Martingales and Rates of Presence in Homogeneous Fragmentations

Nathalie Krell, Alain Rouault

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Abstract

In mass-conservative homogeneous fragmentations, sizes of the fragments decrease at **asymptotic** exponential rates. Like in branching processes, two situations occur: either the number of such fragments is exponentially growing - the rate is effective -, or the probability of presence of such fragments is exponentially decreasing [3],[12].

In a recent paper [18], N. Krell considers fragments whose sizes decrease at **exact** exponential rates. In this new setting, she characterizes the effective rates and studies Hausdorff dimension. The present paper carries out a detailed analysis of this model and focus on presence probabilities, using the spine method and a suitable martingale. For the sake of completeness, we compare our results with results and methods of the classical model.

Key Words. fragmentation, Lévy process, martingales, probability tilting.

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1 Introduction and main results

Fragmentations are well defined in the book of Bertoin [8] (see also [1] and [3]) and a short overview on this field is given here. We consider a homogenous fragmentation F of intervals, which is a Markov process in continuous time taking its values in the set $\mathcal{U}$ of open sets of $(0,1)$. Informally, each interval component - or *fragment* - splits as time goes on, independently of the others and with the same law, up to a rescaling. We make

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the restriction that the fragmentation is conservative, which means that no mass is lost. In this case, the law F is completely characterized by the so-called dislocation measure ν (corresponding to the jump-component of the process) which is a measure on $\mathcal{U}$ fulfilling the following conditions

$$\nu((0,1)) = 0,$$

$$\int_{\mathcal{U}} (1-u_1)\nu(dU) < \infty, \tag{1}$$

and

$$\sum_{i=1}^{\infty} u_i = 1 \qquad \text{for } \nu - \text{almost every } U \ \in \mathcal{U},$$

where for $U \in \mathcal{U}$,

$$|U|^{\downarrow} := (u_1, u_2, ...)$$

is the decreasing sequence of the lengths of the interval components of U.

It appears quite natural to study the rates of decay of fragments. If we measure the fragments by logarithms of their sizes, a homogeneous fragmentation can be considered as an extension of a classical branching random walk in continuous time. The common feature of many branching models consists in the alternative between exponential growth and extinction. Let us recall some basic facts about a Galton-Watson process ζ_n started from $\zeta_0 = 1$ with finite mean $m = \mathbb{E}\zeta_1$. We have $n^{-1}\log \mathbb{E}(\zeta_n) = \log m$ and

(a) if $m > 1$, and $\mathbb{P}(\zeta_1 \geq 1) = 1$ then $\lim n^{-1} \log Z_n = \log m$ a.s.

(b) if $m < 1$ then $\lim_n n^{-1} \log \mathbb{P}(Z_n \neq 0) = \log m$. More generally, in branching random walks, when the local rate of growth of the population in expectation is (exponentially) positive, it is a.s. the effective local rate of growth of the population, and when it is negative, it is the local rate of decrease of the probability of presence.

The goal of this paper is to present results of the second type, i.e. asymptotic study of presence of abnormally large fragments. Let us first explain known results of the first type - exponential growth - and fix some notation.

For $x \in (0,1)$ let $I_x(t)$ be the component of the interval fragmentation $F(t)$ which contains x, and let $|I_x(t)|$ be its length. Bertoin showed in [6] that if V is a uniform random variable on $[0,1]$ independent of the fragmentation, then

$$\xi(t) := -\log |I_V(t)|$$

is a subordinator whose distribution is entirely determined by the characteristics of the fragmentation. Its Laplace exponent is given by

$$\mathbb{E}\mathrm{e}^{-q\xi(t)} = \mathrm{e}^{-t\kappa(q)}$$

where κ is the concave positive function :

$$\kappa(q) := \int_{\mathcal{U}} \left(1 - \sum_{j=1}^{\infty} u_j^{q+1}\right) \nu(dU) \qquad \forall q > \underline{p} \tag{2}$$

and $\underline{p}$ is the smallest real number for which κ remains finite :

$$\underline{p} := \inf\left\{p \in \mathbb{R} : \int_{\mathcal{U}} \sum_{j=2}^{\infty} u_j^{p+1} \nu(dU) < \infty\right\}.$$

The SLLN tells us that a.s. $\xi(t)/t \to \kappa'(0) =: v_{typ}$, so that a.s.

$$\lim_{t\to\infty} -t^{-1} \log |I_V(t)| = v_{typ}\,.$$

In fact, there is an interval $(v_{\min}, v_{\max})$ straddling v_{typ} of effective asymptotic exponential rates of decreasing of fragments which we describe now. Let $\overline{p}$ be the unique solution of the equation

$$\kappa(q) = (q+1)\kappa'(q), \quad q > \underline{p}\,.$$

We define $v_{\min} = \kappa'(\overline{p})$ and $v_{\max} := \kappa'(\underline{p}^+)$.

