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Antoine Mellet, Julien Vovelle

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Abstract

We study a class of semi-linear mean curvature equations $\mathcal{M}u = H + \lambda f(u)$ where $\mathcal{M}$ is the mean curvature operator. We show that there exists an extremal parameter λ^* such that this equation admits a minimal weak solutions for all $\lambda \in [0, \lambda^*]$, while no weak solutions exists for $\lambda > \lambda^*$. In the radial case, we then show that minimal solutions are classical solutions for all $\lambda \in [0, \lambda^*]$ and that another branch of solution exists in a neighborhood $[\lambda_* - \eta, \lambda^*]$ of λ^*.

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The aim of this paper is to study the existence and regularity of non-negative solutions for the following mean-curvature problem:

$$\begin{cases} -\mathrm{div}(Tu) = H + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

$$Tu := \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}$$

and

$$f(u) = |u|^{p-1}u, \qquad p \geq 1.$$

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Formally, Equation (1) is the Euler-Lagrange equation for the minimization of the functional

$$\mathscr{F}_\lambda(u) := \int_\Omega \sqrt{1+|\nabla u|^2} - \int_\Omega (Hu + \lambda F(u))\, dx + \int_{\partial\Omega} |u|\, dH^{n-1}(x) \qquad (2)$$

with $F(u) = \frac{1}{p+1}|u|^{p+1}$ (convex function).

When $\lambda = 0$, Problem (1) reduces to a simple mean-curvature problem, which has been studied by many mathematicians (see for instance Bernstein [Ber10], Finn [Fin65], Giaquinta [Gia74], Massari [Mas74] or Giusti [Giu76, Giu78]). In particular, it is well known that a necessary condition for the existence of a minimizer in that case is

$$\left|\int_A H\, dx\right| < P(A), \text{ for all proper subset } A \text{ of } \Omega, \qquad (3)$$

where $P(A)$ is the perimeter of A (see (6) for the definition of the perimeter), and Giaquinta [Gia74] proves that the following is a sufficient condition:

$$\left|\int_A H\, dx\right| \leq (1-\varepsilon_0)P(A), \text{ for all measurable set } A \subset \Omega, \qquad (4)$$

for some $\varepsilon_0 > 0$.

In that respect, the mean-curvature equation (1) is very different from the Laplace equation (or any other uniformly elliptic equations), which would have a solution for any $H \in L^1(\Omega)$.

Equation (1) has also been studied for $\lambda < 0$ and $p = 1$ ($f(u) = u$), in particular in the framework of capillary surfaces (in that case, the Dirichlet condition is often replaced by a Neumann condition, see R. Finn [Fin86]). The existence of minimizers of (2) when $\lambda < 0$ is proved, for instance, by Giusti [Giu76] and M. Miranda [Mir64].

In this paper, we are interested in the case $\lambda > 0$, which, to our knowledge, has not been thoroughly investigated. One of the main difficulty in that case is that the functional $\mathscr{F}_\lambda$ is no longer convex (in fact, it is not bounded below and global minimizers clearly do not exists). However, under certain assumptions on H (which guarantee the existence of a solution for $\lambda = 0$), it is still possible to show that solutions of (1) exist for small values of λ. We will then show that there exists an extremal parameter λ^* such that (1) admits a minimal weak solutions u_λ for all $\lambda \in [0, \lambda^*]$, while no weak solutions exists for $\lambda > \lambda^*$. Here minimal solution means smallest and weak solutions will be defined as critical points of the energy functional that satisfy the boundary condition (see (2.1) for a precise definition). We will also show that minimal solutions are uniformly bounded in L^∞ by a constant depending only on Ω and the dimension.

We also investigate the regularity of the minimal solutions, and prove that in the radially symmetric case, then $\{u_\lambda\, ;\, 0 \leq \lambda \leq \lambda^*\}$ is a branch of classical solutions (here classical means Lipschitz). We stress out the fact that, the extremal solution u_{λ^*}, which is the increasing limit of u_λ as $\lambda \to \lambda^*$, is itself a classical solution. This implies the existence of another branch of (non-minimal) solutions for λ in a neighborhood $[\lambda^* - \eta, \lambda^*]$ of λ^*.

Problem (1) has to be compared with the following classical problem:

$$\begin{cases} -\Delta u = g_\lambda(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

It is well known that if $g_\lambda(u) = \lambda f(u)$, with f superlinear, then there exists a critical value $\lambda^* \in (0, \infty)$ for the parameter λ such that one (or more) solution exists for $\lambda < \lambda^*$, a unique weak solution u^* exists for $\lambda = \lambda^*$ and there is no solution for $\lambda > \lambda^*$ (see [CR75]). And one of the key issue in the study of (5) is whether the extremal solution u^* is a classical solution (in that case, classical means bounded) or u_λ blows up when $\lambda \to \lambda^*$ (see [KK74, BCMR96, MR96, Mar97]).

Classical examples that have been extensively studied include power growth $g_\lambda(u) = \lambda(1+u)^p$ and the celebrated Gelfand problem $g_\lambda(u) = \lambda e^u$ (see [JL73, MP80, BV97]). For such nonlinearities, the minimal solutions, including the extremal solution u^* can be proved to be classical, at least in low dimension. In particular, for $g_\lambda(u) = \lambda(1+u)^p$, u^* is regular if

$$n - 2 < F(p) := \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$$

(see Mignot-Puel [MP80]) while when $\Omega = B_1$ and $n-2 \geq F(p)$, it can be proved that $u^* \sim Cr^{-2}$ (see Brezis-Vázquez [BV97]). For very general nonlinearities of the form $g_\lambda(u) = \lambda f(u)$ with f superlinear, Nedev [Ned00] proves the regularity of u^* in low dimension while Cabré [Cab06] and Cabré-Capella [CC06, CC07] obtain optimal regularity results for u^* for radially symmetric solutions.

Other examples of nonlinearity have been studied, such as $g_\lambda(x,u) = f_0(x,u) + \lambda\varphi(x) + f_1(x)$ (see Berestycki-Lions [BL81]) or $g_\lambda(x,u) = \lambda f(x)/(1-u)^2$ (see Ghoussoub et al. [GG07, EGG07, GG08]).

Despite all this interest for (5), the corresponding problem with the mean-curvature operator does not seem to have been investigated (not even in the case $p = 1$), which is fairly surprising in view of the importance of the minimal surface equation. As it turns out, the behavior of (1) is very different from the analogous Laplace equation. In particular we will see that the extremal solution u^* is always bounded in L^∞ for any values of n and p. Furthermore, u^* is a classical solution (for any values of n and p), at least in the radially symmetric case.

Note finally that the analysis of the corresponding *evolution* problem, at least when $p = 1$, has been performed by Ecker [Eck82]. Note also the analysis in [Ser09] of the Dirichlet Problem for an equation $\mathcal{M}u = f(u, \nabla u)$ where $\mathcal{M}$ is the mean curvature operator.

2 Definitions and Main theorems

We recall that $\mathrm{BV}(\Omega)$ denotes the set of functions in $L^1_{\mathrm{loc}}(\Omega)$ with bounded variation over Ω, i.e.

$$\int_\Omega |Du| := \sup\left\{\int_\Omega u(x)\mathrm{div}(g)(x)\,dx;\, g\in C^1_c(\Omega)^N, |g(x)|\le 1\right\} < +\infty.$$

If A is a Lebesgue subset of $\mathbb{R}^n$, its perimeter $P(A)$ is defined as the variation of its characteristic function φ_A:

$$P(A) := \int_{\mathbb{R}^n} |D\varphi_A|, \quad \varphi_A(x) = \begin{cases} 1 & \text{if } x\in A,\\ 0 & \text{otherwise.}\end{cases} \tag{6}$$

For $u\in\mathrm{BV}(\Omega)$, we define the "area" of the graph of u by

$$\mathscr{A}(u) := \int_\Omega \sqrt{1+|Du|^2} = \sup\left\{\int_\Omega g_0(x) + u(x)\mathrm{div}(g)(x)\,dx\right\}, \tag{7}$$

where the supremum is taken over all functions $g_0\in C^1_c(\Omega), g\in C^1_c(\Omega)^N$ such that $|g_0|+|g|\le 1$ in Ω. An alternative definition is $\mathscr{A}(u) = \int_{\Omega\times\mathbb{R}}|D\varphi_U|$ where U is the subgraph of u. We have, in particular

$$\int_\Omega |Du| \le \int_\Omega \sqrt{1+|Du|^2} \le |\Omega| + \int_\Omega |Du|.$$

A major difficulty, when developing a variational approach to (1), is to deal with the boundary condition. It is well known that even when $\lambda=0$, minimizers of $\mathscr{F}_\lambda$ may not satisfy the homogeneous Dirichlet condition (we need an additional condition on H and the curvature of $\partial\Omega$, see below). Furthermore, the usual technics to handle this issue, which work when $\lambda<0$ do not seem to generalize easily to the case $\lambda>0$. For this reason, we will not use the functional $\mathscr{F}_\lambda$ in our analysis. Instead, we will define the solutions of (1) as the "critical points" (the definition is made precise below) of the functional

$$\mathscr{J}_\lambda(u) := \int_\Omega \sqrt{1+|\nabla u|^2} - \int_\Omega H(x)u + \lambda F(u)\,dx \tag{8}$$

which satisfy $u=0$ on $\partial\Omega$.

By a "critical point" of $\mathscr{J}_\lambda$, we mean a function $u\in L^{p+1}\cap\mathrm{BV}(\Omega)$ solution of the equation

$$\mathcal{L}(u)(\varphi) = \int_\Omega (H+\lambda f(u))\varphi$$

for all $\varphi\in\mathcal{C}^\infty_c(\Omega)$, where

$$\mathcal{L}(u)(\varphi) := \lim_{t\downarrow 0}\frac{1}{t}\left(\mathscr{A}(u+t\varphi)-\mathscr{A}(u)\right) \tag{9}$$

(note that this limit exists for every $\varphi \in \mathrm{BV}(\Omega)$ since the area functional is convex).

It is readily seen that $\mathcal{L}(u)$ is locally bounded on $\mathrm{BV}(\Omega)$. However it is not linear on $\mathrm{BV}(\Omega)$, since, for any set A with finite perimeter, we have $\mathcal{L}(u)(\varphi_A) = \mathcal{L}(u)(-\varphi_A) = P(A)$. Nevertheless, the application $\varphi \mapsto \mathcal{L}(u)(\varphi)$ is linear if u and φ have enough regularity. For instance, if $u \in W^{1,1}(\Omega)$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then we have:

$$\mathcal{L}(u)(\varphi) = \int_\Omega Tu \cdot \nabla\varphi dx, \quad Tu := \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}.$$

Finally, we note that $\mathcal{L}(u)$ is also convex and locally Lipschitz continuous on $L^1 \cap \mathrm{BV}(\Omega)$.

With this definition of $\mathcal{L}(u)$, it is readily seen that local minimizers of $\mathscr{A}(u) - \int_\Omega Hu\,dx$ in $L^1 \cap \mathrm{BV}(\Omega)$ satisfy

$$\mathcal{L}(u)(\varphi) \geq \int_\Omega H\varphi \qquad \text{for all } \varphi \in L^1 \cap \mathrm{BV}(\Omega)$$

with equality if u and φ are smooth enough (but with strict inequality, for instance, if $\varphi = \varphi_A$). We thus take the following definition:

Definition 2.1. *A measurable function $u\colon \Omega \to [0,+\infty]$ is said to be a weak solution of Problem (1) if it satisfies*

$$\begin{cases} u \in L^{p+1} \cap \mathrm{BV}(\Omega), & u \geq 0 \\ \mathcal{L}(u)(\varphi) \geq \int_\Omega (H+\lambda f(u))\varphi, & \forall \varphi \in L^{p+1} \cap \mathrm{BV}(\Omega) \text{ with } \varphi = 0 \text{ on } \partial\Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

When u and φ are smooth enough (so that $\mathcal{L}(u)(-\varphi) = -\mathcal{L}(u)(\varphi)$), then the following equality holds:

$$\mathcal{L}(u)(\varphi) = \int_\Omega (H+\lambda f(u))\varphi.$$

Note that the boundary condition makes sense here because functions in $L^1 \cap \mathrm{BV}(\Omega)$ have a unique trace in $L^1(\partial\Omega)$ provided $\partial\Omega$ is Lipschitz (see [Giu84]).

