Diagonal hyperbolic systems with large and monotone data Part I: Global continuous solutions

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April 11, 2009

Abstract

In this paper, we study diagonal hyperbolic systems in one space dimension. Based on a new gradient entropy estimate, we prove the global existence of a continuous solution, for large and non-decreasing initial data. We remark that these results cover the case of systems which are hyperbolic but not strictly hyperbolic. Physically, this kind of diagonal hyperbolic systems appears naturally in the modelling of the dynamics of dislocation densities.

AMS Classification: 35L45, 35Q35, 35Q72, 74H25.
Key words: Global existence, system of Burgers equations, system of non-linear transport equations, non-linear hyperbolic system, dynamics of dislocation densities.

1 Introduction and main result

1.1 Setting of the problem

In this paper we are interested in continuous solutions to hyperbolic systems in dimension one. Our work will focus on solutions $u(t,x) = (u^i(t,x))_{i=1,\dots,d}$, where d is an integer, of hyperbolic systems which are diagonal, i.e.

$$\partial_t u^i + \lambda^i(u)\partial_x u^i = 0 \quad \text{on} \quad (0,+\infty)\times\mathbb{R}, \quad \text{for} \quad i = 1,\dots,d, \tag{1.1}$$

with the initial data:

$$u^i(0,x) = u_0^i(x), \qquad x \in \mathbb{R}, \quad \text{for} \quad i = 1,\dots,d. \tag{1.2}$$

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Here $\partial_t = \dfrac{\partial}{\partial t}$ and $\partial_x = \dfrac{\partial}{\partial x}$. Such systems are (sometimes) called $(d \times d)$ hyperbolic systems. Our study of system (1.1) is motivated by consideration of models describing the dynamics of dislocation densities (see the Appendix, Section 5), which is

$$\partial_t u^i + \left(\sum_{j=1,\dots,d} A_{ij} u^j \right) \partial_x u^i = 0 \quad \text{for} \quad i = 1, \dots, d,$$

where $(A_{ij})_{i,j=1,\dots,d}$ is a non-negative symmetric matrix. This model can be seen as a special case of system (1.1).

For real numbers $\alpha^i \leq \beta^i$, let us consider the box

$$U = \Pi_{i=1}^d [\alpha^i, \beta^i]. \tag{1.3}$$

We consider a given function $\lambda = (\lambda^i)_{i=1,\dots,d} : U \to \mathbb{R}^d$, which satisfies the following regularity assumption:

$$(H1) \quad \begin{cases} \text{the function } \lambda \in C^\infty(U), \\ \\ \text{there exists} \quad M_0 > 0 \quad \text{such that for} \quad i = 1, ..., d, \\ |\lambda^i(u)| \leq M_0 \quad \text{for all} \quad u \in U, \\ \\ \text{there exists} \quad M_1 > 0 \quad \text{such that for} \quad i = 1, ..., d, \\ |\lambda^i(v) - \lambda^i(u)| \leq M_1 |v - u| \quad \text{for all} \quad v, u \in U, \end{cases}$$

where $|w| = \sum_{i=1,\dots,d} |w^i|$, for $w = (w^1, \dots, w^d)$. Given any Banach space $(E, \|\cdot\|_E)$, in the rest of the paper we consider the norm on E^d:

$$\|w\|_{E^d} = \sum_{i=1,\dots,d} \|w^i\|_E, \quad \text{for} \quad w = (w^1, \dots, w^d) \in E^d.$$

We assume, for all $u \in \mathbb{R}^d$, that the matrix

$$(\lambda^i_{,j}(u))_{i,j=1,\dots,d}, \quad \text{where} \quad \lambda^i_{,j} = \frac{\partial \lambda^i}{\partial u^j},$$

is non-negative in the positive cone, namely

$$(H2) \quad \left| \begin{array}{l} \text{for all} \quad u \in U, \quad \text{we have} \\ \\ \displaystyle\sum_{i,j=1,\dots,d} \xi_i \xi_j \lambda^i_{,j}(u) \geq 0 \quad \text{for every} \quad \xi = (\xi_1, ..., \xi_d) \in [0, +\infty)^d. \end{array} \right.$$

In (1.2), each component u_0^i of the initial data $u_0 = (u_0^1, \dots, u_0^d)$ is assumed satisfy the following property:

$$(H3) \quad \left\{ \begin{array}{l} \alpha^i \leq u_0^i \leq \beta^i, \\ \\ u_0^i \text{ is non-decreasing,} \\ \\ \partial_x u_0^i \in L\log L(\mathbb{R}), \end{array} \right| \text{ for } i = 1, \ldots, d,$$

where $L\log L(\mathbb{R})$ is the following Zygmund space:

$$L\log L(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) \text{ such that } \int_{\mathbb{R}} |f| \ln(e + |f|) < +\infty \right\}.$$

This space is equipped by the following norm:

$$\|f\|_{L\log L(\mathbb{R})} = \inf \left\{ \mu > 0 : \int_{\mathbb{R}} \frac{|f|}{\mu} \ln\left(e + \frac{|f|}{\mu}\right) \leq 1 \right\},$$

This norm is due to Luxemburg (see Adams [1, (13), Page 234]).

Our purpose is to show the existence of a continuous solution $u = (u^1, \ldots, u^d)$ such that, for $i = 1, \ldots, d$, the function $u^i(t, \cdot)$ satisfies $(H3)$ for all time.

1.2 Main result

It is well-known that for the classical scalar Burgers equation $\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0$, the solution stays continuous when the initial data is Lipschitz-continuous and non-decreasing. We want somehow to generalize this result to the case of diagonal hyperbolic systems. In particular, we say that a function $u_0 = (u_0^1, \ldots, u_0^d)$ is non-decreasing if each component u_0^i is non-decreasing for $i = 1, \ldots, d$.

Theorem 1.1 ***(Global existence of a non-decreasing solution)***
Assume $(H1)$, $(H2)$ and $(H3)$. Then, there exists a function u which satisfies for all $T > 0$:

i) **Existence of a weak solution:**
The function u is solution of (1.1)-(1.2), where

$$u \in [L^\infty((0, +\infty) \times \mathbb{R})]^d \cap [C([0, +\infty); L\log L(\mathbb{R}))]^d \text{ and } \partial_x u \in [L^\infty((0, T); L\log L(\mathbb{R}))]^d,$$

such that for a.e. $t \in [0, T)$ the function $u(t, \cdot)$ is non-decreasing in x and satisfies the following L^∞ estimate:

$$\|u^i(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \quad \text{for} \quad i = 1, \ldots, d, \tag{1.4}$$

and the gradient entropy estimate:

$$\int_{\mathbb{R}} \sum_{i=1,\dots,d} f\left(\partial_x u^i(t,x)\right) dx + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,d} \lambda^i_{,j}(u)\partial_x u^i(s,x)\partial_x u^j(s,x)\, dx\, ds \le C_1, \tag{1.5}$$

where

$$0 \le f(x) = \begin{cases} x\ln(x) + \frac{1}{e} & \text{if} \quad x \ge 1/e, \\ 0 & \text{if} \quad 0 \le x \le 1/e, \end{cases} \tag{1.6}$$

and $C_1\left(T, d, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^d}, \|\partial_x u_0\|_{[L\log L(\mathbb{R})]^d}\right)$.

ii) **Continuity of the solution:**
The solution u constructed in (i) belongs to $[C([0,+\infty)\times\mathbb{R})]^d$ and there exists a modulus of continuity $\omega(\delta,h)$, such that for all $\delta, h \ge 0$ and all $(t,x) \in (0, T-\delta)\times\mathbb{R}$, we have:

$$|u(t+\delta, x+h) - u(t,x)| \le C_2\, \omega(\delta,h) \ \text{ with } \ \omega(\delta,h) = \frac{1}{\ln(\frac{1}{\delta}+1)} + \frac{1}{\ln(\frac{1}{h}+1)}, \tag{1.7}$$

where $C_2\left(T, d, M_0, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^d}, \|\partial_x u_0\|_{[L\log L(\mathbb{R})]^d}\right)$.

The key point to establish Theorem 1.1 is the gradient entropy estimate (1.5). We first consider the parabolic regularization of the system (1.1) and we show that the smooth solution admits the L^∞ bound (1.4) and the fundamental gradient entropy inequality (1.5). Then, these *a priori* estimates will allow us to pass to the limit when the regularization vanishes, which will provide the existence of a solution. Let us mention that a similar gradient entropy inequality was introduced in Cannone et al. [5] to prove the existence of a solution of a two-dimensional system of two coupled transport equations.

Remark 1.2 *Remark that assumption (H2) implies that the second term on the left hand side of (1.5) is non-negative. This will imply the $L\log L$ bound on the gradient of the solutions.*

Up to our knowledge, the result stated in Theorem 1.1 seems new. In relation with our result, we can cite the paper of Poupaud [24], where a result of existence and uniqueness of Lipschitz solutions is proven for a particular quasi-linear hyperbolic system.

Hyperbolic systems (1.1) in the case $d = 2$ are called strictly hyperbolic if and only if we have:

$$\lambda^1(u^1,u^2) < \lambda^2(u^1,u^2). \tag{1.8}$$

In this case, a result of Lax [19] implies the existence of Lipschitz monotone solutions of (1.1)-(1.2). This result was also extended by Serre [25, Vol II] in the case of $(d\times d)$ rich hyperbolic systems (see also Subsection 1.4 for more related references). Their results are limited to the case of strictly hyperbolic systems. On the contrary, in Theorem 1.1, we do not assume that the hyperbolic system is strictly hyperbolic. See the following remark for a quite detailed example.

