (9) implies (11). ■

4 Even inner product.

In the case of even inner product the space F, on which the equation of the quantum A_∞−algebra is defined, is

$$F = Symm(\oplus_{j=0}^{\infty} \Pi((\Pi V)^{\otimes j})^{\mathbb{Z}/j\mathbb{Z}})$$

with components

$$F_n = ((\Pi V)^{\otimes n} \otimes k[\mathbb{S}_n]')^{\mathbb{S}_n}$$

where $k[\mathbb{S}_n]'$ is the vector space with the basis indexed by elements (σ, ρ_σ), where $\sigma \in \mathbb{S}_n$ is a permutation with i_σ cycles σ_α and $\rho_\sigma = \sigma_1 \wedge \dots \wedge \sigma_{i_\sigma}$, $\rho_\sigma \in Det(Cycle(\sigma))$, $Det(Cycle(\sigma)) = Symm^{i_\sigma}(k^{0|i_\sigma})$, is one of the generators of the one-dimensional determinant of the set of cycles of σ, i.e. ρ_σ is an order on the set of cycles defined up to even reordering, and $(\sigma, -\rho_\sigma) = -(\sigma, \rho_\sigma)$.

If the space V has an even inner product, then F has canonically defined differential Δ, see ([B1],[B2]).

Let us consider again the $\mathbb{Z}/2\mathbb{Z}$-graded vector space U which has even and odd components of the same dimension. Let us denote by $p \in End(U)$, $p^2 = 1$, an odd involution. It acts by interchanging isomorphically $U^{1|0}$ with $U^{0|1}$. Our basic algebra with trace in the case of even inner product is the subalgebra of $End(U)$, of operators commuting with p:

$$\theta(U) = \{G \in End(U) |\, [G, p] = 0\}.$$

It looks as follows in the standard block decomposition of supermatrices:

$$G = \begin{pmatrix} X & \Xi \\ -\Xi & X \end{pmatrix}$$

We have isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces: $\theta(U) = \Pi T^* End(U^{1|0})$,

$$\theta(U) = End(U^{1|0}) \oplus \Pi End(U^{1|0}).$$

The basic property of $\theta(U)$ is that it has an *odd* analog of trace functional:

$$otr(G) = \frac{1}{2} tr(Gp) = \frac{1}{2} tr(pG) = tr\Xi$$

which gives canonical *odd* invariant inner product on $\theta(U)$:

$$\langle G, G' \rangle = otr(G \circ G') = tr(X\Xi') + tr(\Xi X').$$

This odd inner product on $\theta(U)$ together with even inner product on V defines the natural odd inner product on the tensor product $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$\theta(U) \bigotimes V$$

Therefore, as in the previoius case, this space is an affine space with constant odd symplectic structure and therefore its algerbra of symmetric tensors $\oplus_{n=0}^{\infty} S^n(\theta(U) \otimes V)$ has natural structure of Batalin-Vilkovisky algebra.

The canonical super group acting on $\theta(U)$ and its tensor powers is the subgroup $\Theta(U) \subset GL(U)$, preserving the odd involution p:

$$\Theta(U) = \{g \in GL(U) | gpg^{-1} = p\}.$$

Proposition 8 *We have*

$$Hom(\theta(U)^{\otimes n}, k)_{\Theta(U)} = k[\mathbb{S}_n]'$$

for $\dim U = (N|N)$ *sufficiently big,* $N > n$

Proof. The basis for coinvariants of the $\Theta(U)$ action on $U \otimes Hom(U,k)$ is $\sum_a e_a \otimes e^a$, $\sum_a (pe_a) \otimes e^a$, $\sum_a e_a \otimes (e^a p)$. Therefore on the space of tensors $(U \otimes Hom(U,k))^{\otimes n}$ the space of $\Theta(U)-$coinvariants is spanned by all possible combinations of these three elements of the form

$$\sum_{a_1,\ldots,a_n} (e_{a_{\sigma(1)}} \otimes e^{a_1}) \otimes \ldots \otimes (pe_{a_{\sigma(i)}} \otimes e^{a_i}) \otimes \ldots \otimes (e_{a_{\sigma(j)}} \otimes (e^{a_j} p)) \otimes \ldots,$$