Before we state our main result, we recall the following theorem concerning the case $\lambda = 0$, which plays an important role in the sequel:

Theorem 2.2 (Giaquinta [Gia74])**.**

(i) Let Ω be a bounded domain with Lipschitz boundary and assume that $H(x)$ is a measurable function such that (4) holds for some $\varepsilon_0 > 0$. Then the functional

$$\mathscr{F}_0(u) := \int_\Omega \sqrt{1+|\nabla u|^2} - \int_\Omega H(x)u(x)\,dx + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1}$$

has a minimizer u in $\mathrm{BV}(\Omega)$.

(ii) Furthermore, if $\partial\Omega$ is $\mathcal{C}^1$, $H(x) \in \mathrm{Lip}(\overline{\Omega})$ and

$$|H(y)| \leq (n-1)\Gamma(y) \quad \text{for all } y \in \partial\Omega \tag{10}$$

where $\Gamma(y)$ denotes the mean curvature of $\partial\Omega$ (with respect to the inner normal), then the unique minimizer of $\mathscr{F}_0$ belongs to $\mathcal{C}^{2,\alpha}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ for all $\alpha \in [0,1)$ and is a classical solution of

$$\begin{cases} -\mathrm{div}(Tu) = H & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

(iii) Finally, if $\partial\Omega$ is $\mathcal{C}^3$ and the hypotheses of (ii) hold, then $u \in \mathrm{Lip}(\overline{\Omega})$.

The key in the proof of *(i)* is the fact that (4) and the coarea formula for BV functions yield

$$\varepsilon_0 \|u\|_{\mathrm{BV}(\Omega)} \leq \int_\Omega |Du| - \int_\Omega H(x)u(x)\,dx$$

for all $u \in \mathrm{BV}(\Omega)$. This is enough to guarantee the existence of a minimizer. The condition (10) is a sufficient condition for the minimizer to satisfy $u = 0$ on $\partial\Omega$. In the sequel, we assume that Ω is such that (4) holds, as well as the following strong version of (10):

$$|H(y)| \leq (1-\varepsilon_0)(n-1)\Gamma(y) \quad \text{for all } y \in \partial\Omega \tag{12}$$

Remark 2.3. *When $H(x) = H_0$ is constant, Serrin proves in [Ser69] that (10) is necessary for the equation $-\mathrm{div}(Tu) = H$ to have a solution for any smooth boundary data. However, it is easy to see that (10) is not always necessary for (11) to have a solution: when $\Omega = B_R$ and $H = \frac{n}{R}$, (11) has an obvious solution given by an upper half sphere, even though (10) does not hold since $(n-1)\Lambda = (n-1)/R < H = n/R$.*

Several results in this paper only require Equation (11) to have a solution with $(1+\delta)H$ in the right hand side instead of H. In particular, this is enough to guarantee the existence of a minimal branch of solutions and the existence of an extremal solution. When $\Omega = B_R$, we can thus replace (12) with

$$|H(y)| \leq (1-\varepsilon_0)n\Gamma(y) \quad \text{for all } y \in \partial B_R.$$

However, the regularity theory for the extremal solution will require the stronger assumption (12).

Finally, we assume that there exists a constant $H_0 > 0$ such that:

$$H \in \mathrm{Lip}(\overline{\Omega}) \text{ and } H(x) \geq H_0 > 0 \text{ for all } x \in \Omega. \tag{13}$$

This last condition will be crucial in the proof of Lemma 4.2 to prove the existence of a non-negative solution for small values of λ. Note that P. Pucci and J. Serrin [PS86] proved that if $H = 0$ and $p \geq (n+2)/(n-2)$, then (1) has no non-trivial solutions for any values of $\lambda \geq 0$ when Ω is star-shaped.

Our main theorem is the following:

Theorem 2.4. *Let Ω be a bounded subset of $\mathbb{R}^n$ such that $\partial\Omega$ is $\mathcal{C}^3$. Assume that $H(x)$ satisfies conditions (4), (12) and (13). Then, there exists $\lambda^* > 0$ such that:*

1. *if $\lambda > \lambda^*$, there is no (weak) solution to Problem (1),*
2. *if $\lambda \leq \lambda^*$, there is at least one minimal (weak) solution to Problem (1).*

The proof of Theorem 2.4 is done in two steps: First we show that the set of λ for which a weak solution exists is a non empty bounded interval (see Section 4). Then we prove the existence of the extremal solution for $\lambda = \lambda^*$ (see Section 6). The key result in this second step is the following uniform L^∞ estimate:

Proposition 2.5. *There exists a constant C depending only on Ω and H, such that the minimal solution u_λ of (P_λ) satisfies*

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C \qquad \textit{for all } \lambda \in [0, \lambda^*].$$

Next we investigate the regularity of minimal solutions: We want to show that minimal solutions are classical solutions of (1). In the case of the Laplace operator (5), this would be an immediate consequence of the L^∞-bound. In our case, classical results of the calculus of variation (see [Mas74]), imply that for $n \leq 6$, the surface $(x, u_\lambda(x))$ is regular (analytic if H is analytic) and that u_λ is continuous almost everywhere in Ω. However, to get further regularity, we need to show that u_λ is Lipschitz continuous in $\overline{\Omega}$.

This, it seems, is a much more challenging problem and we obtain some results only in the radially symmetric case. More precisely, we show the following:

Theorem 2.6. *Assume that $\Omega = B_R \subset \mathbb{R}^n$ $(n \geq 1)$, $H = H(r)$, and that the conditions of Theorem 2.4 hold. Then the minimal solution of (P_λ) is radially symmetric, and there exists a constant C such that*

$$|\nabla u_\lambda(x)| \leq \frac{C}{\lambda^* - \lambda} \qquad \textit{in } \Omega \ \forall \lambda \in [0, \lambda^*). \tag{14}$$

In particular u_λ is a classical solution of (1), and if $H(x)$ is analytic in Ω, then u_λ is analytic in Ω for all $\lambda < \lambda^*$.

Note that the Lipschitz constant in (14) blows up as $\lambda \to \lambda^*$. However, we are then able to show the following:

Theorem 2.7. *Assume that the conditions of Theorem 2.6 hold. Then there exists a constant C such that for any $\lambda \in [0, \lambda^*]$, the minimal solution u_λ satisfies*

$$\|\nabla u_\lambda\|_{L^\infty(\Omega)} \leq C.$$

In particular the extremal solution u^* is a classical solution of (1).

It is now fairly classical to show, using Crandall-Rabinowitz's continuation theory [CR75], that there exists a second branch of solution in the neighborhood of λ^*:

Theorem 2.8. *If the extremal solution u_{λ^*} is Lipschitz continuous, then there exists $\lambda_* \in (0, \lambda^*)$ such that for $\lambda_* < \lambda < \lambda^*$ there is at least two solutions to Problem (P_λ).*

In the radially symmetric case, we can thus summarize our results in the following corollary:

Corollary 2.9. *Assume that $\Omega = B_R \subset \mathbb{R}^n$ ($n \geq 1$), $H = H(r)$, and that the conditions of Theorem 2.4 hold. Then there exists $\lambda^* > \lambda_* > 0$ such that*

1. *if $\lambda > \lambda^*$, there is no weak solution to Problem (P_λ),*
2. *if $\lambda \leq \lambda^*$, there is at least one minimal classical solution to Problem (P_λ).*
3. *if $\lambda_* < \lambda < \lambda^*$, there is at least two classical solutions to Problem (P_λ).*

Finally, we point out that numerical computation suggest that for some values of n and H, a third branch of solutions may arise (and possibly more).

The paper is organized as follows: In Section 3, we give some a priori properties of weak solutions. In Section 4 we show the existence of a branch of minimal solutions for $\lambda \in [0, \lambda^*)$. We then establish, in Section 5, a uniform L^∞ bound for these minimal solutions (Proposition 2.5), which we use, in Section 6, to show the existence of an extremal solution as $\lambda \to \lambda^*$ (thus completing the proof of Theorem 2.4). In the last Section 7 we prove the regularity of the minimal solutions, including that of u_{λ^*}, in the radial case (Theorems 2.6 and 2.7). In appendix, we prove a comparison lemma that is used several times in the paper.

Remark 2.10. *It might be interesting to consider other right hand sides in (1):*

(i) For right hand sides of the form $H + \lambda f(u)$, all the results presented here still holds (with the same proofs) if f is a $\mathcal{C}^2$ function satisfying:

(H1) $f(0) = 0$, $f'(u) \geq 0$ *for all* $u \geq 0$.

(H2) There exists C and $\alpha > 0$ such that $f'(u) \geq \alpha$ for all $u \geq C$.

(H3) If $u \in L^q(\Omega)$ for all $q \in [0, \infty)$ then $f(u) \in L^n(\Omega)$.

The last condition, which is used to prove the L^∞ bound (and the Lipschitz regularity near $r = 0$) of the extremal solution u_{λ^*} is the most restrictive. It excludes in particular nonlinearities of the form $f(u) = e^u - 1$. However, similar results hold also for such nonlinearities, though the proof of Proposition 2.5 has to be modified in that case. This will be developed in a later work.

(ii) We can also consider right hand sides of the form $\lambda(1+u)^p$. In that case, every results above hold with very similar proofs, except for the boundary regularity of the extremal solution u_{λ^*} (Lemma 7.3) which relies heavily on condition (12).

3 Properties of weak solutions

3.1 Weak solutions as global minimizers

Non-negative minimizers of $\mathscr{J}_\lambda$ that satisfy $u = 0$ on $\partial\Omega$ are in particular critical points of $\mathscr{J}_\lambda$, and thus solutions of (P_λ). But not all critical points are minimizers. However, the convexity of the perimeter yields the following result:

Lemma 3.1. *Assume that $\partial\Omega$ is $\mathcal{C}^1$ and let u be a non-negative function in $L^{p+1} \cap \mathrm{BV}(\Omega)$. The following propositions are equivalent:*

(i) u is a weak solution of (P_λ),

(ii) $u = 0$ on $\partial\Omega$ and for every $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$, we have

$$\mathscr{A}(u) - \int_\Omega (H + \lambda f(u))\, u\, dx \le \mathscr{A}(v) - \int_\Omega (H + \lambda f(u))\, v\, dx + \int_{\partial\Omega} |v|\, d\mathscr{H}^{N-1},$$

(iii) $u = 0$ on $\partial\Omega$ and for every $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$, we have

$$\mathscr{J}_\lambda(u) \le \mathscr{J}_\lambda(v) + \int_\Omega \lambda G(u,v)\, dx + \int_{\partial\Omega} |v|\, d\mathscr{H}^{N-1}$$

where

$$G(u,v) = F(v) - F(u) - f(u)(v-u) \ge 0.$$

In particular, any weak solution u of (P_λ) is a global minimizer in $L^{p+1} \cap \mathrm{BV}(\Omega)$ of the functional (which depends on u)

$$\mathscr{F}_\lambda^{[u]}(v) := \mathscr{J}_\lambda(v) + \int_{\partial\Omega} |v|\, d\mathscr{H}^{N-1} + \int_\Omega \lambda G(u,v)\, dx.$$

Proof. The last two statements are clearly equivalent and *(iii)* immediately implies *(i)* since $\mathscr{F}_\lambda^{[u]}(u) = \min_{v \in L^2 \cap \mathrm{BV}(\Omega)} \mathscr{F}_\lambda^{[u]}(v)$ implies $\mathscr{J}_\lambda'(u) = (\mathscr{F}_\lambda^{[u]})'(u) = 0$.