Remark 1.3 (Crossing eigenvalues)
Condition (1.8) on the eigenvalues is not required in our framework (Theorem 1.1). Here is a simple example of a (2×2) *hyperbolic but not strictly hyperbolic system. We consider solution* $u=(u^1,u^2)$ *of*

$$\left\{\begin{array}{l} \partial_t u^1 + cos(u^2)\partial_x u^1 = 0, \\ \\ \partial_t u^2 + u^1 sin(u^2)\partial_x u^2 = 0, \end{array}\right. \quad on \quad (0,+\infty)\times\mathbb{R}. \tag{1.9}$$

Assume:

i) $u^1(-\infty)=1$, $u^1(+\infty)=2$ *and* $\partial_x u^1 \geq 0$,

ii) $u^2(-\infty)=-\frac{\pi}{2}$, $u^2(+\infty)=\frac{\pi}{2}$ *and* $\partial_x u^2 \geq 0$.

Here the eigenvalues $\lambda^1(u^1,u^2)=cos(u^2)$ *and* $\lambda^2(u^1,u^2)=u^1 sin(u^2)$ *cross each other at the initial time (and indeed for any time). Nevertheless, we can compute*

$$(\lambda^i_{,j}(u^1,u^2))_{i,j=1,2} = \left(\begin{array}{cc} 0 & -sin(u^2) \\ sin(u^2) & u^1 cos(u^2) \end{array}\right),$$

which satisfies $(H2)$ *(under assumptions (i) and (ii)). Therefore Theorem 1.1 gives the existence of a solution to (1.9) with in particular (i) and (ii).*

Remark 1.4 (A generalization of Theorem 1.1)
In Theorem 1.1 we have considered a particular system in order to simplify the presentation. Our approach can be easily extended to the following generalized system:

$$\partial_t u^i + \lambda^i(u,x,t)\partial_x u^i = h^i(u,x,t) \quad on \quad (0,+\infty)\times\mathbb{R}, \quad for \quad i=1,...,d, \tag{1.10}$$

with the following conditions:

- $\lambda^i \in W^{1,\infty}(U\times\mathbb{R}\times[0,+\infty))$ *and the matrix* $(\lambda^i_{,j}(u,x,t))_{i,j=1,...,d}$ *is positive in the positive cone for all* $(u,x,t)\in U\times\mathbb{R}\times[0,+\infty)$ *(i.e. a condition analogous to* $(H2)$*).*

- $h^i \in W^{1,\infty}(U\times\mathbb{R}\times[0,+\infty))$, $\partial_x h^i \geq 0$ *and* $h^i_{,j}\geq 0$ *for all* $j\neq i$.

Let us remark that our system (1.1)-(1.2) does not create shocks because the solution (given in Theorem 1.1) is continuous. In this situation, it seems very natural to expect the uniqueness of the solution. Indeed the notion of entropy solution (in particular designed to deal with the discontinuities of weak solutions) does not seem so helpful in this context. Even for such a simple system, we then ask the following:

Open question: Is there uniqueness of the continuous solution given in Theorem 1.1 ?

In a companion paper (El Hajj, Monneau [11]), we will provide some partial answers to this question.

1.3 Application to diagonalizable systems

Let us first consider a smooth function $u = (u^1, \dots, u^d)$, solution of the following non-conservative hyperbolic system:

$$\begin{cases} \partial_t u(t,x) + F(u)\partial_x u(t,x) = 0, \quad u \in U, \quad x \in \mathbb{R}, \quad t \in (0, +\infty), \\ \\ u(x,0) = u_0(x) \qquad\qquad\qquad\qquad x \in \mathbb{R}, \end{cases} \tag{1.11}$$

where the space of states U is now an open subset of $\mathbb{R}^d$, and for each u, $F(u)$ is a $(d \times d)$-matrix and the map F is of class $C^1(U)$. The system (1.11) is said $(d \times d)$ hyperbolic, if $F(u)$ has d real eigenvalues and is diagonalizable for any given u on the domain under consideration. By definition, such a system is said to be diagonalizable, if there exists a smooth transformation $w = (w^1(u), \dots, w^d(u))$ with non-vanishing Jacobian such that (1.11) can be equivalently rewritten (for smooth solutions) as the following system

$$\partial_t w^i + \lambda^i(w)\partial_x w^i = 0 \quad \text{for} \quad i = 1, \dots, d,$$

where λ^i are smooth functions of w. Such functions w^i are called strict i-Riemann invariant.
Our approach can give continuous solutions to the diagonalized system, which provided continuous solution to the original system (1.11).

1.4 A brief review of some related literature

For a scalar conservation law, which corresponds to system (1.11) in the case $d = 1$ where $F(u) = h'(u)$ is the derivative of some flux function h, the global existence and uniqueness of BV solutions has been established by Oleinik [23] in one space dimension. The famous paper of Kruzhkov [18] covers the more general class of L^∞ solutions, in several space dimensions. For an alternative approach based on the notion of entropy process solutions, see for instance Eymard et al. [12]. For a different approach based on a kinetic formulation, see also Lions et al. [22].

We now recall some well-known results for a class of (2×2) strictly hyperbolic systems in one space dimension. This means that $F(u)$ has two real, distinct eigenvalues satisfying (1.8). As mentioned above, Lax [19] proved the existence and uniqueness of non-decreasing and smooth solutions for diagonalized (2×2) strictly hyperbolic systems. In the case of some (2×2) strictly hyperbolic systems, DiPerna [6, 7] showed the global existence of a L^∞ solution. The proof of DiPerna relies on a compensated compactness argument, based on the representation of the weak limit in terms of Young measures, which must reduce to a Dirac mass due to the presence of a large family of entropies.

This result is based on an the idea of Tartar [27].

For general $(d \times d)$ strictly hyperbolic systems; i.e. where $F(u)$ has d real, distinct eigenvalues

$$\lambda^1(u) < \cdots < \lambda^d(u), \tag{1.12}$$

Bianchini and Bressan proved in a very complete paper [3], a striking global existence and uniqueness result of solutions to system (1.11), assuming that the initial data has small total variation. This approach is mainly based on a careful analysis of the vanishing viscosity approximation. An existence result has first been proved by Glimm [15] in the special case of conservative equations, i.e. $F(u) = Dh(u)$ is the Jacobian of some flux function h. Let us mention that an existence result has been also obtained by LeFloch and Liu [20, 21] in the non-conservative case.

We can also mention that, our system (1.1) is related to other similar models in dimension $N \geq 1$, such as scalar transport equations based on vector fields with low regularity. Such equations were for instance studied by Diperna and Lions in [8]. They have proved the existence (and uniqueness) of a solution (in the renormalized sense), for given vector fields in $L^1((0,+\infty); W^{1,1}_{loc}(\mathbb{R}^N))$ whose divergence is in $L^1((0,+\infty); L^\infty(\mathbb{R}^N))$. This study was generalized by Ambrosio [2], who considered vector fields in $L^1((0,+\infty); BV_{loc}(\mathbb{R}^N))$ with bounded divergence. In the present paper, we work in dimension $N = 1$ and prove the existence (and some uniqueness results) of solutions of the system (1.1)-(1.2) with a velocity vector field $\lambda^i(u)$, $i = 1, \ldots, d$. Here, in Theorem 1.1, the divergence of our vector field is only in $L^\infty((0,+\infty), L\log L(\mathbb{R}))$. In this case we proved the existence result thanks to the gradient entropy estimate (1.5), which gives a better estimate on the solution.

Let us also mention that for hyperbolic and symmetric systems in dimension $N \geq 1$, Gårding has proved in [13] a local existence and uniqueness result in $C([0,T); H^s(\mathbb{R}^N)) \cap C^1([0,T); H^{s-1}(\mathbb{R}^N))$, with $s > \frac{N}{2} + 1$ (see also Serre [25, Vol I, Th 3.6.1]), this result being only local in time, even in dimension $N = 1$.

1.5 Organization of the paper

This paper is organized as follows: in Section 2, we approximate the system (1.1), by adding the viscosity term $(\varepsilon \partial_{xx} u^i)$. Then we show a global in time existence for this approximated system. Moreover, we show that these solutions are regular and non-decreasing in x for all $t > 0$. In Section 3, we prove the gradient entropy inequality and some other ε-uniform *a priori* estimates. In Section 4, we prove the main result (Theorem 1.1) passing to the limit as ε goes to 0. Finally, in the appendix (Section 5), we derive a model for the dynamics of dislocation densities.