In all this article, we fix a and b such that $0 < a < 1 < b$.

If we set

$$\widetilde{G}_{v,a,b}(t) = \{I_x(t) : x \in (0,1) \;\; ae^{-vt} < |I_x(t)| < be^{-vt}\}$$

then it is known ([3], [12] Corollary 3) that the asymptotic growth of $\widetilde{G}_{v,a,b}(t)$ is ruled by the concave function C defined for $v < v_{\max}$ by

$$C(v) = \inf_{q>\underline{p}} (q+1)v - \kappa(q)\,, \tag{3}$$

or

$$C(v) = (\Upsilon_v + 1)v - \kappa(\Upsilon_v) \quad , \quad \kappa'(\Upsilon_v) = v\,. \tag{4}$$

More precisely we have:

- for $v \in (v_{\min}, v_{\max})$, $C(v)$ is strictly positive and

$$\lim_{t\to\infty} t^{-1} \log \sharp\widetilde{G}_{v,a,b}(t) = C(v) \text{ a.s.} \tag{5}$$

- for $v \le v_{\min}$, $C(v)$ is strictly negative and the set $\widetilde{G}_{v,a,b}(t)$ is a.s. empty for t large enough.

Let us stress that $C(v)$ depends only on v and not on a, b.

The latter setting will be referred as *classical*.

In a recent paper [18], Krell studied the more constrained set

$$G_{v,a,b}(t) = \{I_x(t) : x \in (0,1) \text{ and } ae^{-vs} < |I_x(s)| < be^{-vs} \;\; \forall\, s \le t\}\,,$$

and proved a result of the same kind. In particular, Proposition 3 (p.908) [18] tells us that there exists[1] a positive number $\rho(v, a, b)$ depending upon v, a, b such that

[1] see the forthcoming Section 2 for a precise definition

- for $v > \rho(v,a,b)$, conditionally on $\{\inf\{t : G_{v,a,b}(t) \neq \emptyset\} = \infty\}$

$$\lim_{t\to\infty} t^{-1} \log \sharp G_{v,a,b}(t) = v - \rho(v,a,b)\,, \text{ a.s.} \tag{6}$$

- for $v < \rho(v,a,b)$, $\lim_{t\to\infty} \sharp G_{v,a,b}(t) = 0$ a.s..

This result holds under the following assumption A, which comes from [21] and [5], and ensures the absolute continuity of the marginals of the underlying Lévy process.

Assumption A. The image ν_1 of the measure ν by the mapping $U \mapsto u_1$ satisfies

$$\nu_1^{ac}([0,\epsilon)) = \infty \quad \text{for any} \quad \epsilon > 0\,, \tag{7}$$

where ν_1^{ac} be the absolutely continuous part of ν_1.

Referring to the above general comments on branching models, we can say that the above assertions (5) and (6) are of the first type. Our main aim here is to present results of the second type.

For the classical model, an assumption is needed. A fragmentation is called r-lattice with $r > 0$, if $\xi(t)$ is a compound Poisson process whose jump measure has a support carried by a discrete subgroup of $\mathbb{R}$ and r is the mesh. If there is no such r, the fragmentation is called non-lattice.

Assumption B. Either the fragmentation is non-lattice, or it is r-lattice and a, b satisfy $b > a\mathrm{e}^r$.

Theorem 1.1. *[11] Under Assumption B, if $v < v_{\min}$, then*

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) = C(v) \tag{8}$$

In [11], the result of Theorem 5 is more precise since it gives sharp (i.e. non logarithmic) estimates of the latter probability.

For the more constrained set $G_{v,a,b}(t)$, the corresponding result is the following.