So we only have to show that *(i)* implies $\mathscr{F}_\lambda^{[u]}(u) = \min_{v \in L^2 \cap \mathrm{BV}(\Omega)} \mathscr{F}_\lambda^{[u]}(v)$. By definition of weak solutions, (i) implies $\mathscr{J}_\lambda'(u) = 0$, *i.e.*

$$\mathcal{L}(u)(\varphi) \ge \int_\Omega (H + \lambda f(u))\varphi\, dx$$

for all $\varphi \in L^{p+1} \cap \mathrm{BV}(\Omega)$ with $\varphi = 0$ on $\partial\Omega$. Furthermore, by convexity of $\mathscr{A}(u)$ and the definition of $\mathcal{L}(u)$ (see (9)), we have

$$\mathscr{A}(u) + \mathcal{L}(u)(v-u) \leq \mathscr{A}(v),$$

for every $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$ with $v = 0$ on $\partial\Omega$. We deduce (using 2.1):

$$\mathscr{A}(u) + \int_\Omega (H + \lambda f(u))(v-u)\, dx \leq \mathscr{A}(v),$$

which implies

$$\mathscr{F}_\lambda^{[u]}(u) \leq \mathscr{F}_\lambda^{[u]}(v) \tag{15}$$

for all $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$ satisfying $v = 0$ on $\partial\Omega$.

It thus only remains to show that (15) holds even when $v \neq 0$ on $\partial\Omega$. For that, the idea is to apply (15) to the function $v - w^\varepsilon$ where (w^ε) is a sequence of functions in $L^{p+1} \cap \mathrm{BV}(\Omega)$ converging to 0 in $L^{p+1}(\Omega)$ such that $w^\varepsilon = v$ on $\partial\Omega$. Heuristically the mass of w^ε concentrates on the boundary $\partial\Omega$ as ε goes to zero, and so $\mathscr{A}(v - w^\varepsilon)$ converges to $\mathscr{A}(v) + \int_{\partial\Omega} |v| d\mathscr{H}^{N-1}$. This type of argument is fairly classical, but we give a detailed proof below, in particular to show how one can pass to the limit in the non-linear term.

First, we consider $v \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$. Then, for every $\varepsilon > 0$, there exists $w^\varepsilon \in \mathrm{BV}(\Omega)$ such that $w^\varepsilon = v$ on $\partial\Omega$ and satisfying the estimates:

$$\|w^\varepsilon\|_{L^1(\Omega)} \leq \varepsilon \int_{\partial\Omega} |v| d\mathscr{H}^{N-1}, \qquad \int_\Omega |Dw^\varepsilon| \leq (1+\varepsilon) \int_{\partial\Omega} |v| d\mathscr{H}^{N-1}$$

and $\|w^\varepsilon\|_{L^\infty(\Omega)} \leq 2\|v\|_{L^\infty(\Omega)}$ (see Theorem 2.16 in [Giu84]). In particular we note that

$$\|w^\varepsilon\|_{L^{p+1}(\Omega)}^{p+1} \leq 2^p \|v\|_{L^\infty(\Omega)}^p \|w^\varepsilon\|_{L^1(\Omega)} \to 0 \tag{16}$$

when $\varepsilon \to 0$. Using (15) and the fact that $\mathscr{A}(v - w^\varepsilon) \leq \mathscr{A}(v) + \int_\Omega |Dw^\varepsilon|$, we deduce:

$$\begin{aligned}
\mathscr{F}_\lambda^{[u]}(u) &\leq \mathscr{F}_\lambda^{[u]}(v - w^\varepsilon) \\
&\leq \mathscr{J}_\lambda(v) + \int_\Omega |Dw| + \lambda \int_\Omega F(v) - F(v - w^\varepsilon)\, dx + \int_\Omega \lambda G(u, v - w^\varepsilon) \\
&= \mathscr{J}_\lambda(v) + \int_\Omega \lambda G(u,v) + \int_{\partial\Omega} |v| d\mathscr{H}^{N-1} + \int_\Omega f(u) w^\varepsilon\, dx \\
&= \mathscr{F}_\lambda^{[u]}(v) + \int_\Omega f(u) w^\varepsilon\, dx.
\end{aligned}$$

Taking the limit $\varepsilon \to 0$, using (16) and the fact that $f(u) \in L^{\frac{p+1}{p}}$ if $u \in L^{p+1}$, we obtain (15) for any $v \in L^\infty \cap \mathrm{BV}(\Omega)$.

We now take $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$. Then, the computation above shows that for every $M > 0$ we have:

$$\mathscr{F}_\lambda^{[u]}(u) \leq \mathscr{F}_\lambda^{[u]}(T_M(v)),$$

where T_M is the truncation operator $T_M(s) := \min(M, \max(s, -M))$. Clearly, we have $T_M(v) \to v$ in $L^{p+1}(\Omega)$. Furthermore, one can show that $\mathscr{A}(T_M(v)) \to \mathscr{A}(v)$. As a matter of fact, the lower semi-continuity of the perimeter gives $\mathscr{A}(v) \le \liminf \mathscr{A}(T_M(v))$, and the coarea formula implies:

$$\begin{aligned}\mathscr{A}(T_M(v)) &\le \mathscr{A}(v) + \int_\Omega |D(v - T_M(v))| \\ &= \mathscr{A}(v) + \int_0^{+\infty} P(\{v - T_M(v) > t\})dt \\ &= \mathscr{A}(v) + \int_M^{+\infty} P(\{v > t\})dt \\ &\longrightarrow \mathscr{A}(v) \text{ when } M \to +\infty.\end{aligned}$$

We deduce that $\mathscr{F}_\lambda^{[u]}(T_M(v)) \longrightarrow \mathscr{F}_\lambda^{[u]}(v)$, and the proof is complete. □

3.2 A priori bounds

Next, we want to derive some a priori bounds satisfied by any weak solutions u of (P_λ).

First, we have the following lemma:

Lemma 3.2. *Let u be a weak solution of (P_λ), then*

$$\int_A H + \lambda f(u)\, dx \le P(A)$$

for all measurable sets $A \subset \Omega$.

Proof. When u is smooth, this lemma can be proved by integrating (1) over the set A and noticing that $|\frac{\nabla u \cdot \nu}{\sqrt{1+|\nabla u|^2}}| \le 1$ on ∂A. If u is not smooth, we use Lemma 3.1 (ii): For all $A \subset \Omega$, we get (with $v = \varphi_A$):

$$\mathscr{A}(u) - \int_\Omega [H + \lambda f(u)]u \le \mathscr{A}(u + \varphi_A) - \int_\Omega [H + \lambda f(u)](u + \varphi_A) + \mathscr{H}^1(\partial\Omega \cap A).$$

We deduce

$$0 \le \int_\Omega |D\varphi_A| + \mathscr{H}^1(\partial\Omega \cap A) - \int_A H + \lambda f(u)\, dx.$$

and so

$$0 \le P(A) - \int_A H + \lambda f(u)\, dx.$$ □

Lemma 3.2 suggests that λ can not be too large for (P_λ) to have a solution. In fact, we can get a bound on λ, provided we can show that $\int_\Omega u\, dx$ is bounded below. This is done in the next lemma:

Lemma 3.3. *Let u be a weak solution of (P_λ) for some $\lambda \geq 0$. Then*

$$u \geq \underline{u} \quad \text{in } \Omega$$

where $\underline{u}$ is the solution corresponding to $\lambda = 0$:

$$\begin{cases} -\operatorname{div}(T\underline{u}) &= H \quad \text{in } \Omega \\ \underline{u} &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{P_0}$$

Proof. For $\delta > 0$, let u_δ be the solution to the problem

$$\begin{cases} -\operatorname{div}(Tu) &= (1-\delta)H \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{P_δ}$$

Problem (P_δ) has a solution $u_\delta \in \mathrm{Lip}(\Omega) \cap L^\infty \cap \mathrm{BV}(\Omega)$ (by Theorem 2.2) and (u_δ) is increasing to $\underline{u}$ when $\delta \to 0$. We also recall [Giu76] that the function u_δ is the unique minimizer in $L^2 \cap \mathrm{BV}(\Omega)$ of the functional

$$\mathscr{F}_\delta(u) = \int_\Omega \sqrt{1+|\nabla u|^2} - \int_\Omega (1-\delta)H(x)u(x)\,dx + \int_{\partial\Omega} |u|.$$

The lemma then follows easily from the comparison principle, Lemma A.1, with $G_-(x,s) = -(1-\delta)H(x)s$, $G_+(x,s) = -H(x)s - \lambda F(s) + \lambda G(u(x),s)$, $K_- = K_+ = L^{p+1} \cap \mathrm{BV}(\Omega)$. More precisely, we get:

$$\begin{aligned} 0 &\leq \int_\Omega -\delta H(\max(u_\delta,u)-u) + \lambda[F(u) - F(\max(u,u_\delta)) + G(u,\max(u,u_\delta))] \\ &= -\int_\Omega (\delta H + \lambda f(u))\,(u_\delta - u)_+, \end{aligned}$$

where $v_+ = \max(v,0)$. Since $H > 0$ and $u \geq 0$ in Ω, this implies $u_\delta \leq u$ a.e. in Ω. Taking the limit $\delta \to 0$, we obtain $\underline{u} \leq u$ a.e. in Ω. $\square$

Finally, we note that $\underline{u}$ is a classical solution of (P_0), and (13) implies $\underline{u} > 0$ in Ω. We deduce

$$\int_\Omega u\,dx \geq \int_\Omega \underline{u}\,dx > 0$$

and so Lemma 3.2 yields the following a priori bounds on λ:

Lemma 3.4. *If (P_λ) has a solution for some $\lambda \geq 0$, then*

$$\lambda \leq \frac{P(\Omega) - \int_\Omega H\,dx}{\int_\Omega \underline{u}\,dx}$$

with $\underline{u}$ solution of (P_0).

4 Existence of minimal solutions for $\lambda \in [0, \lambda^*)$

In this section, we prove the following proposition:

Proposition 4.1. *Suppose that (4), (12) and (13) hold. Then, there exists $\lambda^* \in (0, +\infty)$ such that*

(i) For all $\lambda \in [0, \lambda^*)$, (P_λ) has one minimal solution u_λ.

(ii) For $\lambda > \lambda^*$, (P_λ) has no solution.

(iii) The application $\lambda \mapsto u_\lambda$ is non-decreasing.

To prove Proposition 4.1, we will first show that solutions exist for small values of λ. Then, we will prove that the set of the values of λ for which solutions exist is an interval.

4.1 Existence of u_λ for small values of λ

We start with the following lemma:

Lemma 4.2. *Suppose that (4), (12) and (13) hold. Then there exists $\lambda_0 > 0$ such that (P_λ) has at least one minimal solution for all $\lambda < \lambda_0$.*

Proof. We will show that for small λ, the functional $\mathscr{J}_\lambda$ has a local minimizer in $\mathrm{BV}(\Omega)$ that satisfies $u = 0$ on $\partial\Omega$. Such a minimizer is a critical point for $\mathscr{J}_\lambda$, and thus (see Definition 2.1) a solution of (P_λ).

Let δ be a small parameter such that $(1+\delta)(1-\varepsilon_0) < 1$ where ε_0 is defined by the conditions (4) and (12). Then there exists $\varepsilon' > 0$ such that

$$\left| \int_A (1+\delta) H \, dx \right| \leq (1+\delta)(1-\varepsilon_0)\mathscr{H}^{n-1}(\partial A) \leq (1-\varepsilon')P(A),$$

and

$$|(1+\delta)H(y)| \leq (1-\varepsilon')(n-1)\Gamma(y) \qquad \forall y \in \partial\Omega.$$

Theorem 2.2 thus gives the existence of $w \geq 0$ local minimizer of

$$\mathscr{G}_\delta(u) = \mathscr{A}(u) - \int_\Omega (1+\delta)H(x)u \, dx + \int_{\partial\Omega} |u| \, d\sigma(x).$$

with $w \in \mathcal{C}^{2,\alpha}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ and $w = 0$ on $\partial\Omega$.