2 Local existence of an approximated system

The system (1.1) can be written as:

$$\partial_t u + \lambda(u) \diamond \partial_x u = 0, \tag{2.13}$$

where $u := (u^i)_{1,\dots,d}$, $\lambda(u) = (\lambda^i(u))_{1,\dots,d}$ and $u \diamond v$ is the "component by component product" of the two vectors $u = (u^1, \dots, u^d)$, $v = (v^1, \dots, v^d) \in \mathbb{R}^d$. This is the vector in $\mathbb{R}^d$ whose coordinates are given by $(u \diamond v)^i := u^i v^i$. We now consider the following parabolic regularization of system (2.13), for all $0 < \varepsilon \leq 1$:

$$\begin{cases} \partial_t u^\varepsilon + \lambda(u^\varepsilon) \diamond \partial_x u^\varepsilon = \varepsilon \partial_{xx} u^\varepsilon \\ \\ u^\varepsilon(x,0) = u_0^\varepsilon(x), \quad \text{with} \quad u_0^\varepsilon(x) := u_0 * \eta_\varepsilon(x), \end{cases} \tag{2.14}$$

where $\partial_{xx} = \dfrac{\partial^2}{\partial x^2}$ and η_ε is a mollifier verify, $\eta_\varepsilon(\cdot) = \frac{1}{\varepsilon}\eta(\frac{\cdot}{\varepsilon})$, such that $\eta \in C_c^\infty(\mathbb{R})$ is a non-negative function satisfying $\int_{\mathbb{R}} \eta = 1$.

Remark 2.1 *By classical properties of the mollifier $(\eta_\varepsilon)_\varepsilon$ and the fact that $u_0^\varepsilon \in [L^\infty(\mathbb{R})]^d$, then $u_0 \in [C^\infty(\mathbb{R})]^d \cap [W^{2,\infty}(\mathbb{R})]^d$. Moreover using the non-negativity of η_ε, the second equation of (2.14) we get that*

$$\|u_0^{\varepsilon,i}\|_{L^\infty(\mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \quad \textit{for} \quad i = 1, \dots, d,$$

and $(H3)$ also implies that u_0^ε is non-decreasing.

The following theorem is a global existence result for the regularized system (2.14).

Theorem 2.2 (Global existence of non-decreasing smooth solutions) *Assume $(H1)$ and that the initial data u_0^ε is non-decreasing and satisfies $u_0^\varepsilon \in [C^\infty(\mathbb{R})]^d \cap [W^{2,\infty}(\mathbb{R})]^d$. Then the system (2.14), admits a solution $u^\varepsilon \in [C^\infty([0,+\infty) \times \mathbb{R})]^d \cap [W^{2,\infty}((0,+\infty) \times \mathbb{R})]^d$ such that the function $u^\varepsilon(t,\cdot)$ is non-decreasing for all $t > 0$. Moreover, for all $t > 0$, we have the a priori bounds:*

$$\|u^{\varepsilon,i}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_0^{\varepsilon,i}\|_{L^\infty(\mathbb{R})}, \quad \textit{for} \quad i = 1, \dots, d, \tag{2.15}$$

$$\left\|\partial_x u^{\varepsilon,i}\right\|_{L^\infty([0,+\infty);L^1(\mathbb{R}))} \leq 2\|u_0^{\varepsilon,i}\|_{L^\infty(\mathbb{R})}, \quad \textit{for} \quad i = 1, \dots, d. \tag{2.16}$$

The lines of the proof of this theorem are very standard (see for instance Cannone et al. [5] for a similar problem). For this reason, we skip the details of the proof. First of all we remark that the estimate (2.15) is a direct application of the Maximum Principle Theorem for parabolic equations (see Gilbarg-Trudinger [14, Th.3.1]). The regularity of the solution follows from a bootstrap argument. The monotonicity of the solution is a consequence of the maximum principle for scalar parabolic equations applied to $w^\varepsilon = \partial_x u^\varepsilon$ satisfying

$$\partial_t w^\varepsilon + \lambda(u^\varepsilon) \diamond \partial_x w^\varepsilon + \partial_x(\lambda(u^\varepsilon)) \diamond w^\varepsilon = \varepsilon \partial_{xx} w^\varepsilon.$$

Since $\partial_x u^\varepsilon \geq 0$ this implies easily the second estimate (2.16).

3 ε-uniform *a priori* estimates

In this section, we show some ε-uniform estimates on the solutions of system (2.14).

Before going into the proof of the gradient entropy inequality defined in (1.5), we announce the main idea to establish this estimate. Now, let us set for $w \geq 0$ the entropy function

$$\bar{f}(w) = w \ln w.$$

For any *non-negative* test function $\varphi \in C_c^1([0, +\infty) \times \mathbb{R})$, let us define the following *"gradient entropy"* with $w^i := \partial_x u^i$:

$$\bar{S}(t) = \int_{\mathbb{R}} \varphi(t, \cdot) \left(\sum_{i=1,\dots,d} \bar{f}(w^i(t, \cdot)) \right) \, dx.$$

It is very natural to introduce such quantity $\bar{S}(t)$ which in the case $\varphi \equiv 1$, appears to be nothing else than the total entropy of the system of d type of particles of non-negative densities $w^i \geq 0$. Then after two integration by parts, it is formally possible to deduce from (1.1) the equality in the following *gradient entropy inequality* for all $t \geq 0$

$$\frac{d\bar{S}(t)}{dt} + \int_{\mathbb{R}} \varphi \left(\sum_{i,j=1,\dots,d} \lambda^i_{,j} w^i w^j \right) \, dx \leq R(t), \qquad \text{for} \quad t \geq 0, \tag{3.17}$$

with the rest

$$R(t) = \int_{\mathbb{R}} \left\{ (\partial_t \varphi) \left(\sum_{i=1,\dots,d} \bar{f}(w^i) \right) + (\partial_x \varphi) \left(\sum_{i=1,\dots,d} \lambda^i \bar{f}(w^i) \right) \right\} \, dx,$$

where we do not show the dependence on t in the integrals. We remark in particular that this rest is formally equal to zero if $\varphi \equiv 1$.

To guarantee the existence of continuous solutions when $\varepsilon = 0$, we will assume later $(H2)$ which guarantees the non-negativity on the second term of the left hand side of inequality (3.17).

Coming back to a rigorous statement, we will prove the following result.

Proposition 3.1 (Gradient entropy inequality)
Assume $(H1)$ *and consider a function* $u_0 \in [L^\infty(\mathbb{R})]^d$ *satisfying* $(H3)$. *For any*

$0 < \varepsilon \leq 1$, we consider the solution u^ε of the system (2.14) given in Theorem 2.2 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$. Then for any $T > 0$, there exists a constant $C\left(T, d, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^d}, \|\partial_x u_0\|_{[L\log L(\mathbb{R})]^d}\right)$ such that

$$S(t) + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,d} \lambda^i_{,j}(u^\varepsilon) w^{\varepsilon,i} w^{\varepsilon,j} \leq C, \quad \text{with} \quad S(t) = \int_{\mathbb{R}} \sum_{i=1,\dots,d} f(w^{\varepsilon,i}(t,\cdot))dx. \tag{3.18}$$

where f is defined in (1.6) and $w^\varepsilon = (w^{\varepsilon,i})_{i=1,\dots,d} = \partial_x u^\varepsilon$.

For the proof of Proposition 3.1, we need the following technical lemma:

***Lemma* 3.2 ($L\log L$ estimate)**
Let $(\eta_\varepsilon)_{\varepsilon\in(0,1]}$ be a non-negative mollifier satisfying $\int_{\mathbb{R}} \eta_\varepsilon = 1$, let f be the function defined in (1.6) and $h \in L^1(\mathbb{R})$ be a non-negative function. Then

i) $\int_{\mathbb{R}} f(h) < +\infty$ if and only if $h \in L\log L(\mathbb{R})$. Moreover we have the following estimates:

$$\int_{\mathbb{R}} f(h) \leq 1 + \|h\|_{L\log L(\mathbb{R})} + \|h\|_{L^1(\mathbb{R})} \ln\left(1 + \|h\|_{L\log L(\mathbb{R})}\right), \tag{3.19}$$

$$\|h\|_{L\log L(\mathbb{R})} \leq 1 + \int_{\mathbb{R}} f(h) + \ln(1+e^2)\|h\|_{L^1(\mathbb{R})}. \tag{3.20}$$

ii) If $h \in L\log L(\mathbb{R})$, then for every $\varepsilon \in (0,1]$ the function $h_\varepsilon = h * \eta_\varepsilon \in L\log L(\mathbb{R})$ and satisfies

$$\|h_\varepsilon\|_{L\log L(\mathbb{R})} \leq C\|h\|_{L\log L(\mathbb{R})} \quad \text{and} \quad \|h - h_\varepsilon\|_{L\log L(\mathbb{R})} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where C is a universal constant.

Proof of Lemma 3.2:
The proof of (i) is trivial. To prove estimate (3.19), we first remark that, for all $h \geq 0$ and $\mu \in (0,1]$, we have

$$\left(h\ln(h) + \frac{1}{e}\right) \mathbb{1}_{\{h \geq \frac{1}{e}\}} \leq h\ln(h+e) \leq h\ln(e+\mu h) + |\ln(\mu)|h.$$

We apply this inequality with $\mu = \dfrac{1}{\max(1, \|h\|_{L\log L(\mathbb{R})})}$ and integrate, we get

$$\begin{aligned}\int_{\mathbb{R}} f(h) &\leq \frac{1}{\mu}\int_{\mathbb{R}} \mu h \ln(e+\mu h) + |\ln(\mu)|\|h\|_{L^1(\mathbb{R})} \\ &\leq \frac{1}{\mu} + |\ln(\mu)|\|h\|_{L^1(\mathbb{R})},\end{aligned}$$

where we have used the definition of $\|h\|_{L\log L(\mathbb{R})}$. This gives (3.19) using the fact that $\mu \geq \dfrac{1}{1+\|h\|_{L\log L(\mathbb{R})}}$.
To prove (3.20), we remark that, for $h \geq \frac{1}{e}$, we have $e \leq e^2 h$ and

$$h\ln(e+h) \leq h\ln(h) + h\ln(1+e^2) \leq f(h) + h\ln(1+e^2).$$

However, for $0 \leq h \leq \frac{1}{e}$, we have in particular

$$h\ln(e+h) \leq h\ln(1+e^2).$$

Therefore

$$\int_{\mathbb{R}} h\ln(e+h) \leq \int_{\mathbb{R}} f(h) + \ln(1+e^2)\|h\|_{L^1(\mathbb{R})}.$$

From the definition of $\|h\|_{L\log L(\mathbb{R})}$, we deduce in particular (3.20). For the proof of (ii) see Adams [1, Th 8.20].