Theorem 1.2. *Under assumption A, if $v - \rho(v,a,b) < 0$, then*

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}(G_{v,a,b}(t) \neq \emptyset) = v - \rho(v,a,b) \tag{9}$$

Theorem 1.2 is the main result of the present paper. The crucial tool consists in first introducing additive martingales to make a change of probability and then using a decomposition according to the spine method. For the sake of completeness, a direct short proof of Theorem 1.1 in the same spirit is given to illustrate the common feature of both models (it should be noted that a similar method and result hold for the branching Brownian motion in [16]).

Let us remark that since $G_{v,a,b} \subset \tilde{G}_{v,a,b}$, the limits (5), (8), (6) and (9) are comparable. In fact we have the following general result m

Proposition 1.3. *Under assumption A, for all $v < v_{max}$*

$$C(v) > v - \rho(v, a, b). \tag{10}$$

In Section 2, we present all the tools on fragmentations and Lévy processes. Section 3 is devoted to the study of the two martingales and their asymptotic properties. In particular, a mistake in the proof of Theorem 2.1 in [18] is corrected. In Section 4, we give the proofs of the theorems on the presence probabilities and the proofs of the results on martingales. Section 5 is devoted to a proof of Proposition 1.3 only based on properties of Lévy processes.

2 Background on fragmentations and Lévy processes.

2.1 Partition fragmentations and interval fragmentations

Let $\mathcal{P}$ the space of partition of $\mathbb{N}$, and for every integer k, the block $\{1, ..., k\}$ is denoted by $[k]$. As in [10], we call discrete point measure on the space $\Omega := \mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, any measure:

$$w = \sum_{(t,\pi,k)\in\mathcal{D}}^{\infty} \delta_{(t,\pi,k)},$$

where $\mathcal{D}$ is a subset of $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ such that

$$\forall t' \geq 0 \;\; \forall n \in \mathbb{N} \;\; \sharp \left\{ (t, \pi, k) \in \mathcal{D} \mid t \leq t', \pi_{|[n]} \neq ([n], \emptyset, \emptyset, ...), k \leq n \right\} < \infty$$

and for all $t \in \mathbb{R}$

$$w(\{t\} \times \mathcal{P} \times \mathbb{N}) \in \{0, 1\}.$$

Starting from an arbitrary discrete point measure ω on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, we will construct a nested partition $\Pi = (\Pi(t), t \geq 0)$ (which means that for all $t \geq t'$ $\Pi(t)$ is a finer partition of $\mathbb{N}$ than $\Pi(t')$). We fix $n \in \mathbb{N}$, the assumption that the point measure ω is discrete enables us to construct a step path $(\Pi(t, n), t \geq 0)$ with values in the space of partitions of $[n]$, which only jumps at times t at which the fiber $\{t\} \times \mathcal{P} \times \mathbb{N}$ carries an atom of ω, say (t, π, k), such that $\pi_{|[n]} \neq ([n], \emptyset, \emptyset, ...)$ and $k \leq n$. In that case, $\Pi(t, n)$ is the partition obtained by replacing the $k-th$ block of $\Pi(t-, n)$, denoted $\Pi_k(t-, n)$, by the restriction $\pi_{|\Pi_k(t-,n)}$ of π to this block, and leaving the other blocks unchanged. Of course for all $t \geq 0$, $(\Pi(t, n), n \geq 0)$ is compatible (i.e. for every n, $\Pi(n, t)$ is a partition of $[n]$ such that the restriction of $\Pi(n+1, t)$ to $[n]$ coincide with $\Pi(n, t)$), as a consequence, there exists a unique partition $\Pi(t)$, such that for all $n \geq 0$ we have $\Pi(t)_{|[n]} = \Pi(t, n)$. With the terminology of [6], it is shown in [10] that this process Π is a partition-valued homogeneous fragmentation.