It is readily seen that the functional $\mathscr{J}_\lambda$ has a global minimizer u in

$$K = \{v \in L^{p+1} \cap \mathrm{BV}(\Omega)\,;\, 0 \leq v \leq w + 1\}.$$

We are now going to show that if λ is small enough, then u satisfies

$$u(x) \leq w(x) \quad \text{ in } \Omega. \tag{17}$$

For this, we use the comparison principle (Lemma A.1) with $G_-(x,s) = -H(x)s - \lambda F(s)$ and $G_+(x,s) = -(1+\delta)H(x)s$ (i.e. $\mathscr{F}_- = \mathscr{J}_\lambda$ and $\mathscr{F}_+ = \mathscr{G}_\delta$), and $K_- = L^{p+1} \cap \mathrm{BV}(\Omega)$, $K_+ = K$. Since $\max(u,w) \in K$, we obtain

$$\begin{aligned} 0 &\leq \int_\Omega -\delta H(\max(u,w) - w) + \lambda(F(\max(u,w)) - F(w))\,dx \\ &\leq \int_\Omega -\delta H(\max(u,w) - w) + \lambda \sup_{s\in[0,\|w\|_\infty+1]} |f(s)|(\max(u,w) - w)\,dx \\ &\leq \int_\Omega -(u-w)_+ \left[\delta H - f(\|w\|_\infty + 1)\right]\,dx. \end{aligned}$$

Therefore, if we take λ small enough such that $\lambda < \delta \frac{\inf H}{f(\|w\|_\infty+1)} = \delta \frac{H_0}{f(\|w\|_\infty+1)}$, we deduce (17).

Finally, (17) implies that $u = 0$ on $\partial\Omega$ and that u is a critical point of $\mathscr{J}_\lambda$ in $\mathrm{BV}(\Omega)$, which completes the proof. □

4.2 Existence of u_λ for $\lambda < \lambda^*$

We now define

$$\lambda^* = \sup\{\lambda\,;\, (P_\lambda) \text{ has a weak solution}\}.$$

Lemmas 3.4 and 4.2 imply

$$0 < \lambda^* < \infty.$$

In order to complete the proof of Proposition 4.1, we need to show:

Proposition 4.3. *For all $\lambda \in [0,\lambda^*)$ there exists a minimal solution u_λ of (P_λ). Furthermore, the application $\lambda \mapsto u_\lambda$ is nondecreasing.*

Proof of Proposition 4.3. Let us fix $\lambda_0 \in [0,\lambda^*)$. By definition of λ^*, there exists $\overline{\lambda} \in (\lambda_0, \lambda^*]$ such that (P_λ) has a solution $\overline{u} \in L^{p+1} \cap \mathrm{BV}(\Omega)$ for $\lambda = \overline{\lambda}$.

Let also $\underline{u}$ be the solution to (P_0). We then define the sequence u_n as follows: We take

$$u_0 = \underline{u}$$

and for any $n \geq 1$, we set

$$I_n(v) = \mathscr{A}(v) - \int_\Omega [H + \lambda_0 f(u_{n-1})] v\,dx + \int_{\partial\Omega} |v|$$

and let u_n be the unique minimizer of I_n in $L^{p+1} \cap \mathrm{BV}(\Omega)$.

In order to prove Proposition 4.3, we will show that this sequence (u_n) is well defined (i.e. that u_n exists for all n), and that it converges to a solution of (P_λ) with $\lambda = \lambda_0$.

This will be a consequence of the following Lemma:

Lemma 4.4. *For all $n \geq 1$, u_n is well defined (i.e. the functional I_n admits a global minimizer u_n on $L^{p+1} \cap \mathrm{BV}(\Omega)$). Moreover, $u_n \in \mathrm{Lip}(\overline{\Omega})$ and u_n satisfies*

$$\underline{u} \leq u_{n-1} < u_n \leq \overline{u} \text{ in } \Omega. \tag{18}$$

We can now complete the proof of Proposition 4.3: By Lebesgue's monotone convergence Theorem, we have that (u_n) converges almost everywhere and in L^{p+1} to a function u_∞ satisfying

$$0 \leq u_\infty \leq \overline{u}.$$

In particular, it satisfies $u_\infty = 0$ on $\partial\Omega$. Furthermore, for every $n \geq 0$, we have

$$I_n(u_n) \leq I_n(0) = |\Omega|$$

and so

$$\|u_n\|_{\mathrm{BV}} \leq 2|\Omega| + \sup(H)\|\overline{u}\|_{L^1} + \lambda_0 \|\overline{u}\|^{p+1}_{L^{p+1}(\Omega)}$$

hence $u_\infty \in \mathrm{BV}(\Omega)$. Finally, for all $v \in \mathrm{BV}(\Omega)$ and for all $n \geq 1$, we have

$$I_n(u_n) \leq I_n(v)$$

and using the lower semi-continuity of the perimeter, and the strong L^{p+1} convergence, we deduce

$$\begin{aligned}\int_\Omega \sqrt{1+|\nabla u_\infty|^2}\,dx - \int H u_\infty + \lambda f(u_\infty) u_\infty\,dx \\ \leq \int_\Omega \sqrt{1+|\nabla v|^2}\,dx - \int Hv + \lambda f(u_\infty) v\,dx\end{aligned}$$

We conclude, using Lemma 3.1 (ii), that u_∞ is a solution of (P_λ). □

The rest of this section is devoted to the proof of Lemma 4.4:

Proof of Lemma 4.4. We recall that $\underline{u}$ denotes the unique minimizer of $\mathscr{F}_0$ in $\mathrm{BV}(\Omega)$. Note that, by Lemma 3.3, we have the inequality $\underline{u} \leq \overline{u}$ a.e. on Ω.

Assume now that we constructed u_{n-1} satisfying $u_{n-1} \in \mathrm{Lip}(\overline{\Omega})$ and

$$\underline{u} \leq u_{n-1} \leq \overline{u}.$$

We are going to show that u_n exists and satisfies (18) (this implies Lemma 4.4 by first applying the result to $n = 1$ and proceeding from there by induction).

First of all, Lemma 3.2 implies

$$\int_A H + \overline{\lambda} f(\overline{u})\,dx \leq P(A)$$

for all measurable sets $A \subset \Omega$. Since $u_{n-1} \leq \overline{u}$ and $\lambda_0 < \overline{\lambda}$, we deduce that

$$\int_A H + \lambda_0 f(u_{n-1})\,dx < P(A) \tag{19}$$

for all measurable sets $A \subset \Omega$. Following Giusti [Giu78], we can then prove (a proof of this lemma is given at the end of this section):

Lemma 4.5. *There exists $\varepsilon > 0$ such that*

$$\int_A H + \lambda_0 f(u_{n-1})\,dx < (1-\varepsilon)P(A)$$

for all measurable sets $A \subset \Omega$. In particular (4) holds with $\overline{H} = H + \lambda f(u_{n-1})$ instead of H

This lemma easily implies the existence of a minimizer u_n of I_n (using Theorem 2.2 with $\overline{H}$ instead of H). Furthermore, since $u_{n-1} \in \mathrm{Lip}(\overline{\Omega})$ and $u_{n-1} = 0$ on $\partial\Omega$ condition (10) is satisfied with $\overline{H}$ instead of H and so (by Theorem 2.2):

$$u_n = 0 \text{ on } \partial\Omega$$

and

$$u_n \in \mathrm{Lip}(\overline{\Omega}).$$

Finally, we check that the minimizer u_n satisfies

$$\underline{u} \leq u_n \leq \overline{u}.$$

Indeed, the first inequality is a consequence of the comparison Lemma A.1 applied to $\mathscr{F}_- = \mathscr{F}_0$, $\mathscr{F}_+ = I_n$, $K_+ = K_- = L^{p+1} \cap \mathrm{BV}(\Omega)$, which gives

$$0 \leq -\int_\Omega \lambda_0 f(u_{n-1})(\max(\underline{u}, u_n) - u_n)\,dx.$$

The second inequality is obtained by applying Lemma A.1 to $\mathscr{F}_- = I_n$, $\mathscr{F}_+ = \mathscr{F}_{\overline{\lambda}}^{[\overline{u}]}$, $K_+ = K_- = L^2 \cap \mathrm{BV}(\Omega)$:

$$0 \leq \int_\Omega (\lambda_0 f(u_{n-1}) - \overline{\lambda} f(\overline{u}))(\max(\overline{u}, u_n) - \overline{u})\,dx$$

and using the fact that $u_{n-1} \leq \overline{u}$ and $\lambda_0 < \overline{\lambda}$.

Since u_n is Lipschitz continuous in Ω, in particular $W^{1,1}(\Omega)$, it satisfies the Euler-Lagrange equation associated to the minimization of I_n: $-\mathrm{div}(T(\nabla u_n)) = H + \lambda_0 f(u_{n-1})$. If $n \geq 2$ and $u_{n-1} \geq u_{n-2}$, we then obtain the inequality $u_n > u_{n-1}$ by the strong maximum principle (37) for Lipschitz continuous functions. $\square$

Proof of Lemma 4.5. The proof of the lemma is similar to the proof of Lemma 1.1 in [Giu78]: Assuming that the conclusion is false, we deduce that there exists a sequence A_k of (non-empty) subsets of Ω satisfying $\displaystyle\int_{A_k} \overline{H} \geq (1-k^{-1})P(A_k)$, $\overline{H} := H + \lambda_0 f(u_{n-1})$. In particular $P(A_k) = \displaystyle\int_{\mathbb{R}^N} |D\varphi_{A_k}|$ is bounded, so there exists a Borel subset A of Ω such that, up to a subsequence, $\varphi_{A_k} \to \varphi_A$ in $L^1(\Omega)$

and, by lower semi-continuity of the perimeter, $\displaystyle\int_A \overline{H} \geq P(A)$. This is a contradiction to the strict inequality (19) except if A is empty. But the isoperimetric inequality gives

$$|A_k|^{\frac{n}{n-1}} \leq P(A_k) \leq (1-k^{-1})^{-1} \int_{A_k} \overline{H} \leq (1-k^{-1})^{-1} \|\overline{H}\|_{L^n(A_k)} |A_k|^{\frac{n}{n-1}}$$

hence

$$(1-k^{-1}) \leq \|\overline{H}\|_{L^n(A_k)} \qquad \text{for all } k \geq 2.$$

Since $\overline{H}$ is bounded (remember that u_{n-1} is Lipschitz in Ω), we deduce

$$\frac{1}{2} \leq C|A_k|^{1/n}$$

and so $|A| > 0$ (since $\varphi_{A_k} \to \varphi_A$ in $L^1(\Omega)$). Consequently, A cannot be empty, and we have a contradiction. □

5 Uniform L^∞ bound for minimal solutions

The goal of this section is to establish the following L^∞ estimate, which will be used in the next section to show that u_λ converges to a weak solution of (1) as $\lambda \to \lambda^*$:

Proposition 5.1. *There exists a constant C depending only on Ω and H such that, for every $0 \leq \lambda < \lambda^*$, the minimal solution u_λ to (P_λ) satisfies*

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C.$$

The proof relies on an energy method à la DeGiorgi. Note that, in general, weak solutions are not minimizers (not even local ones) of the energy functional $\mathscr{J}_\lambda$. But it is classical that the minimal solutions u_λ enjoy some semi-stability properties. More precisely, we will show that u_λ is a global minimizer of $\mathscr{J}_\lambda$ with respect to non-positive perturbations. We will then use classical calculus of variation methods to prove Proposition 5.1.