□

Proof of Proposition 3.1:
First we want to check that $S(t)$ is well defined. To this end, we remark that if $w \geq 0$, then

$$0 \leq f(w) \leq \frac{1}{e}\mathbb{1}_{\{w\geq\frac{1}{e}\}} + w\ln(1+w).$$

Which gives that

$$\int_{\mathbb{R}} f(w) \leq \|w\|_{L^1(\mathbb{R})}\ln\left(1+\|w\|_{L^\infty(\mathbb{R})}\right) + \int_{\mathbb{R}} \frac{1}{e}\mathbb{1}_{\{w\geq\frac{1}{e}\}} \leq \|w\|_{L^1(\mathbb{R})}\left(1+\ln\left(1+\|w\|_{L^\infty(\mathbb{R})}\right)\right).$$

Now by Theorem 2.2, we have $\partial_x u^\varepsilon = w^\varepsilon \in \left[L^\infty((0,+\infty);L^1(\mathbb{R}))\right]^d \cap \left[W^{2,\infty}((0,+\infty)\times\mathbb{R})\right]^d$. This implies that $S \in L^\infty(0,+\infty)$. We compute

$$\begin{aligned}
\frac{d}{dt}S(t) &= \int_{\mathbb{R}} \sum_{i=1,\dots,d} f'(w^{\varepsilon,i})(\partial_t w^{\varepsilon,i}),\\
&= \int_{\mathbb{R}} \sum_{i=1,\dots,d} f'(w^{\varepsilon,i})\partial_x\left(-\lambda^i(u^\varepsilon)w^{\varepsilon,i} + \varepsilon\partial_x w^{\varepsilon,i}\right),\\
&= \overbrace{\int_{\mathbb{R}} \sum_{i=1,\dots,d} \lambda^i(u^\varepsilon)w^{\varepsilon,i}f''(w^{\varepsilon,i})\partial_x w^{\varepsilon,i}}^{J_1} \overbrace{- \varepsilon\int_{\mathbb{R}} \sum_{i=1,\dots,d} \left(\partial_x w^{\varepsilon,i}\right)^2 f''(w^{\varepsilon,i})}^{J_2}.
\end{aligned}$$

Remark that these computations (and the integration by parts) are justified because on the one hand $w^{\varepsilon,i}$, its derivatives and λ^i are bounded, and on the other hand $w^{\varepsilon,i}$ is in $L^\infty((0,+\infty);L^1(\mathbb{R}))$. We know that $J_2 \leq 0$ because f is convex. To control J_1, we rewrite it under the following form

$$J_1 = \int_{\mathbb{R}} \sum_{i=1,\dots,d} \lambda^i(u^\varepsilon) g'(w^{\varepsilon,i}) \partial_x w^{\varepsilon,i},$$

where

$$g(x) = \begin{cases} x - \frac{1}{e} & \text{if} \quad x \geq 1/e, \\ 0 & \text{if} \quad 0 \leq x \leq 1/e. \end{cases}$$

Then, we deduce that

$$\begin{aligned} J_1 &= \int_{\mathbb{R}} \sum_{i=1,\dots,d} \lambda^i(u^\varepsilon) \partial_x (g(w^{\varepsilon,i})) \\ &= -\int_{\mathbb{R}} \sum_{i,j=1,\dots,d} \lambda^i_{,j}(u^\varepsilon) w^{\varepsilon,j} g(w^{\varepsilon,i}), \\ &= \overbrace{-\int_{\mathbb{R}} \sum_{i,j=1,\dots,d} \lambda^i_{,j}(u^\varepsilon) w^{\varepsilon,j} w^{\varepsilon,i}}^{J_{11}} \overbrace{-\int_{\mathbb{R}} \sum_{i,j=1,\dots,d} \lambda^i_{,j}(u^\varepsilon) w^{\varepsilon,j} (g(w^{\varepsilon,i}) - w^{\varepsilon,i})}^{J_{12}}. \end{aligned}$$

We use the fact that $|g(x) - x| \leq \frac{1}{e}$ for all $x \geq 0$ and $(H1)$, to deduce that

$$\begin{aligned} |J_{12}| &\leq \frac{1}{e} d M_1 \left\| w^\varepsilon \right\|_{[L^\infty((0,+\infty),L^1(\mathbb{R}))]^d} \\ &\leq \frac{2}{e} d M_1 \|u_0\|_{[L^\infty(\mathbb{R})]^d} := C_0(\|u_0\|_{[L^\infty(\mathbb{R})]^d}, d, M_1) \end{aligned}$$

where we have use

$$\left\| w^{\varepsilon,i} \right\|_{L^\infty((0,+\infty),L^1(\mathbb{R}))} \leq 2 \|u_0^i\|_{L^\infty(\mathbb{R})}, \quad \text{for} \quad i = 1, \dots, d, \tag{3.21}$$

which follows from Remark 2.1 and Theorem 2.2. Finally, we deduce that

$$\begin{aligned} \frac{d}{dt} S(t) &\leq J_{11} + J_{12} + J_2 \\ &\leq J_{11} + C_0. \end{aligned}$$

Integrating in time on $(0,t)$, for $0 < t < T$, we get that, there exists a positive constant $C\left(T, d, M_1, \|u_0\|_{[L^\infty(\mathbb{R})]^d}, \|\partial_x u_0\|_{[L \log L(\mathbb{R})]^d}\right)$ which is independent of ε by (3.19) and Lemma 3.2 (ii) such that

$$S(t) + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,d} \lambda^i_{,j}(u^\varepsilon) w^{\varepsilon,j} w^{\varepsilon,i} \leq C_0 T + S(0) \leq C.$$

□

***Lemma* 3.3 ($W^{-1,1}$ estimate on the time derivative of the solutions)**
Assume (H1) and that the function $u_0 \in [L^\infty(\mathbb{R})]^d$ satisfies (H3). Then for any $0 < \varepsilon \leq 1$, the solution u^ε of the system (2.14) given in Theorem 2.2 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$, satisfies the following ε-uniform estimate for all $T > 0$:

$$\|\partial_t u^\varepsilon\|_{[L^2((0,T);W^{-1,1}(\mathbb{R}))]^d} \leq C\|u_0\|_{[L^\infty(\mathbb{R})]^d}.$$

where $C = C(T, M_0) > 0$ and $W^{-1,1}(\mathbb{R})$ is the dual of the space $W^{1,\infty}(\mathbb{R})$.

Proof of Lemma 3.3:
The idea to bound $\partial_t u^\varepsilon$ is simply to use the available bounds on the right hand side of the equation (2.14). We will give a proof by duality. We multiply the equation (2.14) by $\phi \in [L^2((0,T), W^{1,\infty}(\mathbb{R}))]^d$ and integrate on $(0,T) \times \mathbb{R}$, which gives

$$\int_{(0,T)\times\mathbb{R}} \phi \cdot \partial_t u^\varepsilon = \overbrace{\varepsilon \int_{(0,T)\times\mathbb{R}} \phi \cdot \partial^2_{xx} u^\varepsilon}^{I_1} \overbrace{- \int_{(0,T)\times\mathbb{R}} \phi \cdot (\lambda(u^\varepsilon) \diamond \partial_x u^\varepsilon)}^{I_2}.$$

We integrate by parts the term I_1, and obtain:

$$\begin{aligned} |I_1| \leq \left| \int_{(0,T)\times\mathbb{R}} \partial_x \phi \cdot \partial_x u^\varepsilon \right| &\leq \|\partial_x \phi\|_{[L^2((0,T),L^\infty(\mathbb{R}))]^d} \|\partial_x u^\varepsilon\|_{[L^2((0,T),L^1(\mathbb{R}))]^d}, \\ &\leq 2T^{\frac{1}{2}} \|\phi\|_{[L^2((0,T),W^{1,\infty}(\mathbb{R}))]^d} \|u_0\|_{[L^\infty(\mathbb{R})]^d}, \end{aligned} \tag{3.22}$$

where we have used inequality (3.21). Similarly, for the term I_2, we have:

$$\begin{aligned} |I_2| &\leq M_0 \|\phi\|_{[L^2((0,T),L^\infty(\mathbb{R}))]^d} \|\partial_x u^\varepsilon\|_{[L^2((0,T),L^1(\mathbb{R}))]^d}, \\ &\leq 2T^{\frac{1}{2}} M_0 \|u_0\|_{[L^\infty(\mathbb{R})]^d} \|\phi\|_{[L^2((0,T),W^{1,\infty}(\mathbb{R}))]^d}. \end{aligned} \tag{3.23}$$

Finally, collecting (3.22) and (3.23), we get that there exists a constant $C = C(T, M_0)$ independent of $0 < \varepsilon \leq 1$ such that:

$$\left| \int_{(0,T)\times\mathbb{R}} \phi \cdot \partial_t u^\varepsilon \right| \leq C\|u_0\|_{[L^\infty(\mathbb{R})]^d} \|\phi\|_{[L^2((0,T),W^{1,\infty}(\mathbb{R}))]^d}$$

which gives the announced result. □

Corollary 3.4 (ε-uniform estimates)
Assume (H1) and that the function $u_0 \in [L^\infty(\mathbb{R})]^d$ satisfies (H3). Then for any $0 < \varepsilon \leq 1$, the solution u^ε of the system (2.14) given in Theorem 2.2 with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$, satisfies the following ε-uniform estimate for all $T > 0$:

$$\|\partial_x u^\varepsilon\|_{[L^\infty((0,+\infty),L^1(\mathbb{R}))]^d} + \|u^\varepsilon\|_{[L^\infty((0,+\infty)\times\mathbb{R})]^d} + \|\partial_t u^\varepsilon\|_{[L^2((0,T);W^{-1,1}(\mathbb{R}))]^d} \leq C, \tag{3.24}$$

where $C = C(T, M_0, \|u_0\|_{[L^\infty(\mathbb{R})]^d})$.

