A trivariate non-Gaussian copula having 2-dimensional Gaussian copulas as margins

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Testing Gaussian copula hypothesis for all pairs of assets is not the same as testing higher-dimensional Gaussian copula hypothesis for the whole portfolio

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Abstract

Arthur Charpentier (see Arthur's blog) was recently contacted by some researchers willing to test if a multivariate copula is - or not - Gaussian. They use a test proposed in Malevergne and Sornette (2003) stating that one should simply test for pairwise normality. This test may be of importance in finance, in actuarial science, and in risk management in general: for example, given 120 financial assets, in order to test whether or not some 120-dimensional random vector of interest in finance admits a Gaussian copula, can one restrict the Gaussian copula hypothesis test to pairs of assets? This short note proves that it is not the case, and provides a simple counter-example based on some multivariate EFGM copula. This confirms the intuition that one cannot only consider all pairs of the studied random variables and that one cannot avoid to study the full vector to test whether a random vector admits a Gaussian copula.

Keywords : Gaussian copula; trivariate copulas with fixed bivariate copulas; pairwise and global normality.

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On Arthur Charpentier's blog, one can read: *I was just contacted by some researchers willing to test if a multivariate copula is - or not - Gaussian. They use a test proposed in an unpublished paper by Malevergne and Sornette, stating that one should simply test for pairwise normality. In this paper, the following result is mentioned: if (a, b), (b, c) and (c, a) have a Gaussian copula, then the triplet (a, b, c) has also a Gaussian copula. Unfortunately, this result is (probably) not correct (and if it is valid, it is nontrivial). It should be possible to construct a counterexample (thanks to Roger Nelsen for the idea) by letting the correlation in each pair be close to -1 (i.e. global pairwise counter-comonotonicity). Then the correlation matrix of the triplet would fail to be positive definite (which is a requirement for Gaussian vectors).*
Arthur told me about this problem a few weeks ago and I am pleased to provide an answer for his blog and for the researchers who contacted him. I use here an approach that is different from the one suggested by Roger B. Nelsen, which might lead to other kinds of counter-examples.

In this note, for simplicity, we only consider the case of a trivariate distribution with symmetric correlation structure. For any $0 < \rho < 1$, we construct a vector (Z_1, Z_2, Z_3) whose trivariate copula is not Gaussian, and such that (Z_1, Z_2), (Z_1, Z_3) and (Z_2, Z_3) have the same 2-dimensional Gaussian copula with correlation parameter ρ.

To build a counter-example, it is natural to look for a triplet whose components are pairwise independent but not mutually independent. This is the case for some Eyraud-Farlie-Gumbel-Morgenstern [1] (EFGM) 3-copulas (see Nelsen (2006) page 108).

Consider ϵ such that $0 < \epsilon < 1 - \rho$. Let (X_1, X_2, X_3) be a Gaussian vector with mean vector $(0, 0, 0)$ and covariance matrix $\begin{pmatrix} 1 & \rho+\epsilon & \rho+\epsilon \\ \rho+\epsilon & 1 & \rho+\epsilon \\ \rho+\epsilon & \rho+\epsilon & 1 \end{pmatrix}$.

Let (Y_1, Y_2, Y_3) be a random vector with standard normal (univariate) marginals that admits a 3-EFGM copula with parameter $\theta \in (-1, 1) \setminus \{0\}$, defined for $(u, v, w) \in [0, 1]^3$ by

$$C_\theta(u, v, w) = uvw\left[1 + \theta(1-u)(1-v)(1-w)\right].$$

[1] This copula is often called FGM, but as noted by Nelsen (2006), an early reference is a paper by my wife's ancester (see Eyraud (1938)), which I must absolutely mention, at least for family reasons. Cambanis (1977) even calls this copula Eyraud-Gumbel-Morgenstern (EGM).

Proposition 1 *Assume that* (X_1, X_2, X_3) *and* (Y_1, Y_2, Y_3) *are independent and defined as above, and define*

$$(Z_1, Z_2, Z_3) = \sqrt{\frac{\rho}{\rho+\epsilon}}(X_1, X_2, X_3) + \sqrt{\frac{\epsilon}{\rho+\epsilon}}(Y_1, Y_2, Y_3),$$

with $0 < \epsilon < 1-\rho$*. Then* (Z_1, Z_2)*,* (Z_1, Z_3) *and* (Z_2, Z_3) *have Gaussian copula with correlation parameter* ρ*, but the* 3*-copula of* (Z_1, Z_2, Z_3) *is not Gaussian! Note that EFGM copulas with non-zero parameters furnishes directly a counter-example in the case where* $\rho = 0$*.*

Proof:
All three pairs (Z_1, Z_2), (Z_1, Z_3) and (Z_2, Z_3) play the same role, so let's focus on (Z_1, Z_2) without loss of generality. The copula of (Y_1, Y_2) evaluated at (u, v) is $C_\theta(u, v, 1) = uv$, which shows that Y_1 and Y_2 are independent (or admit the Gaussian copula with null correlation parameter).
From standard properties of sums of independent Gaussian vectors, (Z_1, Z_2) is a Gaussian vector with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.
The 3-copula of (Z_1, Z_2, Z_3) is not Gaussian: if it were, then (Z_1, Z_2, Z_3) would be a Gaussian vector, and so would be (Y_1, Y_2, Y_3) using characteristic functions. But (Y_1, Y_2, Y_3) has the 3-EFGM copula, which (for $\theta \in (-1, 1) \setminus \{0\}$) is different from any Gaussian copula. This leads to the desired contradiction. $\square$.
A more general study of those counter-examples is left for further research.

A much earlier counter-example is indeed available in Romano and Siegel (1986); thanks to Christian Genest for pointing this out! This earlier counter-example has now been added at the end of this document.

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Arthur Charpentier's blog:
http://blogperso.univ-rennes1.fr/arthur.charpentier/index.php/post/2009/02/11/Pariwise-and-global-dependence...-some-pitfalls

2.12. Three pairwise independent, Gaussian random variables that are not trivariate Gaussian.

Three mutually independent Gaussian random variables must be jointly trivariate Gaussian; however, pairwise independence is not sufficient to guarantee this. Let X, Y, and Z_0 be independent Gaussian random variables with mean zero and variance one. Define

$$Z = |Z_0| \operatorname{sgn}(XY)$$

where

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0. \end{cases}$$

Then Z has a Gaussian distribution because $\operatorname{sgn}(XY)$ is 1 or -1 with probability $\frac{1}{2}$, independent of Z_0. To verify the pairwise independence of X and Z, for example, observe that $\operatorname{sgn}(XY)$ is statistically independent of X. The other two pairs are also seen to be independent. However, (X, Y, Z) is not trivariate Gaussian because if it were, then (due to pairwise independence) X, Y, and Z would have to be mutually independent. This is impossible because

$$P(X > 0,\ Y > 0, \text{and } Z < 0) = 0 \neq \tfrac{1}{8} = P(X > 0) \cdot P(Y > 0) \cdot P(Z < 0).$$

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