5.1 Minimal solutions as one-sided global minimizers

We now show the following lemma:

Lemma 5.2. *The minimal solution u_λ of (P_λ) is a global minimizer of the functional $\mathscr{J}_\lambda$ over the set $K_\lambda = \{v \in \mathrm{BV}(\Omega); 0 \leq v \leq u_\lambda\}$. Furthermore, u_λ is a semi-stable solution in the sense that $\mathscr{J}_\lambda''(u) \geq 0$: if u_λ is Lipschitz continuous in Ω, then, for all φ in $\mathcal{C}^1(\Omega)$ satisfying $\varphi = 0$ on $\partial\Omega$, we have:*

$$Q_\lambda(\varphi) = \int_\Omega \frac{|\nabla \varphi|^2}{(1+|\nabla u_\lambda|^2)^{1/2}} - \frac{|\nabla \varphi \cdot \nabla u_\lambda|^2}{(1+|\nabla u_\lambda|^2)^{3/2}} - \lambda f'(u)\varphi^2 \, dx \geq 0. \qquad (20)$$

Proof. It is readily seen that the functional $\mathscr{J}_\lambda$ admits a global minimizer $\tilde{u}_\lambda$ on K_λ. We are going to show that $\tilde{u}_\lambda = u_\lambda$ by proving, by recursion on n, that $\tilde{u}_\lambda \geq u_n$ for all n, where (u_n) is the sequence used to construct the minimal solution u_λ in the proof of Proposition 4.3, that is $u_0 = \underline{u}$ and $I_n(u_n) = \min_{v\in \mathrm{BV}(\Omega)} I_n(v)$ with, we recall,

$$I_n(v) = \mathscr{A}(v) - \int_\Omega (H + \lambda f(u_{n-1}))v + \int_{\partial\Omega} |v| d\mathscr{H}^{N-1}.$$

Set $u_{-1} = 0$, so that $u_0 = \underline{u}$ is the minimizer of I_0. Let $n \geq 0$. Applying Lemma A.1 to $\mathscr{F}_- = I_n$, $\mathscr{F}_+ = \mathscr{J}_\lambda$, $K_- = L^2 \cap \mathrm{BV}(\Omega)$, $K_+ = K_\lambda$, we obtain

$$0 \quad \leq \quad \lambda \int_\Omega F(\tilde{u}_\lambda) - F(\max(u_n, \tilde{u}_\lambda)) + f(u_{n-1})(\max(u_n, \tilde{u}_\lambda) - \tilde{u}_\lambda)\, dx \tag{21}$$

For $n = 0$, (21) reduces to:

$$0 \leq -\int_\Omega F(\max(\underline{u}, \tilde{u}_\lambda)) - F(\tilde{u}_\lambda)\, dx$$

with $\tilde{u}_\lambda \geq 0$ in Ω. We thus have $F(\max(\underline{u}, \tilde{u}_\lambda)) \geq F(\tilde{u}_\lambda)$ and so

$$F(\max(\underline{u}, \tilde{u}_\lambda)) = F(\tilde{u}_\lambda) \qquad \text{a.e. in } \Omega$$

which implies $\underline{u} \leq \tilde{u}_\lambda$ a.e. in Ω.

For $n \geq 1$ and assuming that we have proved that $u_{n-1} \leq \tilde{u}_\lambda$ a.e. in Ω, then $f(u_{n-1}) \leq f(\tilde{u}_\lambda)$ and (21) implies

$$\begin{aligned} 0 \quad &\leq \quad -\lambda \int_\Omega F(\max(u_n, \tilde{u}_\lambda)) - F(\tilde{u}_\lambda) - f(\tilde{u}_\lambda)(\max(u_n, \tilde{u}_\lambda) - \tilde{u}_\lambda)\, dx && (22)\\ &\leq \quad -\lambda \int_\Omega G(\tilde{u}_\lambda, \max(u_n, \tilde{u}_\lambda))\, dx && (23) \end{aligned}$$

The strict convexity of F implies $\tilde{u}_\lambda = \max(u_n, \tilde{u}_\lambda)$ and thus $u_n \leq \tilde{u}_\lambda$ a.e. in Ω.

Passing to the limit $n \to \infty$, we deduce

$$u_\lambda \leq \tilde{u}_\lambda \qquad \text{in } \Omega$$

and thus $u_\lambda = \tilde{u}_\lambda$, which completes the proof that u_λ is a one sided minimizer.

Next, we note that if φ is a non-positive smooth function satisfying $\varphi = 0$ on $\partial\Omega$, then $\mathscr{J}_\lambda(u_\lambda + t\varphi) \geq \mathscr{J}_\lambda(u_\lambda)$ for all $t \geq 0$. Letting t go to zero, and assuming that u_λ is Lipschitz continuous in Ω, we deduce that the second variation $Q_\lambda(\varphi)$ is non negative. Since $Q_\lambda(\varphi) = Q_\lambda(-\varphi)$, is is readily seen that (20) holds also for non-negative function. Finally decomposing φ into its positive and negative part, we deduce (20) for any φ. □

5.2 L^∞ estimate

We now prove:

Proposition 5.3. *Let $\lambda \in (0, \lambda^*)$. There exists a constant C_1 depending on λ^{-1} and Ω such that*

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C_1.$$

Note that this implies Proposition 5.1: Proposition 5.3 gives the existence of C depending only on Ω such that $\|u_\lambda\|_{L^\infty(\Omega)} \leq C$ for every $\min(1, \lambda^*/2) \leq \lambda < \lambda^*$. And since $0 \leq u_\lambda \leq u_{\lambda'}$ if $\lambda < \lambda'$, the inequality is also satisfied when $0 \leq \lambda \leq \min(1, \lambda^*/2)$.

Proof. This proof is essentially a variation of the proof of Theorem 2.2 in Giusti [Giu76]. We fix $\lambda \in (0, \lambda^*)$ and set $u = u_\lambda$

For some fixed $k > 1$, we set $v_k = \min(u, k)$ and $w_k = u - v_k = (u - k)_+$. The difference between the areas of the graphs of u and v_k can be estimated by below as follows:

$$\int_\Omega |Dw_k| - |\{u > k\}| \leq \mathscr{A}(u) - \mathscr{A}(v_k).$$

On the other hand, since $0 \leq v_k \leq u$, Lemma 3.1 gives $\mathscr{J}_\lambda(u) \leq \mathscr{J}_\lambda(v_k)$, which implies

$$\mathscr{A}(u) - \mathscr{A}(v_k) \leq \int_\Omega H(u - v_k) + \lambda[F(u) - F(v_k)]\, dx.$$

Writing

$$F(u) - F(v_k) = \int_0^1 f(su + (1 - s)v_k)\, ds\, (u - v_k),$$

We deduce the following inequality

$$\int_\Omega |Dw_k| \leq |\{u > k\}| + \int_\Omega \left(H + \lambda \int_0^1 f(su + (1 - s)v_k)\, ds \right) w_k\, dx. \tag{24}$$

First, we will show that (24) implies the following estimate:

$$\|u\|_{L^q(\Omega)} \leq C_1(q), \tag{25}$$

for every $q \in [1, +\infty)$, where $C_1(q)$ depends on q, Ω, λ^{-1}.

Indeed, by Lemma 3.2, we have $\displaystyle\int_A H + \lambda f(u)\, dx \leq P(A)$ for all Caccioppoli subset A of Ω. We deduce (using the coarea formula):

$$\begin{aligned}
\int_\Omega (H + \lambda f(u)) w_k\, dx &= \int_0^{+\infty} \int_{\{w_k > t\}} H + \lambda f(u)\, dx\, dt \\
&\leq \int_0^{+\infty} P(w_k > t) dt \\
&\leq \int_\Omega |Dw_\lambda|.
\end{aligned}$$

We deduce, with (24), that

$$0 \leq |\{u > k\}| - \lambda \int_{\{u \geq k\}} \left[f(u) - \int_0^1 f(su + (1-s)v_k)\, ds \right] w_k\, dx.$$

Since $u \geq 1$ and $v_k \geq 1$ on $\{u \geq k\}$, and since $f'(s) \geq 1$ for $s \geq 1$, we have furthermore

$$\begin{aligned} f(u) &\geq f(su + (1-s)v_k) + (u - su - (1-s)v_k) \\ &= f(su + (1-s)v_k) + (1-s)(u - v_k). \end{aligned}$$

We deduce (recall that $w_k = u - v_k = (u-k)_+$):

$$\int_\Omega [(u-k)_+]^2\, dx \leq \frac{2}{\lambda} |\{u > k\}|$$

which implies, in particular, (25) for $q = 2$. Furthermore, integrating this inequality with respect to $k \in (k', +\infty)$, we get:

$$\int_\Omega [(u-k)_+]^3\, dx \leq 3 \cdot \frac{2}{\lambda} \int_\Omega (u-k)_+\, dx,$$

and by repeated integration we obtain:

$$\int_\Omega [(u-k)^+]^q\, dx \leq q(q-1) \frac{1}{\lambda} \int_\Omega [(u-k)^+]^{q-2}\, dx$$

for every $q \geq 3$, which implies (25) by induction on q.

Note however, that the constant $C_1(q)$ blows up as $q \to \infty$, and so we cannot obtain the L^∞ estimate that way. We thus go back to (24): Using Poincaré's inequality for BV functions and (24), we get

$$\begin{aligned} \|w_k\|_{L^{\frac{n}{n-1}}(\Omega)} &\leq C(\Omega) \int_\Omega |Dw_k| \\ &\leq C(\Omega) \left(|\{u > k\}| + \int_\Omega (H + \lambda f(u))\, w_k \right) \\ &\leq C(\Omega) \left(|\{u > k\}| + \|H + \lambda f(u)\|_{L^n(\{w_k > 0\})} \|w_k\|_{L^{\frac{n}{n-1}}(\Omega)} \right) \end{aligned}$$

Inequality (25) implies in particular that $H + \lambda f(u) \in L^n(\Omega)$ (with bound depending on Ω, λ^{-1}), so there exists $\varepsilon > 0$ such that $C(\Omega)\|H + \lambda f(u)\|_{L^n(A)} \leq 1/2$ for any subset $A \subset \Omega$ with $|A| < \varepsilon$. Moreover, Lemma 3.2 gives $\|u\|_{L^1(\Omega)} \leq P(\Omega)/\lambda$ and therefore

$$|\{w_k > 0\}| = |\{u > k\}| \leq \frac{1}{k} \frac{P(\Omega)}{\lambda}.$$

It follows that there exists k_0 depending on Ω, λ^{-1} such that

$$C(\Omega)\|H + \lambda f(u)\|_{L^n(\{w_k > 0\})} \leq 1/2$$

for $k \geq k_0$. For $k \geq k_0$, we deduce

$$\|w_k\|_{L^{\frac{n}{n-1}}(\Omega)} = \|(u-k)_+\|_{L^{\frac{n}{n-1}}(\Omega)} \leq 2C(\Omega)|\{u > k\}|.$$

Finally, for $k' > k$, we have $1_{|\{u>k'\}} \leq \left(\frac{(u-k)_+}{k'-k}\right)^{\frac{n}{n-1}}$ and so

$$|\{u > k'\}| \leq \frac{1}{(k'-k)^{\frac{n}{n-1}}} \|(u-k)_+\|^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(\Omega)} \leq \frac{2C(\Omega)}{(k'-k)^{\frac{n}{n-1}}} |\{u > k\}|^{\frac{n}{n-1}}$$

which implies, by classical arguments (see [Sta66]) that $|\{u_\lambda > k\}|$ is zero for k large (depending on $|\Omega|$ and λ^{-1}). The proposition follows. $\square$

As a consequence, we have:

Corollary 5.4. *There exists a constant C depending only on Ω such that*

$$\|u_\lambda\|_{\mathrm{BV}(\Omega)} \leq C.$$

Proof. By Lemma 3.1 (ii) and Proposition 5.3, we get:

$$\begin{aligned} \mathcal{A}(u_\lambda) \quad &\leq \quad \mathcal{A}(v) - \int_\Omega (H + \lambda f(u_\lambda)) v \, dx + \int_\Omega (H + \lambda f(u_\lambda)) u_\lambda \, dx \\ &\leq \quad \mathcal{A}(v) + C \int_\Omega |v| \, dx + C \end{aligned}$$

for any function $v \in \mathrm{BV}(\Omega) \cap L^1(\Omega)$ such that $v = 0$ on $\partial\Omega$. Taking $v = 0$, the result follows immediately. $\square$

6 Existence of the extremal solution

We can now complete the proof of Theorem 2.4. The only missing piece is the existence of a weak solution for $\lambda = \lambda^*$, which is given by the following proposition:

Proposition 6.1. *There exists a function $u^* \in L^{p+1}(\Omega) \cap \mathrm{BV}(\Omega)$ such that*

$$u_\lambda \to u^* \quad \text{in } L^{p+1}(\Omega) \quad \text{as } \lambda \to \lambda^*.$$

Furthermore, u^* is a weak solution of (P_λ) for $\lambda = \lambda^*$.