This Corollary is a straightforward consequence of Remark 2.1, Theorem 2.2, estimate (3.21) and Lemma 3.3.

4 Passage to the limit and proof of Theorem 1.1

In this section, we prove that the system (1.1)-(1.2) admits solutions u in the distributional sense. They are the limits of u^ε given by Theorem 2.2 when $\varepsilon \to 0$. To do this, we will justify the passage to the limit as ε tends to 0 in the system (2.14) by using some compactness tools that are presented in a first subsection.

4.1 Preliminary results

First, for all open interval I of $\mathbb{R}$, we denote by

$$L\log L(I) = \left\{ f \in L^1(I) \text{ such that } \int_I |f| \ln(e + |f|) < +\infty \right\}.$$

***Lemma* 4.1 (Simon's Lemma)**
Let X, B, Y be three Banach spaces, such that we have the following injections

$$X \hookrightarrow B \text{ with compact embedding and } B \hookrightarrow Y \text{ with continuous embedding.}$$

Let $T > 0$. If $(u^\varepsilon)_\varepsilon$ is a sequence such that,

$$\|u^\varepsilon\|_{L^\infty((0,T);X)} + \|\partial_t u^\varepsilon\|_{L^q((0,T);Y)} \leq C,$$

where $q > 1$ and C is a constant independent of ε, then $(u^\varepsilon)_\varepsilon$ is relatively compact in $L^p((0,T);B)$ for all $1 \leq p < q$.

For the proof, see Simon [26, Corollary 4, Page 85].

In order to show the existence of a solution to system (1.1) in Subsection 4.2, we will apply this lemma to each scalar component of u^ε in the particular case where $X = W^{1,1}(I)$, $B = L^1(I)$ and $Y = W^{-1,1}(I) := (W_0^{1,\infty}(I))'$.

We denote by $K_{exp}(I)$ the class of all measurable function u, defined on I, for which,

$$\int_I \left(e^{|u|} - 1\right) < +\infty.$$

The space $EXP(I) = \{\mu u : \quad \mu \geq 0 \quad \text{and} \quad u \in K_{exp}(I)\}$ the linear hull of $K_{exp}(I)$. This space is supplemented with the following Luxemburg norm (see Adams [1, (13), Page 234]):

$$\|u\|_{EXP(I)} = \inf \left\{ \lambda > 0 : \int_I \left(e^{\frac{|u|}{\lambda}} - 1\right) \leq 1 \right\}.$$

Let us recall some useful properties of this space.

***Lemma* 4.2 (Generalized Hölder inequality, Adams [1, 8.11, Page 234])**
Let $h \in EXP(I)$ and $g \in L\log L(I)$. Then $hg \in L^1(I)$, with

$$\|hg\|_{L^1(I)} \leq 2\|h\|_{EXP(I)}\|g\|_{L\log L(I)}.$$

***Lemma* 4.3 (Continuity)**
Let $T > 0$. Assume that $u \in L^\infty((0,+\infty)\times\mathbb{R})$ such that

$$\|\partial_x u\|_{L^\infty((0,T);L\log L(\mathbb{R}))} + \|\partial_t u\|_{L^\infty((0,T);L\log L(\mathbb{R}))} \leq C_2$$

Then that for all $\delta, h \geq 0$ and all $(t,x) \in (0,T-\delta)\times\mathbb{R}$, we have:

$$|u(t+\delta,x+h) - u(t,x)| \leq 6C_2\left(\frac{1}{\ln(\frac{1}{\delta}+1)} + \frac{1}{\ln(\frac{1}{h}+1)}\right).$$

Proof of Lemma 4.3:
For all $h > 0$ and $(t,x) \in (0,T)\times\mathbb{R}$, we have:

$$\begin{aligned}
|u(t,x+h) - u(t,x)| &\leq \left|\int_x^{x+h} \partial_x u(t,y)dy\right| \\
&\leq 2\|1\|_{EXP(x,x+h)}\|\partial_x u(t,\cdot)\|_{L\log L(x,x+h)}, \\
&\leq 2\frac{1}{\ln(\frac{1}{h}+1)}\|\partial_x u\|_{L^\infty((0,T);L\log L(\mathbb{R}))}, \\
&\leq 2C_2\frac{1}{\ln(\frac{1}{h}+1)},
\end{aligned} \tag{4.25}$$

where we have used in the second line the generalized Hölder inequality (Lemma 4.2). Now, we prove the continuity in time, for all $\delta > 0$ and $(t,x) \in (0,T-\delta)\times\mathbb{R}$, we have:

$$\begin{aligned}
&\delta|u(t+\delta,x) - u(t,x)| \\
&= \int_x^{x+\delta} |u(t+\delta,x) - u(t,x)|dy, \\
&\leq \overbrace{\int_x^{x+\delta} |u(t+\delta,x) - u(t+\delta,y)|dy}^{K_1}, + \overbrace{\int_x^{x+\delta} |u(t+\delta,y) - u(t,y)|dy}^{K_2}, + \overbrace{\int_x^{x+\delta} |u(t,y) - u(t,x)|dy}^{K_3}.
\end{aligned}$$

Similarly, as in the last estimate (4.25), we get that:

$$K_1 + K_3 \leq \delta \int_x^{x+\delta} |\partial_x u(t+\delta, y)| dy, + \delta \int_x^{x+\delta} |\partial_x u(t, y)| dy,$$

$$\leq 4C_2 \frac{\delta}{\ln(\frac{1}{\delta}+1)}.$$

Now, we use that $\partial_t u$ is bounded in $L^\infty((0,T); L\log L(\mathbb{R}))$, to obtain that:

$$K_2 \leq \int_x^{x+\delta} \int_t^{t+\delta} |\partial_t u(s,y)| ds\, dy,$$

$$\leq 2\delta \|1\|_{EXP(x,x+\delta)} \|\partial_t u\|_{L^\infty((0,T);L\log L(\mathbb{R}))} \leq 2C_2 \frac{\delta}{\ln(\frac{1}{\delta}+1)}.$$

Collecting the estimates of K_1, K_2 and K_3, we get that:

$$|u(t+\delta, x) - u(t,x)| \leq \frac{1}{\delta}(K_1 + K_2 + K_3) \leq 6C_2 \frac{1}{\ln(\frac{1}{\delta}+1)}.$$

This last inequality joint to (4.25) implies the result.

□

4.2 Proof of Theorem 1.1

The authors would like to thank T. Gallouët for fruitful remarks that helped to simplify of the proof of Theorem 1.1. Before to prove Theorem 1.1, we first prove the following result.

Theorem 4.4 (Passage to the limit)
Assume that u^ε is a solution of system (2.14) given by Theorem 2.2, with initial data $u_0^\varepsilon = u_0 * \eta_\varepsilon$ where u_0 satisfies (H3). If we assume that for all $T > 0$, there exists a constant $C > 0$ independent on ε, such that:

$$\|\partial_x u^\varepsilon\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d} \leq C, \tag{4.26}$$

then up to extract a subsequence, the function u^ε converges, as ε goes to zero, to a function u weakly-$\star$ in $[L^\infty((0,+\infty)\times\mathbb{R})]^d$. Moreover, u is a solution of (1.1)-(1.2), and satisfies

$$\begin{cases} \|u\|_{[L^\infty((0,+\infty)\times\mathbb{R})]^d} \leq \|u_0\|_{[L^\infty(\mathbb{R})]^d}, \\ \|\partial_x u\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d} \leq C, \\ \|\partial_t u\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d} \leq M_0 C, \end{cases}$$

and $u(t,\cdot)$ is non-decreasing in x, for all $t > 0$ and satisfies

$$\|u^i\|_{L^\infty((0,+\infty);L^1(\mathbb{R}))} \leq 2\|u_0^i\|_{L^\infty(\mathbb{R})} \quad for \quad i = 1, \ldots, d. \tag{4.27}$$

Proof of Theorem 4.4:

Step 1 (u solution of (1.1)): First, we remark that by estimate (3.24) we know that for any $T > 0$, the solutions u^ε of the system (2.14) obtained with the help of Theorem 2.2, are ε-uniformly bounded in $[L^\infty((0,+\infty)\times\mathbb{R})]^d$. Hence, as ε goes to zero, we can extract a subsequence still denoted by u^ε, that converges weakly-$\star$ in $[L^\infty((0,+\infty)\times\mathbb{R})]^d$ to some limit u. Then we want to show that u is a solution of system (1.1). Indeed, since the passage to the limit in the linear terms is trivial in $[\mathcal{D}'((0,+\infty)\times\mathbb{R})]^d$, it suffices to pass to the limit in the non-linear term

$$\lambda(u^\varepsilon) \diamond \partial_x u^\varepsilon.$$

According to estimate (3.24) we know that for all open and bounded interval I of $\mathbb{R}$ there exists a constant C independent on ε such that:

$$\|u^\varepsilon\|_{[L^\infty((0,T);W^{1,1}(I))]^d} + \|\partial_t u^\varepsilon\|_{[L^2((0,T);W^{-1,1}(I))]^d} \leq C.$$

From the compactness of $W^{1,1}(I) \hookrightarrow L^1(I)$, we can apply Simon's Lemma (i.e. Lemma 4.1), with $X = [W^{1,1}(I)]^d$, $B = [L^1(I)]^d$ and $Y = [W^{-1,1}(I)]^d$, which shows in particular that

$$u^\varepsilon \text{ is relatively compact in } [L^1((0,T)\times I)]^d. \tag{4.28}$$

Then, we can see that (up to extract a subsequence)

$$\lambda(u^\varepsilon) \to \lambda(u) \quad \text{a.e.}$$

Moreover, from Lemma 4.2, similarly as in (4.25), we can get, for all $t \in (0,T)$ the following estimates:

$$\left| \int_I \partial_x u^\varepsilon(t,y)dy \right| \leq 2C \frac{1}{\ln(\frac{1}{|I|}+1)},$$

where C is given in (4.26). By the previous estimate and the fact that $\lambda(u^\varepsilon)$ is uniformly bounded in $[L^\infty((0,+\infty)\times\mathbb{R})]^d$ and converges a.e. to $\lambda(u)$, we can apply the Dunford-Pettis Theorem (see Brezis [4, Th IV.29]) and prove that

$$\lambda(u^\varepsilon) \diamond \partial_x u^\varepsilon \to \lambda(u) \diamond \partial_x u$$

weakly in $[L^1((0,T)\times I)]^d$. Because this is true for any bounded open interval I, then we can pass to the limit in (2.14) and get that,

$$\partial_t u + \lambda(u) \diamond \partial_x u = 0 \quad \text{in} \quad \mathcal{D}'((0,+\infty)\times\mathbb{R}).$$

Step 2 (*A priori* bounds): By weakly-$\star$ convergence and from the fact that $L^\infty((0,T);L\log L(\mathbb{R}))$ is the dual of $L^1((0,T);E_{exp}(\mathbb{R}))$ (see Adams [1] for the definition of the Banach space $E_{exp}(\mathbb{R})$), we can check that u satisfies the following estimates:

$$\|\partial_x u\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d} \leq \liminf_{\varepsilon\to 0} \|\partial_x u^\varepsilon\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d} \leq C,$$

$$\|u\|_{[L^\infty((0,+\infty)\times\mathbb{R})]^d} \leq \liminf_{\varepsilon\to 0} \|u^\varepsilon\|_{[L^\infty((0,+\infty)\times\mathbb{R})]^d} \leq \|u_0\|_{[L^\infty(\mathbb{R})]^d}. \tag{4.29}$$

Thanks to these two estimates, we obtain that

$$\begin{aligned}\|\partial_t u\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d} &\leq \|\lambda(u)\diamond\partial_x u\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d}\\ &\leq M_0\|\partial_x u\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d} \leq M_0 C.\end{aligned}$$

Moreover (4.27) follows from (4.29) and the fact that $u(t,\cdot)$ is non-decreasing in x (as it was the case for u^ε).

Step 3 (Recovering the initial data): Now we prove that the initial conditions (1.2) coincides with $u(0,\cdot)$. Indeed, by the ε-uniformly estimate given in Corollary 3.4, we can prove easily that, we have

$$\|u^\varepsilon(t) - u_0^\varepsilon\|_{[W^{-1,1}(\mathbb{R})]^d} \leq Ct^{\frac{1}{2}}.$$

Then, we get

$$\begin{aligned}\|u(t)-u_0\|_{[W^{-1,1}(\mathbb{R})]^d} &\leq \|u-u_0\|_{[L^\infty((0,t);W^{-1,1}(\mathbb{R}))]^d}\\ &\leq \liminf_{\varepsilon\to 0}\|u^\varepsilon - u_0^\varepsilon\|_{[L^\infty((0,t);W^{-1,1}(\mathbb{R}))]^d} \leq Ct^{\frac{1}{2}},\end{aligned}$$

where we have used the weakly-$\star$ convergence in $L^\infty((0,t);W^{-1,1}(\mathbb{R}))$ in the second line. This proves that $u(0,\cdot)=u_0$ in $[\mathcal{D}'(\mathbb{R})]^d$.

□

Proof of Theorem 1.1:

Step 1 (Existence): Remark that by assumption $(H2)$ and estimate (3.18), we deduce from (3.20) joint to (3.21) that, the solution u^ε given by Corollary 3.4 satisfies the following estimate:

$$\|\partial_x u^\varepsilon\|_{[L^\infty((0,T);L\log L(\mathbb{R}))]^d} \leq C, \tag{4.30}$$

where $C = C\left(T,d,M_1,\|u_0\|_{[L^\infty(\mathbb{R})]^d},\|\partial_x u_0\|_{[L\log L(\mathbb{R})]^d}\right)$. Now, we apply Theorem 4.4 to prove that, up to extract a subsequence, the function u^ε converges, as ε goes to zero, to a function u weakly-$\star$ in $[L^\infty((0,+\infty)\times\mathbb{R})]^d$, with u is beeing solution to (1.1)-(1.2). Moreover, from Lemma 4.3, we deduce that the function u satisfies the continuity estimate (1.7).

Step 2 (Justification of (1.5)): Let

$$\begin{cases} \Gamma_{ij}(u^\varepsilon) = \frac{1}{2}\left(\lambda^i_{,j}(u^\varepsilon) + \lambda^j_{,i}(u^\varepsilon)\right), \quad \text{for} \quad i,j = 1,\dots,d, \\ \\ w^\varepsilon = \partial_x u^\varepsilon. \end{cases}$$

For a general matrix Γ, where ${}^t\Gamma = \Gamma \geq 0$, let us introduce the square root $B = \sqrt{\Gamma}$ of Γ, uniquely defined by

$${}^tB = B \geq 0 \quad \text{and} \quad B^2 = \Gamma.$$

Remark that for non-negative symmetric matrices, the map $\Gamma \longmapsto \sqrt{\Gamma}$ is continuous. Then we can rewrite

$$\int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,d} \lambda^i_{,j}(u^\varepsilon) w^{\varepsilon,i} w^{\varepsilon,j} = \int_0^t \int_{\mathbb{R}} \left|\sqrt{\Gamma(u^\varepsilon)} w^\varepsilon\right|^2 \leq C,$$

where C is given in (3.18). Therefore

$$\sqrt{\Gamma(u^\varepsilon)} w^\varepsilon \to q \quad \text{weakly in } [L^2((0,t)\times\mathbb{R})]^d.$$

Applying the same argument as in Step 1, of the proof of Theorem 4.4, for the convergence of $\lambda(u^\varepsilon) \diamond \partial_x u^\varepsilon$, we see that

$$\sqrt{\Gamma(u^\varepsilon)} \partial_x u^\varepsilon \to \sqrt{\Gamma(u)} \partial_x u = q \quad \text{weakly in } [L^1((0,t)\times\mathbb{R})]^d.$$

Therefore, using the weakly convergence in $L^2((0,t)\times\mathbb{R})$, we get

$$\int_0^t \int_{\mathbb{R}} \sum_{i,j=1,\dots,d} \lambda^i_{,j}(u) \partial_x u^i \partial_x u^j = \int_0^t \int_{\mathbb{R}} q^2 \leq \liminf_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}} \left|\sqrt{\Gamma(u^\varepsilon)} \partial_x u^\varepsilon\right|^2 \leq C. \quad (4.31)$$

Remark also that for $w^i = \partial_x u^i$, we have

$$\begin{aligned} \sup_{0\leq t\leq T} \int_{\mathbb{R}} f(w^i) \quad &\leq 1 + \|w^i\|_{L^\infty((0,T);L\log L(\mathbb{R}))} + \|w^i\|_{L^\infty((0,T);L^1(\mathbb{R}))} \ln\left(1 + \|w^i\|_{L^\infty((0,T);L\log L(\mathbb{R}))}\right) \\ &\leq 1 + \|w^i\|_{L^\infty((0,T);L\log L(\mathbb{R}))} + 2\|u^i_0\|_{L^\infty(\mathbb{R})} \ln\left(1 + \|w^i\|_{L^\infty((0,T);L\log L(\mathbb{R}))}\right) := g[w^i] \\ &\leq \liminf_{\varepsilon\to 0} g[w^{\varepsilon,i}] \\ &\leq 1 + C + 2\|u^i_0\|_{L^\infty(\mathbb{R})} \ln(1+C) := C', \end{aligned}$$

where in the first line we have used (3.19), in the second line we have used (4.27), in the third line we have used the weakly-$\star$ convergence of $w^{\varepsilon,i}$ towards w^i in $L^\infty((0,T);L\log L(\mathbb{R}))$ and in the fourth line, we have used (4.30). Putting this result together with (4.31), we get (1.5) with $C_1 = C + C'$.