where $\sigma \in \mathbb{S}_n$. Such element corresponds to an arbitrary permutation $\sigma \in \mathbb{S}_n$ and the marking, which associates one of the three types of tensors $(e_{a_{\sigma(i)}} \otimes e^{a_i})$, $(pe_{a_{\sigma(i)}} \otimes e^{a_i})$, or $(e_{a_{\sigma(i)}} \otimes e^{a_i} p)$ to every $i \in \{1,\ldots,n\}$. If the cycle decomposition of $\sigma$is denoted by $(\rho_1 \ldots \rho_r) \ldots (\tau_1 \ldots \tau_t)$, then such an element gives on $End(U)^{\otimes n}$ the linear functional of the following type:

$$tr(A_{\rho_1} \ldots A_{\rho_{i-1}} p A_{\rho_i} \ldots A_{\rho_{j-1}} p A_{\rho_j} \ldots A_{\rho_r}) \cdot \ldots \cdot tr(A_{\tau_1} \ldots A_{\tau_t})$$

consisting of products of traces of compositions of the endomorphisms with arbitrary inclusions of the operator p. The operator p commutes with any matrix $A_{\rho_j} \in \theta(U)$. Therefore on $\theta(U)^{\otimes n}$ all inclusions of p inside the given trace cancel with each other except for possibly one:

$$tr(A_{\rho_1} \ldots A_{\rho_{i-1}} p A_{\rho_i} \ldots A_{\rho_r}) = tr(p^l A_{\rho_1} \ldots A_{\rho_r})$$

where $l = 0$ or $l = 1$ depending on the parity of the total number of inclusions of p. Notice now that for any $A \in \theta(U)$, $tr(A) = 0$. And therefore the traces with even number of inclusions of p vanish on $\theta(U)^{\otimes n}$. The trace with odd number of inclusions of p becomes the odd trace otr when restricted to $\theta(U)$. We see that the $\Theta(U)-$coinvariants in $\mathrm{Hom}(\theta(U)^{\otimes n}, k)$ are spanned by the products of odd traces:

$$otr(A_{\rho_1} \ldots A_{\rho_r}) \cdot \ldots \cdot otr(A_{\tau_1} \ldots A_{\tau_t}), \quad A_i \in \theta(U).$$

It follows easily from the corresponding result for gl, that these products of odd traces are linearly independent for $N > n$. ■

The super group $\Theta(U)$ preserves the odd trace otr and therefore the invariants subspace $\oplus_{n=0}^{\infty} S^n(\theta(U) \otimes V)^{\Theta(U)}$ inherits the natural Batalin-Vilkovisky algebra structure.We have now the following analogs of the propositions 1, 5 and of the theorem 4. The proofs are completely analogous to the proofs in the odd inner product case.

Proposition 9 *The vector space F_n is canonically identified with $\Theta(U)$-invariant subspace of $n-$symmetric powers of the vector space $\theta(U) \otimes V$:*

$$F_n \simeq \left(S^n(\theta(U) \otimes V)\right)^{\Theta(U)}, \tag{12}$$

where U is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with $\dim_k U = (N|N)$, $N \geq n$.

Theorem 10 *The operator* Δ *defined on the* $\Theta(U)-$*invariant*

$$\Delta : (S^n(\theta(U)\otimes V))^{\Theta(U)} \to \left(S^{n-2}(\theta(U)\otimes V)\right)^{\Theta(U)},$$

where $\dim_k U = (N|N)$ *is sufficiently big* ($N \geq n$)*, coincides with the differential* $\Delta : F_n \to F_{n-2}$ *defined in ([B1],[B2]).*

Proposition 11 *The odd symplectic bracket on the* $\Theta(U)-$*invariant subspaces*

$$S^n(\theta(U)\otimes V)^{GL(U)} \otimes S^{n'}(\theta(U)\otimes V)^{GL(U)} \to S^{n+n'-2}(\theta(U)\otimes V)^{GL(U)}$$

coincides with the standard odd symplectic bracket (see references in loc.cit):

$$F_n \otimes F_{n'} \to F_{n+n'-2}.$$

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