Proof. Recalling that the sequence u_λ is nondecreasing with respect to λ, it is readily seen that Proposition 5.1 implies the existence of a function $u^* \in L^\infty(\Omega)$ such that

$$\lim_{\lambda \to \lambda^*} u_\lambda(x) = u^*(x).$$

Furthermore, by the Lebesgue dominated convergence theorem, u_λ converges to u^* strongly in $L^q(\Omega)$ for all $q \in [1, \infty)$.

Next, by lower semi-continuity of the area functional $\mathscr{A}(u)$ and Corollary 5.4, we have

$$\mathscr{A}(u^*) \leq \liminf_{\lambda \to \lambda^*} \mathscr{A}(u_\lambda) < \infty.$$

So, if we write

$$\lambda \int F(u_\lambda)\, dx - \lambda^* \int F(u^*)\, dx = (\lambda - \lambda^*) \int F(u_\lambda)\, dx + \lambda^* \int F(u_\lambda) - F(u^*)\, dx,$$

it is readily seen that

$$\mathscr{J}_{\lambda^*}(u^*) \leq \liminf_{\lambda \to \lambda^*} \mathscr{J}_\lambda(u_\lambda).$$

Furthermore, Lemma 3.1 yields

$$\mathscr{J}_\lambda(u_\lambda) \leq \mathscr{J}_\lambda(u^*) + \lambda \int_\Omega G(u_\lambda, u^*)\, dx$$

and so (using the strong L^{p+1} convergence):

$$\limsup_{\lambda \to \lambda^*} \mathscr{J}_\lambda(u_\lambda) \leq \mathscr{J}_{\lambda^*}(u^*).$$

We deduce the convergence of the functionals:

$$\mathscr{J}_{\lambda^*}(u^*) = \lim_{\lambda \to \lambda^*} \mathscr{J}_\lambda(u_\lambda)$$

which implies in particular that

$$\mathscr{A}(u_\lambda) \to \mathscr{A}(u^*)$$

and so $u_\lambda \to u^*$ in $L^1(\partial\Omega)$. It follows that u^* satisfies the boundary condition $u^* = 0$ on Ω.

Finally, using Lemma 3.1 again, we have, for any $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$ with $v = 0$ on $\partial\Omega$:

$$\mathscr{J}_\lambda(u_\lambda) \leq \mathscr{J}_\lambda(v) + \lambda \int_\Omega G(u_\lambda, v)\, dx$$

which yields, as $\lambda \to \lambda^*$:

$$\mathscr{J}_{\lambda^*}(u^*) \leq \mathscr{J}_{\lambda^*}(v) + \lambda^* \int_\Omega G(u^*, v)\, dx.$$

for any $v \in L^{p+1} \cap \mathrm{BV}(\Omega)$ with $v = 0$ on $\partial\Omega$. Lemma 3.1 implies that u^* is a solution of (P_λ) for $\lambda = \lambda^*$. $\square$

7 Regularity of the minimal solution in the radial case: Proof of Theorem 2.6

Throughout this section, we assume that $\Omega = B_R$ and that H depends on $r = |x|$ only. Then, for any rotation R that leaves B_R invariant, we see that the function $u_\lambda^R(x) = u_\lambda(Rx)$ is a weak solution of (P_λ), and the minimality of u_λ implies

$$u_\lambda \leq u_\lambda^R \text{ in } \Omega.$$

Taking the inverse rotation R^{-1}, we get the opposite inequality and so $u_\lambda^R = u_\lambda$, i.e. u_λ is radially (or spherically) symmetric. Furthermore, equation (1) reads:

$$-\frac{1}{r^{n-1}}\frac{d}{dr}\left(\frac{r^{n-1}u_r}{(1+u_r^2)^{1/2}}\right) = H + \lambda f(u). \tag{26}$$

or

$$-\left[\frac{u_{rr}}{(1+u_r^2)^{3/2}} + \frac{n-1}{r}\frac{u_r}{(1+u_r^2)^{1/2}}\right] = H + \lambda f(u) \tag{27}$$

together with the boundary conditions

$$u_r(0) = 0, \qquad u(R) = 0.$$

Note that, by integration of (26) over $(0, r)$, $0 < r < R$, we obtain

$$\frac{-r^{n-1}u_r(r)}{(1+u_r(r)^2)^{1/2}} = \int_0^r [H + \lambda f(u)]r^{n-1}dr,$$

which gives $u_r \leq 0$, provided u is Lipschitz continuous in Ω at least.

It is classical that the solutions of (5) can blow up at $r = 0$. In our case however, the functions u_λ are bounded in L^∞. We deduce the following result:

Lemma 7.1. *There exists $r_1 \in (0, R)$ and $C_1 > 0$ such that for any $\lambda \in [0, \lambda^*]$, we have*

$$|\nabla u_\lambda(x)| \leq C_1 \textit{ for all } x \textit{ such that } |x| \leq r_1.$$

Proof. First, we assume that u_λ is smooth. Then, integrating (1) over B_r, we get:

$$\int_{\partial B_r} \frac{\nabla u_\lambda \cdot \nu}{\sqrt{1+|\nabla u_\lambda|^2}}\,dx = \int_{B_r} H + \lambda f(u_\lambda)\,dx.$$

Since u_λ is spherically symmetric, this implies:

$$\frac{|u_{\lambda r}|}{\sqrt{1+|u_{\lambda r}|^2}}(r) = \frac{1}{P(B_r)}\int_{B_r} H + \lambda f(u_\lambda)\,dx \tag{28}$$

and the L^∞ bound on u_λ yields:

$$\frac{|u_{\lambda r}|}{\sqrt{1+|u_{\lambda r}|^2}}(r) \leq C\frac{|B_r|}{P(B_r)} \leq Cr.$$

In particular, there exists r_1 such that $Cr \leq 1/2$ for $r \leq r_1$ and so

$$|(u_\lambda)_r|(r) \leq C_1 \qquad \text{for } r \leq r_1. \tag{29}$$

Of course, these computations are only possible if we already know that u is a classical solution of (1). However, it is always possible to perform the above computations with the sequence (u_n) used in the proof of Proposition 4.3 to construct u_λ. In particular, we note that we have $\underline{u} \leq u_n \leq u_\lambda$ for all n and

$$-\mathrm{div}(Tu_n) = H + \lambda f(u_{n-1}) \text{ in } \Omega$$

so the same proof as above implies that there exists a constant C independent of n or λ such that

$$|\nabla u_n| \leq C_1 \text{ for all } x \text{ such that } |x| \leq r_1.$$

The lemma follows by taking the limit $n \to \infty$ (recall that the sequence u_n converges in a monotone fashion to u_λ). $\square$

Proof of Theorem 2.6. We now want to prove the gradient estimate (14). Thanks to Lemma 7.1, we only have to show the result for $r \in [r_1, R]$. We denote $u^* = u_{\lambda^*}$ Since u^* is a weak solution of (P_λ), Lemma 3.2 with $A = B_r$ ($r \in [0, R]$) implies

$$\int_{B_r} H + \lambda^* f(u^*)\, dx \leq P(B_r)$$

and so, using the fact that $u^* \geq u_\lambda \geq \underline{u}$, we have

$$\int_{B_r} H + \lambda f(u_\lambda)\, dx \leq P(B_r) - \int_{B_r} (\lambda^* - \lambda) f(u_\lambda) \leq P(B_r) - (\lambda^* - \lambda) \int_{B_r} f(\underline{u})\, dx.$$

Hence (28) becomes:

$$\frac{|u_{\lambda r}|}{\sqrt{1 + |u_{\lambda r}|^2}}(r) \leq 1 - \frac{(\lambda^* - \lambda)}{r^{n-1}} \int_{B_r} f(\underline{u})\, dx.$$

For $r \in (r_1, R)$, we have

$$\frac{(\lambda^* - \lambda)}{r^{n-1}} \int_{B_r} f(\underline{u})\, dx \geq (\lambda^* - \lambda)\delta > 0$$

for some universal δ and so

$$|(u_\lambda)_r|(r) \leq \frac{C}{\lambda^* - \lambda} \qquad \text{for } r \in [r_1, R].$$

Together with (29), this gives the result.

Note once again that these computations can only be performed rigorously on the function (u_n), which satisfy in particular $\underline{u} \leq u_n \leq u^*$ for all n. So (14) holds for u_n instead of u_λ. The result follows by passing to the limit $n \to \infty$. $\square$

Remark 7.2. *We point out that the Lipschitz regularity near the origin $r = 0$ is a consequence of the L^∞ estimate (it is in fact enough to have $f(u_\lambda) \in L^n$), while the gradient estimate away from the origin only requires $f(u_\lambda)$ to be integrable.*

7.1 Regularity of the extremal solution

In this section, we prove Theorem 2.7, that is the regularity of the extremal solution u^*. The proof is divided in two parts: boundary regularity and interior regularity.

7.1.1 Boundary regularity

We have the following a priori estimate:

Lemma 7.3. *Assume that $\Omega = B_R$, that H depends on r and that condition (4), (12) and (13) are fulfilled. Let u be any solution of (1) in* $\mathrm{Lip}(\overline{\Omega})$. *Then there exists a constant C depending only on R, ε_0 and n such that*

$$|u_r(R)| \leq C(1+\lambda).$$

Since we know that $u_\lambda \in \mathrm{Lip}(\overline{\Omega})$ for $\lambda < \lambda^*$, we deduce

$$|(u_\lambda)_r(R)| \leq C(1+\lambda) \quad \text{for all } \lambda < \lambda^*.$$

Passing to the limit, we obtain:

$$|u^*_r(R)| \leq C(1+\lambda^*) \tag{30}$$

Proof of Lemma 7.3: In this proof, Assumption (12) plays a crucial role. When Ω is a ball of radius R and using the fact that $H \in \mathrm{Lip}(\overline{\Omega})$, it implies:

$$H(r) \leq (1-\varepsilon_0)\frac{n-1}{R}$$

in a neighborhood of $\partial\Omega$ (with a slightly smaller ε_0). The argument is similar to the proof of Theorem 2.2 (ii) (to show that u satisfies the Dirichlet condition), and relies on the construction of an appropriate barrier. We consider a circle of radius ε^{-1} (ε to be determined) centered at (M,δ) with δ small and $M > R$ chosen such that the circle passes through the point $(R,0)$ (see Figure 1). We define the function $h(r)$ in $[M-\varepsilon^{-1}, R]$ such that $(r, h(r))$ lies on the circle (with $h(r) < \delta$).

Then, we note that for $r \in [M-\varepsilon^{-1}, R]$ and $\varepsilon\delta \leq 1$, we have

$$\frac{h'(r)}{(1+h'(r)^2)^{1/2}} \leq \frac{h'(R)}{(1+h'(R)^2)^{1/2}} = -(1-(\delta\varepsilon)^2)^{1/2} \leq -1+(\delta\varepsilon)^2$$

(this quantity can be interpreted as the horizontal component of the normal vector to the circle), and

$$\frac{d}{dr}\left(\frac{h'(r)}{(1+h'(r)^2)^{1/2}}\right) = \varepsilon$$

(this quantity is actually the one-dimensional curvature of the curve $r \mapsto h(r)$). Hence we have:

$$\begin{aligned}
\frac{1}{r^{n-1}}\frac{d}{dr}\left(\frac{r^{n-1}h'(r)}{(1+h'(r)^2)^{1/2}}\right) &= \frac{d}{dr}\left(\frac{h'(r)}{(1+h'(r)^2)^{1/2}}\right) + \frac{n-1}{r}\frac{h'(r)}{(1+h'(r)^2)^{1/2}} \\
&\leq \varepsilon + \frac{n-1}{r}(-1+(\delta\varepsilon)^2) \\
&\leq \varepsilon + \frac{n-1}{R}(-1+(\delta\varepsilon)^2)
\end{aligned}$$

We now use a classical sliding method: Let

$$\eta^* = \inf\{\eta > 0\,;\, u(r) \leq h(r-\eta) \text{ for } r \in [M-\varepsilon^{-1}+\eta, R]\}.$$

If $\eta^* > 0$, then $h(r+\eta^*)$ touches u from above at a point in $(M-\varepsilon^{-1}+\eta, R)$ such that $u < \delta$ (recall that u is Lipschitz continuous so it cannot touch $h(r-\eta)$ at $M-\varepsilon^{-1}+\eta$ since $h=\delta$ and $h'=\infty$ at that point). At that contact point, we must thus have

$$\begin{aligned}
\frac{1}{r^{n-1}}\frac{d}{dr}\left(\frac{r^{n-1}h'(r)}{(1+h'(r)^2)^{1/2}}\right) &\geq \frac{1}{r^{n-1}}\frac{d}{dr}\left(\frac{r^{n-1}u_r(r)}{(1+u_r(r)^2)^{1/2}}\right) \\
&\geq -(H+\lambda f(u)) \\
&\geq -(1-\varepsilon_0)\frac{n-1}{R} - \lambda\delta^p.
\end{aligned}$$

We will get a contradiction if ε and δ are such that

$$\varepsilon + \frac{n-1}{R}(-1+(\delta\varepsilon)^2) < -(1-\varepsilon_0)\frac{n-1}{R} - \lambda\delta^p$$

which is equivalent to

$$\varepsilon + \lambda\delta^p + \frac{n-1}{R}(\varepsilon\delta)^2 < \frac{n-1}{R}\varepsilon_0.$$

This can be achieved easily by choosing ε and δ small enough.