□

5 Appendix: Example of the dynamics of dislocation densities

In this section, we present a model describing the dynamics of dislocation densities. We refer to Hirth et al. [17] for a physical presentation of dislocations which are (moving) defects in crystals. Even if the problem is naturally a three-dimensional problem, we will first assume that the geometry of the problem is invariant by translations in the x_3-direction. This reduces the problem to the study of dislocations densities defined on the plane (x_1, x_2) and moving in a given direction b belonging to the plane (x_1, x_2) (which is called the "Burger's vector").

In Subsection 5.1, we present the 2D-model with multi-slip directions. In the particular geometry where the dislocations densities only depend on the variable $x = x_1 + x_2$, this two-dimensional model reduces to a one-dimensional model which is presented in Subsection 5.2. Finally in Subsection 5.3, we explain how to recover equation (1.1) as a model for dislocation dynamics with

$$\lambda^i(u) = \sum_{j=1,\dots,d} A_{ij} u^j$$

for some particular non-negative and symmetric matrix A.

5.1 The 2D-model

We now present in details the two-dimensional model. We denote by X the vector $X = (x_1, x_2) \in \mathbb{R}^2$. We consider a crystal filling the whole space $\mathbb{R}^2$ and its displacement $v = (v_1, v_2) : \mathbb{R}^2 \to \mathbb{R}^2$, where we have not yet introduced the time dependence.

We introduce the total strain $\varepsilon(v) = (\varepsilon_{ij}(v))_{i,j=1,2}$ which is a symmetric matrix defined by

$$\varepsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

The total strain can be spitted in two parts:

$$\varepsilon_{ij}(v) = \varepsilon^e_{ij} + \varepsilon^p_{ij} \quad \text{with} \quad \varepsilon^p = \sum_{k=1,\dots,d} \varepsilon^{0,k} u^k,$$

where ε^e_{ij} is the elastic strain and ε^p_{ij} is the plastic strain. The scalar function u^k is the plastic displacement associated to the k-th slip system whose matrix $\varepsilon^{0,k}_{ij}$ is defined by

$$\varepsilon^{0,k}_{ij} = \frac{1}{2}\left(b^k_i n^k_j + n^k_i b^k_j\right),$$

where (b^k, n^k) is a family of vectors in $\mathbb{R}^2$, such that n^k is a unit vector orthogonal to the Burger's vector b^k (see Hirth et al. [17] for the definition of the Burger's vector of a dislocation)

To simplify the presentation, we assume the simplest possible periodicity property of the unknowns.

Assumption (H):

i) The function v is $\mathbb{Z}^2$-periodic with $\displaystyle\int_{(0,1)^2} v\, dX = 0.$

ii) For each $k = 1, \ldots, d$, there exists $L^k \in \mathbb{R}^2$ such that $u^k(X) - L^k \cdot X$ is a $\mathbb{Z}^2$-periodic.

iii) The integer d is even with $d = 2N$ and we have for $k = 1, \ldots, N$:

$$L^{k+N} = L^k, \;\; n^{k+N} = n^k, \;\; b^{k+N} = -b^k, \;\; \varepsilon^{0,k+N} = -\varepsilon^{0,k}.$$

iv) We denote by $\tau^k \in \mathbb{R}^2$ a unit vector parallel to b^k such that $\tau^{k+N} = \tau^k$. We require that L^k is chosen such $\tau^k \cdot L^k \geq 0$.

Remark in particular that the plastic strain ε^p_{ij} is $\mathbb{Z}^2$-periodic as a consequence of Assumption (H). The stress matrix is then given by

$$\sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl} \varepsilon^e_{kl} \quad \text{for} \quad i, j = 1, 2,$$

where $\Lambda = (\Lambda_{ijkl})_{i,j,k,l=1,2}$, are the constant elastic coefficients of the material, satisfying for some constant $m > 0$:

$$\sum_{i,j,k,l=1,2} \Lambda_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq m \sum_{i,j=1,2} \varepsilon^2_{ij}, \tag{5.32}$$

for all symmetric matrices $\varepsilon = (\varepsilon_{ij})_{ij}$, *i.e.* such that $\varepsilon_{ij} = \varepsilon_{ji}$.

Then the stress is assumed to satisfy the equation of elasticity

$$\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad \text{for} \quad i = 1, 2.$$

On the other hand the plastic displacement u^k is assumed to satisfy the following transport equation

$$\partial_t u^k = c^k \tau^k . \nabla u^k \quad \text{with} \quad c^k = \sum_{i,j=1,2} \sigma_{ij} \varepsilon^{0,k}_{ij}.$$

This equation can be interpreted, saying that

$$\theta^k = \tau^k.\nabla u^k \geq 0, \tag{5.33}$$

is the density of edge dislocations associated to the Burger's vector b^k moving in the direction τ^k at the velocity c^k. Here c^k is also called the resolved Peach-Koehler force in the physical literature. In particular, we see that the dislocation density θ^k satisfies the following conservation law

$$\partial_t \theta^k = \mathrm{div}(c^k \tau^k \theta^k).$$

Finally, for $k = 1, \ldots, d$, the functions u^k and v are then assumed to depend on $(t, X) \in (0, +\infty) \times \mathbb{R}^2$ and to be solutions of the coupled system (see Yefimov [28, ch. 5.] and Yefimov, Van der Giessen [29]):

$$\left\{ \begin{array}{l} \displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad \text{on } (0,+\infty)\times\mathbb{R}^2, \qquad \text{for } i = 1,2, \\ \left.\begin{array}{ll} \displaystyle\sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl} \left(\varepsilon_{kl}(v) - \sum_{k=1,\ldots,d} \varepsilon_{ij}^{0,k} u^k \right) & \text{on } (0,+\infty)\times\mathbb{R}^2, \\ \displaystyle\varepsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) & \text{on } (0,+\infty)\times\mathbb{R}^2, \end{array} \right| \quad \text{for } i,j = 1,2 \\ \\ \partial_t u^k = \left(\displaystyle\sum_{i,j\in\{1,2\}} \sigma_{ij} \varepsilon_{ij}^{0,k} \right) \tau^k.\nabla u^k \quad \text{on } (0,+\infty)\times\mathbb{R}^2, \qquad \text{for } k = 1,\ldots,d, \end{array} \right. \tag{5.34}$$

where Λ_{ijkl}, $\varepsilon_{ij}^{0,k}$ are fixed parameters previously introduced, and the unknowns of the system are $u = (u^k)_{k=1,\ldots,d}$ and the displacement $v = (v_1, v_2)$. Remark also that our equations are compatible with our periodicity assumptions (H), (i)-(ii).

For a detailed physical presentation of a model with multi-slip directions, we refer to Yefimov, Van der Giessen [29] and Yefimov [28, ch. 5.] and to Groma, Balogh [16] for the case of a model with a single slip direction. See also Cannone et al. [5] for a mathematical analysis of the Groma, Balogh model.

5.2 Derivation of the 1D-model

In this subsection we are interested in a particular geometry where the dislocation densities depend only on the variable $x = x_1 + x_2$. This will lead to a 1D-model. More precisely, we make the following:

Assumption (H'):

i) The functions $v(t,X)$ and $u^k(t,X) - L^k \cdot X$ depend only on the variable $x = x_1 + x_2$.

ii) For $k = 1, \dots, d$, the vector $\tau^k = (\tau_1^k, \tau_2^k)$ satisfies $\tau_1^k + \tau_2^k > 0$ with $\mu^k = \dfrac{1}{\tau_1^k + \tau_2^k}$.

iii) For $k = 1, \dots, d$, the vector $L^k = (L_1^k, L_2^k)$ satisfies $L_1^k = L_2^k = l^k$.

For this particular one-dimensional geometry, we denote by an abuse of notation the function $v = v(t,x)$ which is 1-periodic in x. By assumption (H'), (iii), we can see (again by an abuse of notation) that $u = (u^k(t,x))_{k=1,\dots,d}$ is such that for $k = 1, \dots, d$, $u^k(t,x) - l^k \cdot x$ is 1-periodic in x.

Now, we can integrate the equations of elasticity, *i.e.* the first equation of (5.34). Using the $\mathbb{Z}^2$-periodicity of the unknowns (see assumption (H), (i)-(ii)), and the fact that $\varepsilon^{0,k+N} = -\varepsilon^{0,k}$ (see assumption (H), (iii)), we can easily conclude that the strain

$$\varepsilon^e \text{ is a linear function of } (u^j - u^{j+N})_{j=1,\dots,N} \quad \text{and of } \left(\int_0^1 (u^j - u^{j+N})\, dx \right)_{j=1,\dots,N}. \tag{5.35}$$

This leads to the following Lemma

***Lemma* 5.1 (Stress for the 1D-model)**
Under assumptions (H), (i)-(ii)-(iii) and (H'), (i)-(iii) and (5.32), we have

$$-\sigma : \varepsilon^{0,i} = \sum_{j=1,\dots,d} A_{ij} u^j + \sum_{j=1,\dots,d} Q_{ij} \int_0^1 u^j \, dx, \quad \text{for } i = 1, \dots, N, \tag{5.36}$$

where for $i, j = 1, \dots, N$

$$\begin{cases} A_{i,j} = A_{j,i} \quad \text{and} \quad A_{i+N,j} = -A_{i,j} = A_{i,j+N} = -A_{i+N,j+N}, \\ Q_{i,j} = Q_{j,i} \quad \text{and} \quad Q_{i+N,j} = -Q_{i,j} = Q_{i,j+N} = -Q_{i+N,j+N}. \end{cases} \tag{5.37}$$

Moreover the matrix A is non-negative.