It follows that $\eta^* = 0$ and so $u \leq h$ in the neighborhood of R. Since $u(R) = h(R) = 0$, we deduce:

$$|u'(R)| \leq |h'(R)| \leq C(R,n)(\varepsilon\delta)^{-1} \leq C(R,n)\frac{1+\lambda}{\varepsilon_0^2}.$$

□

Corollary 7.4. *There exist $\eta \in (0,R)$ and $C > 0$ such that*

$$|\nabla u(r)| \leq C \qquad \textit{for all } r \in [R-\eta, R].$$

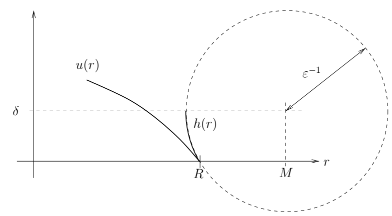

Figure 1: Construction of a barrier

Proof. The same proof as that of Lemma 7.3 shows that there exists $\delta > 0$ and $C > 0$ such that:

$$\text{If } u(r) \leq \delta \text{ for all } r \in [r_0, R] \text{ with } R - r_0 \leq \delta \text{ then } |u_r(r_0)| \leq C. \tag{31}$$

Furthermore, the proof of Lemma 7.3 implies that $u(r) \leq h(r)$ in a neighborhood of R, and so for some small η we have:

$$u(r) \leq \delta \text{ for all } r \in [R - \eta, R].$$

The result follows. $\square$

7.1.2 Interior regularity

We now show the following interior regularity result:

Proposition 7.5. *Let $\eta \in (0, R/2)$. There exists $C_\eta > 0$ depending only on η, n and $\|u_\lambda\|_{\mathrm{BV}(\Omega)}$ such that, for all $0 \leq \lambda < \lambda^*$,*

$$|\nabla u_\lambda(x)| \leq C_\eta \text{ for all } x \text{ in } \Omega \text{ with } \eta < |x| < R - \eta.$$

Using Lemma 7.1 (regularity for r close to 0), Corollary 7.4 (regularity for r close to R), and Proposition 7.5 (together with Corollary 5.4 which give the BV estimate uniformly with respect to λ), we deduce that there exists C depending only on H and n such that

$$|\nabla u_\lambda(x)| \leq C \text{ for all } x \text{ in } \Omega$$

for all $\lambda \in [0, \lambda^*)$. Theorem 2.7 then follows by passing to the limit $\lambda \to \lambda^*$.

Proof of Proposition 7.5. It is sufficient to prove the result for $\frac{\lambda^*}{2} < \lambda < \lambda^*$. Throughout the proof, we fix $\lambda \in (\frac{\lambda^*}{2}, \lambda^*)$, $r_0 \in (\eta, R - \eta)$ and we denote $u = u_\lambda$.

Idea of the proof: Let $\varphi_0 = \varphi_{B_{r_0}}$ (the characteristic function of the set B_{r_0}). Then for all $t \geq 0$, we have:

$$\mathscr{J}(u + t\varphi_0) \quad \leq \quad \mathscr{J}(u) + t\int_\Omega |D\varphi_0| - t\int_\Omega H\varphi_0\, dx - \lambda \int_\Omega F(u + t\varphi_0) - F(u)\, dx$$

Furthermore, since $u \geq \underline{u}$, we have $u \geq \mu > 0$ in B_{r_0} and so

$$F(u + t\varphi_0) - F(u) \geq f(u)t\varphi_0 + \frac{\alpha}{2}t^2\varphi_0^2 \quad \text{for all } x$$

(with α such that $f'(s) \geq \alpha$ for all $s \geq \mu$). It follows:

$$\begin{aligned}
\mathscr{J}(u + t\varphi_0) &\leq \mathscr{J}(u) + t\int_\Omega |D\varphi_0| - t\int_\Omega (H + \lambda f(u))\varphi_0 \, dx - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_0^2 \, dx \\
&= \mathscr{J}(u) + tP(B_{r_0}) - t\int_{B_{r_0}} H + \lambda f(u) \, dx - t^2\frac{\alpha\lambda}{2}|B_{r_0}| \\
&= \mathscr{J}(u) + tP(B_{r_0})\left(1 - \frac{|u_r(r_0)|}{v(r_0)}\right) - t^2\frac{\alpha\lambda}{2}|B_{r_0}|.
\end{aligned}$$

where the last equality follows by integration of (1) over B_{r_0}, which yields:

$$-P(B_{r_0})\frac{u_r(r_0)}{v(r_0)} = \int_{B_{r_0}} H + \lambda f(u) \, dx.$$

This would imply Proposition 7.5 if we had $\mathscr{J}(u) \leq \mathscr{J}(u+t\varphi_0)$ for some $t > 0$. Unfortunately, u is only a minimizer with respect to negative perturbation. However, we will show that u is also almost a minimizer with respect to some positive perturbation.

Step 1: First of all, the function φ_0 above is not smooth, so we need to consider the following piecewise linear approximation of φ_0:

$$\varphi_\varepsilon = \begin{cases} 1 & \text{if } r \leq r_0 - \varepsilon \\ \varepsilon^{-1}(r_0 - r) & \text{if } r_0 - \varepsilon \leq r \leq r_0 \\ 0 & \text{if } r \geq r_0. \end{cases}$$

We then have (using Equation (1)):

$$\begin{aligned}
\mathscr{J}(u + t\varphi_\varepsilon) &= \mathscr{J}(u) + t\int_\Omega |\nabla\varphi_\varepsilon| \, dx - t\int_\Omega (H + \lambda f(u))\varphi_\varepsilon \, dx - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2 \, dx \\
&= \mathscr{J}(u) + t\int_\Omega |\nabla\varphi_\varepsilon| \, dx - t\int_\Omega \frac{(u)_r(\varphi_\varepsilon)_r}{v} \, dx - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2 \, dx \\
&= \mathscr{J}(u) + t\int_\Omega \left(1 - \frac{|u_r|}{v}\right)|\nabla\varphi_\varepsilon| \, dx - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2 \, dx \\
&= \mathscr{J}(u) + t\omega_n\int_{r_0-\varepsilon}^{r_0} \left(1 - \frac{|u_r|}{v}\right)\varepsilon^{-1}r^{n-1} \, dr - t^2\frac{\alpha\lambda}{2}\lambda\frac{t^2}{2}\int_\Omega \varphi_\varepsilon^2 \, dx \\
&\leq \mathscr{J}(u) + t\omega_n\varepsilon^{-1}\int_{r_0-\varepsilon}^{r_0} \frac{1}{v^2}r^{n-1} \, dr - t^2\frac{\alpha\lambda}{2}\int_\Omega \varphi_\varepsilon^2 \, dx
\end{aligned}$$

and so if we denote $\rho(\varepsilon) = \sup_{r\in(r_0-\varepsilon,r_0)} \frac{1}{v^2}$, we deduce:

$$\mathscr{J}(u + t\varphi_\varepsilon) \leq \mathscr{J}(u) + t\omega_n r_0^{n-1}\rho(\varepsilon) - t^2\frac{\alpha\lambda}{2}\omega_n\left(\frac{r_0}{2}\right)^n \tag{32}$$

for all $\varepsilon < r_0/2$.

Step 2: We now want to control $\mathscr{J}(u+t\varphi_\varepsilon)$ from below: for a smooth function φ, we denote

$$\theta(t) = \mathcal{A}(u+t\varphi) = \int_\Omega L(u_r + t\varphi_r),$$

where $L(s) = (1+s^2)^{1/2}$. Then

$$\theta^{(3)}(t) = \int_\Omega L^{(3)}(u_r + t\varphi_r)\varphi_r^3 \, dx$$

where

$$L^{(3)} \colon s \mapsto \frac{-3s}{(1+s^2)^{5/2}}.$$

Since $L^{(4)}(s) = \frac{-3(1-4s^2)}{(1+s^2)^{7/2}}$, $|L^{(3)}| = -L^{(3)}$ is non-decreasing on $[1/2, +\infty)$, and so

$$|L^{(3)}(s)| \leq \frac{3}{(1+s^2)^2}.$$

When $\varphi = \varphi_\varepsilon$, we have $|u_r + t\varphi_r| \geq |u_r|$ for all $t \geq 0$ and therefore:

$$\begin{aligned}
|\theta^{(3)}(t)| &\leq \int_\Omega \frac{3}{v^4}(\varphi_\varepsilon)_r^3 \, dx \\
&\leq \varepsilon^{-3}\omega_n \int_{r_0-\varepsilon}^{r_0} \frac{3}{v^4} r^{n-1} \, dr \\
&\leq \varepsilon^{-2}\omega_n \, \rho(\varepsilon)^2 \, r_0^{n-1}
\end{aligned}$$

for all $t \geq 0$.

Since the second variation $Q_\lambda(\varphi_\varepsilon)$ is non-negative by Lemma 5.2 (recall that u_λ is a semi-stable solution), we deduce that for some $t_0 \in (0,t)$ we have:

$$\begin{aligned}
\mathscr{J}(u+t\varphi_\varepsilon) &= \mathscr{J}(u) + \frac{t^2}{2}Q_\lambda(\varphi_\varepsilon) + \theta^{(3)}(t_0)\frac{t^3}{6} - \lambda \int_\Omega \frac{f''(u+t_0\varphi_\varepsilon)}{6} t^3 \varphi^3 \, dx \\
&\geq \mathscr{J}(u) - \frac{t^3}{2}|\theta^{(3)}(t_0)| - \|f''(u+t_0\varphi_\varepsilon)\|_{L^\infty(B_{r_0})} \lambda t^3 \omega_n r_0^n \\
&\geq \mathscr{J}(u) - \frac{t^3}{2}\varepsilon^{-2}\omega_n \, \rho(\varepsilon)^2 \, r_0^{n-1} - C\lambda t^3 \omega_n r_0^n, \qquad (33)
\end{aligned}$$

where we used the fact that $f''(u+t_0\varphi_\varepsilon) \in L^\infty(B_{r_0})$ (if $p \geq 2$, this is a consequence of the L^∞ bound on u, if $p \in (1,2)$, then this follows from the fact that $u + t_0\varphi_\varepsilon \geq \underline{u} > 0$ in B_{r_0}).

Step 3: Inequalities (32) and (33) yield:

$$\lambda \frac{t^2}{2}\omega_n r_0^n \leq t\omega_n r_0^{n-1}\rho(\varepsilon) + \frac{t^3}{2}\varepsilon^{-2}\omega_n \, \rho(\varepsilon)^2 \, r_0^{n-1} + C\lambda t^3 \omega_n r_0^n$$

and so

$$\frac{\lambda r_0}{2}(1-2Ct)t \le \rho(\varepsilon) + \frac{\varepsilon^{-2}t^2}{2}\,\rho(\varepsilon)^2$$

for all $t \ge 0$. If $t \le 1/(4C)$, we deduce

$$\mu t \le \rho(\varepsilon) + \frac{\varepsilon^{-2}t^2}{2}\,\rho(\varepsilon)^2$$

with $\mu = \lambda r_1/4$ (recall that $r_0 > r_1$).