The proof of Lemma 5.1 will be given at the end of this subsection.

Finally using Lemma 5.1, we can eliminate the stress and reduce the problem to a one-dimensional system of d transport equations only depending on the function u^i, for $i = 1, \dots, d$. Naturally, from (5.36) and (H'), (ii) this 1D-model has the following form

The 1D-model of the dynamics of dislocation densities:

$$\mu^i \partial_t u^i + \left(\sum_{j=1,\dots,d} A_{ij} u^j + \sum_{j=1,\dots,d} Q_{ij} \int_0^1 u^j \, dx \right) \partial_x u^i = 0, \qquad \text{on } (0,+\infty)\times\mathbb{R}, \quad \text{for } i = 1,\dots,d, \tag{5.38}$$

with from (5.33)

$$\partial_x u^i \geq 0 \quad \text{for } i = 1,\dots,d. \tag{5.39}$$

Now, we give the proof of Lemma 5.1.

Proof of Lemma 5.1:
For the 2D-model, let us consider the elastic energy on the periodic cell (using the fact that ε^e is $\mathbb{Z}^2$-periodic)

$$E(u,v) = \frac{1}{2} \int_{(0,1)^2} \sum_{i,j,k,l=1,2} \Lambda_{ijkl} \varepsilon^e_{ij} \varepsilon^e_{kl} \, dX \quad \text{with} \quad \varepsilon^e_{ij} = \varepsilon_{ij}(v) - \sum_{k=1,\dots,d} \varepsilon^{0,k}_{ij} u^k.$$

By definition of σ_{ij} and ε^e_{ij}, we have for $k = 1,\dots,d$

$$\sum_{i,j=1,2} (\sigma_{ij} \varepsilon^{0,k}_{ij}) = -E'_{u^k}(u,v). \tag{5.40}$$

On the other hand using (H'), (i)-(iii), (with $x = x_1 + x_2$) we can check that we can rewrite the elastic energy as

$$E = \frac{1}{2} \int_0^1 \sum_{i,j,k,l=1,2} \Lambda_{ijkl} \varepsilon^e_{ij} \varepsilon^e_{kl} dx.$$

Replacing ε^e_{ij} by its expression (5.35), we get:

$$\begin{aligned} E = \; & \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} (u^j - u^{j+N})(u^i - u^{i+N}) \, dx \\ & + \frac{1}{2} \sum_{i,j=1,\dots,N} Q_{ij} \left(\int_0^1 (u^j - u^{j+N}) \, dx \right) \left(\int_0^1 (u^i - u^{i+N}) \, dx \right), \end{aligned}$$

for some symmetric matrices $A_{ij} = A_{ji}$, $Q_{ij} = Q_{ji}$. In particular, joint to (5.40) this gives exactly (5.36) with (5.37).

Let us now consider the functions $w^i = u^i - u^{i+N}$ such that

$$\int_0^1 w^i \, dx = 0 \quad \text{for i=1,...,N.} \tag{5.41}$$

From (5.32), we deduce that

$$0 \leq E = \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} w^i w^j \, dx.$$

More precisely, for all $i = 1, \dots, N$ and for all $\bar{w}^i \in \mathbb{R}$, we set

$$w^i = \begin{cases} \bar{w}^i & \text{on} \quad [0, \frac{1}{2}], \\ -\bar{w}^i & \text{on} \quad [\frac{1}{2}, 1], \end{cases}$$

which satisfies (5.41). Finally, we obtain that

$$0 \leq E = \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} \bar{w}^i \bar{w}^j \, dx.$$

Because this is true for every $\bar{w} = (\bar{w}^1, \dots, \bar{w}^N) \in \mathbb{R}^N$, we deduce that A a non-negative matrix.

□

We refer the reader to El Hajj [9] and El Hajj, Forcadel [10] for a study in the special case of a single slip direction, i.e. in the case $N = 1$.

5.3 Heuristic derivation of the non-periodic model

Starting from the model (5.38)-(5.39) where for $i = 1, \dots, d$, the function $u^i(t,x) - l^i \cdot x$ is 1-periodic in x, we now want to rescale the unknowns to make the periodicity disappear. More precisely, we have the following Lemma:

***Lemma* 5.2 (Non-periodic model)**
Let u be a solution of (5.38)-(5.39) assuming Lemma 5.1 and $u^i(t,x) - l^i \cdot x$ is 1-periodic in x. Let

$$u^j_\delta(t,x) = u^j(\delta t, \delta x), \quad \textit{for a small } \delta > 0 \textit{ and for } j = 1, \dots, d,$$

such that, for all $j = 1, \dots, d$

$$u^j_\delta(0,\cdot) \to \bar{u}^j(0,\cdot), \quad \textit{as} \quad \delta \to 0, \quad \textit{and} \quad \bar{u}^j(0, \pm\infty) = \bar{u}^{j+N}(0, \pm\infty). \tag{5.42}$$

Then $\bar{u} = (\bar{u}^j)_{j=1,\dots,d}$ is formally a solution of

$$\mu^i \partial_t \bar{u}^i + \left(\sum_{j=1,\dots,d} A_{ij} \bar{u}^j \right) \partial_x \bar{u}^i = 0, \qquad \textit{on } (0, +\infty) \times \mathbb{R}, \tag{5.43}$$

where the symmetric matrix A is non-negative and $\partial_x \bar{u}^i \geq 0$ for $i = 1, \dots, d$.

We remark that the limit problem (5.43) is of type (1.1) when $\mu^i = 1$. In particular, there are no reasons to assume that this system is strictly hyperbolic in general. Nevertheless, the general case $\mu^i > 0$ can be treated with our approach developed in Theorem 1.1 considering the entropy $\displaystyle\int_{\mathbb{R}} \sum_{i=1,\dots,d} \mu^i f\left(\partial_x \bar{u}^i(t,x)\right) dx$ instead of $\displaystyle\int_{\mathbb{R}} \sum_{i=1,\dots,d} f\left(\partial_x \bar{u}^i(t,x)\right) dx$.

Formal proof of Lemma 5.2:
Here, we know that $u^i_\delta - \delta l^i \cdot x$ is $\dfrac{1}{\delta}$-periodic in x, and satisfies for $i = 1, \dots, d$

$$\mu^i \partial_t u^i_\delta + \left(\sum_{j=1,\dots,d} A_{ij} u^j_\delta + \delta \sum_{j=1,\dots,d} Q_{ij} \int_0^{\frac{1}{\delta}} u^j_\delta \, dx \right) \partial_x u^i_\delta = 0, \qquad \text{on } (0,+\infty)\times\mathbb{R}. \quad (5.44)$$

To simplify, assume that the initial data $u_\delta(0,\cdot)$ converge to a function $\bar{u}(0,\cdot)$ such that the function $\partial_x u_\delta(0,\cdot)$ inside the interval $\left(\dfrac{-1}{2\delta}, \dfrac{1}{2\delta}\right)$ has a support in $(-R,R)$, uniformly in δ, where R a positive constant. Because of the antisymmetry property of the matrix Q (see assumption (5.37)), and because of assumption (5.42), we expect heuristically that the velocity in (5.44) remains uniformly bounded as $\delta \to 0$.

Therefore, using the finite propagation speed, we see that, there exists a constant C independent in δ, such that $\partial_x u_\delta(t,\cdot)$ has a support on $(-R-Ct, R+Ct) \subset \left(\dfrac{-1}{2\delta}, \dfrac{1}{2\delta}\right)$. Moreover, from (5.42) and the fact that

$$\sum_{j=1,\dots,d} Q_{ij} \int_0^{\frac{1}{\delta}} u^j_\delta \, dx = \sum_{j=1,\dots,N} Q_{ij} \int_0^{\frac{1}{\delta}} (u^j_\delta - u^{j+N}_\delta) \, dx,$$

we deduce that

$$\sum_{j=1,\dots,d} Q_{ij} \int_0^{\frac{1}{\delta}} u^j_\delta \, dx,$$

remains bounded uniformly in δ. Then formally the non-local term vanishes and we get for $i = 1, \dots, d$

$$\sum_{j=1,\dots,d} A_{ij} u^j_\delta + \delta \sum_{j=1,\dots,d} Q_{ij} \int_0^{\frac{1}{\delta}} u^j_\delta \, dx \to \sum_{j=1,\dots,d} A_{ij} \bar{u}^j, \quad \text{as} \quad \delta \to 0,$$

which proves that $\bar{u}$ is solution of (5.43), with the non-negative symmetric matrix A. □

6 Acknowledgements

The first author would like to thank M. Cannone, T. Gallouët and M. Jazar for fruitful remarks that helped in the preparation of the paper. This work was partially supported

the program “PPF, programme pluri-formations mathématiques financières et EDP”, (2006-2010), Marne-la-Vallée University and École Nationale des Ponts et Chaussées and the ANR MICA "Mouvements d'Interfaces, Calcul et Applications" (2006-2009).

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