Let now $t = M\varepsilon$ (M to be chosen later), then we get

$$\mu M\varepsilon \le \rho(\varepsilon) + \frac{M^2}{2}\,\rho(\varepsilon)^2.$$

If $\rho(\varepsilon) \le \frac{\mu M\varepsilon}{2}$, then

$$\rho(\varepsilon) + \frac{M^2}{2}\,\rho(\varepsilon)^2 \le \frac{\mu M\varepsilon}{2} + \frac{\mu^2 M^4\varepsilon^2}{8}$$

and we get a contradiction if $\frac{\mu^2 M^4\varepsilon^2}{8} < \frac{\mu M\varepsilon}{2}$. It follows that

$$\rho(\varepsilon) \ge \frac{\mu M\varepsilon}{2} \qquad \text{for all } \varepsilon < \frac{4}{\mu M^3}. \tag{34}$$

Step 4: Since $\rho(\varepsilon) = \sup_{r\in(r_0-\varepsilon,r_0)} \frac{1}{v^2}$, (34) yields

$$\inf_{r\in(r_0-\varepsilon,r_0)} v^2 \le \frac{2}{\mu M\varepsilon} \qquad \text{for all } \varepsilon < \frac{4}{\mu M^3}.$$

In order to conclude, we need to use some type of Harnack inequality to control $\sup_{r\in(r_0-\varepsilon,r_0)} v^2$. This will follow from the following lemma:

Lemma 7.6. *Let $v = \sqrt{1+u_r^2}$. Then v solves the following equation in $(0,R)$:*

$$-\frac{1}{r^{n-1}}\left(\frac{r^{n-1}v_r}{v^3}\right)_r + c^2 = H_r\frac{u_r}{v} + \lambda f'(u)\frac{u_r^2}{v}. \tag{35}$$

where

$$c^2 = \frac{n-1}{r^2}\frac{u_r^2}{v^2} + \frac{u_{rr}^2}{v^6}$$

is the sum of the square of the curvatures of the graph of u.

We postpone the proof of this lemma to the end of this section. Clearly, the equation (35) is degenerate elliptic. In order to write a Harnack inequality, we introduce $w = \frac{1}{v^2}$, solution of the following equation

$$\frac{1}{r^{n-1}}\left(r^{n-1}w_r\right)_r = 2H_r\frac{u_r}{v} + 2\lambda f'(u)\frac{u_r^2}{v} - 2c^2$$

which is a nice uniformly elliptic equation in a neighborhood of $r_0 \in (0, R)$. In particular, if $\varepsilon \le R - r_0$, Harnack's inequality [GT01] yields:

$$\sup_{r\in(r_0-\varepsilon,r_0)} w \le C \inf_{r\in(r_0-\varepsilon,r_0)} w + C\varepsilon\|g\|_{L^1(r_0-2\varepsilon,r_0+\varepsilon)} \tag{36}$$

where

$$g = 2H_r \frac{u_r}{v} + 2\lambda f'(u)\frac{u_r^2}{v} - 2c^2.$$

Next, we note that

$$|g| \le 2|H_r| + C\lambda|u_r| + 2c^2.$$

It is readily seen that the first $(n-1)$ curvatures $\frac{1}{r}\frac{u_r}{v}$ are bounded in a neighborhood of $r_0 \neq 0$. Furthermore, since the mean curvature is in L^∞, it is easy to check that the last curvature is also bounded: More precisely, (27) gives

$$\frac{u_{rr}}{v^3} = -H - \lambda f(u) - \frac{n-1}{r}\frac{u_r}{v} \in L^\infty.$$

We deduce that $c^2 \in L^\infty$ and since $u \in \mathrm{BV}(\Omega)$, we get

$$\|g\|_{L^1(r_0-2\varepsilon,r_0+\varepsilon)} \le C\|u\|_{\mathrm{BV}(\Omega)} + C$$

Together with (36) and (34), we deduce:

$$\frac{\mu M\varepsilon}{2} \le C \inf_{r\in(r_0-\varepsilon,r_0)} w + C(\|u\|_{\mathrm{BV}(\Omega)} + 1)\varepsilon \qquad \text{for all } \varepsilon < \frac{4}{\mu M^3}.$$

With M large enough ($M \ge \frac{C}{\lambda r_1}(\|u\|_{\mathrm{BV}(\Omega)} + 1)$), it follows that

$$\frac{\mu M\varepsilon}{4} \le C \inf_{r\in(r_0-\varepsilon,r_0)} w \qquad \text{for all } \varepsilon < \frac{4}{\mu M^3}$$

and thus (with $\varepsilon = \min(\frac{2}{\mu M^3}, (R-r_0)/4, \frac{1}{4MC})$:

$$v(r_0)^2 \le \sup_{r\in(r_0-\varepsilon,r_0)} v^2 \le C((\lambda r_0)^{-1}, (R-r_0)^{-1}, \|u\|_{\mathrm{BV}(\Omega)}, \|u\|_{L^\infty(\Omega)})$$

which completes the proof. □

Proof of Lemma 7.6. Taking the derivative of (26) with respect to r and multiplying by u_r, we get:

$$\frac{n-1}{r^n}\left(\frac{r^{n-1}u_r}{v}\right)_r u_r - \frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_r}{v}\right)_{rr} u_r = H_r u_r + \lambda f'(u)u_r^2.$$

Using the fact that

$$\left(\frac{u_r}{v}\right)_r = \frac{u_{rr}}{v^3} \qquad \text{and} \qquad v_r = \frac{u_r u_{rr}}{v},$$

we deduce:

$$\frac{(n-1)^2}{r^n}\frac{r^{n-2}u_r^2}{v}+\frac{n-1}{r}\frac{u_r u_{rr}}{v^3}-\frac{n-1}{r^{n-1}}\left(\frac{r^{n-2}u_r}{v}\right)_r u_r-\frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_{rr}}{v^3}\right)_r u_r \\ = H_r u_r+\lambda f'(u)u_r^2$$

and so (simplifying and dividing by v):

$$\frac{(n-1)^2}{r^2}\frac{u_r^2}{v^2}-\frac{(n-1)(n-2)}{r^{n-1}}\frac{r^{n-3}u_r^2}{v^2}-\frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_{rr}}{v^3}\right)_r\frac{u_r}{v}=H_r\frac{u_r}{v}+\lambda f'(u)\frac{u_r^2}{v}.$$

This yields

$$\frac{(n-1)}{r^2}\frac{u_r^2}{v^2}-\frac{1}{r^{n-1}}\left(\frac{r^{n-1}u_{rr}u_r}{v^4}\right)_r+\frac{1}{r^{n-1}}\frac{r^{n-1}u_{rr}}{v^3}\left(\frac{u_r}{v}\right)_r=H_r\frac{u_r}{v}+\lambda f'(u)\frac{u_r^2}{v}$$

hence

$$\frac{(n-1)}{r^2}\frac{u_r^2}{v^2}-\frac{1}{r^{n-1}}\left(\frac{r^{n-1}v_r}{v^3}\right)_r+\frac{u_{rr}^2}{v^6}=H_r\frac{u_r}{v}+\lambda f'(u)\frac{u_r^2}{v}$$

which is the desired equation. □

A Comparison principles

It is well known that classical solutions (in $\mathcal{C}^{1,1}(\overline{\Omega})$) of (1) satisfy a strong comparison principle namely, if

$$-\mathrm{div}(Tu)\leq-\mathrm{div}(Tv)\text{ in }\Omega,\qquad u\leq v\text{ on }\partial\Omega$$

with $u\neq v$, then

$$u<v\text{ in }\Omega. \tag{37}$$

If u,v are only in $W^{1,1}(\Omega)$ then we still have a weak comparison principle (see [Giu84]). But no such principle holds for functions that are only in $\mathrm{BV}(\Omega)$ (even if one of the function is smooth). This is due to the lack of strict convexity of the functional $\mathscr{A}$ on $\mathrm{BV}(\Omega)$ ($\mathscr{A}$ is affine on any interval $[0,\varphi_A]$). In particular, we have $\mathcal{L}(\varphi_A)=\mathcal{L}(-\varphi_A)=\mathcal{L}(0)=0$ for any finite perimeter set A.

Throughout the paper, we consider weak solutions of the equation $\mathcal{L}(u)=H+\lambda u$, and most such solutions are in $\mathrm{BV}(\Omega)$. In order to use comparison principles, we must thus use the properties of the functional $\mathscr{J}_\lambda$ rather than the Euler-Lagrange equation. We thus use repeatedly the following lemma:

Lemma A.1 (Comparison principle)**.** *Let $p\geq 1$. Let $G_\pm\colon\Omega\times\mathbb{R}\to\mathbb{R}$ satisfy the growth condition $|G_\pm(x,s)|\leq C_1(x)|s|^p+C_2(x)$ where $C_1\in L^{p'}(\Omega)$ and $C_2\in L^1(\Omega)$. Let $\mathscr{F}_\pm$ be the functional defined on $L^p\cap\mathrm{BV}(\Omega)$ by*

$$\mathscr{F}_\pm(v)=\mathscr{A}(v)+\int_{\partial\Omega}|v|d\mathscr{H}^{N-1}+\int_\Omega G_\pm(x,v)\,dx.$$

Suppose that $u_\pm$ is a global minimizer of $\mathscr{F}_\pm$ on a set $K_\pm$ and suppose that

$$\min(u_+, u_-) \in K_-, \quad \max(u_+, u_-) \in K_+,$$

Then we have

$$0 \le \Delta(\max(u_+, u_-)) - \Delta(u_+), \quad \Delta(v) := \int_\Omega G_+(x, v) - G_-(x, v)\, dx.$$

Proof of Lemma A.1. We need to recall the inequality

$$\int_Q |D\varphi_{E\cup F}| + \int_Q |D\varphi_{E\cap F}| \le \int_Q |D\varphi_E| + \int_Q |D\varphi_F|, \tag{38}$$

which holds for any open set $Q \subset \mathbb{R}^m$ ($m \ge 1$) and any sets E, F with locally finite perimeter in $\mathbb{R}^m$. Applied to $Q = \Omega \times \mathbb{R}$ and to the characteristic functions of the subgraphs of u and v, Inequality (38) gives:

$$\mathscr{A}(\max(u, v)) + \mathscr{A}(\min(u, v)) \le \mathscr{A}(u) + \mathscr{A}(v), \quad u, v \in \mathrm{BV}(\Omega). \tag{39}$$

Since $\int_\Omega |Du| \le \mathscr{A}(u)$, this shows in particular that $\max(u, v)$, $\min(u, v)$ and $(u-v)_+ = \max(u, v) - v = u - \min(u, v) \in \mathrm{BV}(\Omega)$ whenever u and $v \in \mathrm{BV}(\Omega)$.

Since $u \mapsto \int_\Omega G_\pm(u)$ is invariant by rearrangement, we deduce:

$$\mathscr{F}_-(\max(u_+, u_-)) + \mathscr{F}_-(\min(u_+, u_-)) \le \mathscr{F}_-(u_+) + \mathscr{F}_-(u_-). \tag{40}$$

Furthermore, we have $\min(u_+, u_-) \in K_-$, and so $\mathscr{F}_-(u_-) \le \mathscr{F}_-(\min(u_+, u_-))$. Therefore, (40) implies that $\mathscr{F}_-(\max(u_+, u_-)) \le \mathscr{F}_-(u_+)$, which, by definition of Δ also reads:

$$\mathscr{F}_+(\max(u_+, u_-)) - \Delta(\max(u_+, u_-)) \le \mathscr{F}_+(u_+) - \Delta(u_+).$$

Finally, we recall that u_+ is the global minimizer of $\mathscr{F}_+$ on K_+ and that $\max(u_+, u_-) \in K_+$, and so $\mathscr{F}_+(u_+) \le \mathscr{F}_+(\max(u_+, u_-))$. We conclude that $\Delta(\max(u_+, u_-)) - \Delta(u_+) \ge 0$. $\square$

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