Let the set $\mathcal{S}^{\downarrow}$ be

$$\mathcal{S}^{\downarrow} := \left\{ s = (s_1, s_2, ...) \mid s_1 \geq s_2 \geq ... \geq 0 \; , \sum_{i=1}^{\infty} s_i \leq 1 \right\}.$$

A block $B \subset \mathbb{N}$ has an asymptotic frequency, if the limit

$$|B| := \lim_{n\to\infty} n^{-1}\sharp(B \cap [n])$$

exists. When every block of some partition $\pi \in \mathcal{P}$ has an asymptotic frequency, we write $|\pi| = (|\pi_1|, ...)$ and then $|\pi|^{\downarrow} = (|\pi_1|^{\downarrow}, ...) \in \mathcal{S}^{\downarrow}$ for the decreasing rearrangement of the sequence $|\pi|$. In the case where a block of the partition π does not have an asymptotic frequency, we decide that $|\pi| = |\pi|^{\downarrow} = \partial$, where ∂ stands for some extra point added to $\mathcal{S}^{\downarrow}$.

The sigma-field generated by the restriction to $[0, t] \times \mathcal{P} \times \mathbb{N}$ is denoted by $\mathcal{G}(t)$. So $(\mathcal{G}(t))_{t\geq 0}$ is a filtration, and the nested partitions $(\Pi(t), t \geq 0)$ are $(\mathcal{G}(t))_{t\geq 0}$-adapted. We also define the sigma-field $(\mathcal{F}(t))_{t\geq 0}$ generated by the decreasing rearrangement $|\Pi(r)|^{\downarrow}$ of the sequence of the asymptotic frequencies of the blocks of $\Pi(r)$ for $r \leq t$. Of course $(\mathcal{F}(t))_{t\geq 0}$ is a sub-filtration of $(\mathcal{G}(t))_{t\geq 0}$.

Let $\mathcal{G}_1(t)$ the sigma-field generated by the restriction of the discrete point measure w to the fiber $[0, t] \times \mathcal{P} \times \{1\}$. So $(\mathcal{G}_1(t), t \geq 0)$ is a sub-filtration of $(\mathcal{G}(t), t \geq 0)$, and the first block of Π is $(\mathcal{G}_1(t), t \geq 0)$-measurable. Let $\mathcal{D}_1 \subseteq \mathbb{R}_+$ be the random set of times $r \geq 0$ for which the discrete point measure has an atom on the fiber $\{r\} \times \mathcal{P} \times \{1\}$, and for every $r \in \mathcal{D}_1$, denote the second component of this atom by $\pi(r)$.

There is a powerful link between partition fragmentations and interval fragmentations. On the one hand, the $\mathcal{S}^{\downarrow}$-valued process of ranked asymptotic frequencies $|\Pi|^{\downarrow}$ of a partition fragmentation is a so-called ranked (or mass) fragmentation ([2], [6]), and conversely a partition fragmentation can be built from a ranked fragmentation via a "paint-box" process. On the other hand, the interval decomposition $(J_i(t), J_2(t), ...)$ of the open $F(t)$ ranked in decreasing order is a ranked fragmentation, denoted by $X(t) := (|J_i(t)|, |J_2(t)|, ...)^{\downarrow}$. We can then lift this ranked fragmentation to a partition fragmentation. More precisely, if ν is the dislocation measure of F, and $\tilde{\nu}$ its image by the map $U \mapsto |U|^{\downarrow}$, then according to Theorem 2 in [6], there exists a unique measure μ on $\mathcal{P}$ which is exchangeable (i.e. invariant by the action of finite permutations on $\mathcal{P}$), and such that $\tilde{\nu}$ is the image of μ by the map $\pi \mapsto |\pi|^{\downarrow}$ where $|\pi|^{\downarrow}$ is the decreasing rearrangement of the sequence of the asymptotic frequencies of the blocks of π. So, for all measurable function $f : [0, 1] \to \mathbb{R}_+$ such that $f(0) = 0$,

$$\int_{\mathcal{P}} f(|\pi_1|)\mu(d\pi) = \int_{\mathcal{S}^{\downarrow}} \sum_{i=1}^{\infty} s_i f(s_i)\tilde{\nu}(ds) = \int_{\mathcal{U}} \sum_{i=1}^{\infty} u_i f(u_i)\nu(dU).$$

It should be noted that $\{|J_1(t)|, |J_2(t)|, ...\}_{t\geq 0} = \{|\Pi_1(t)|, |\Pi_2(t)|, ...\}_{t\geq 0}$.

In the following sections, Π refers to this partition fragmentation.

2.2 Lévy processes.

A Lévy process is a stochastic process with càdlàg sample paths and stationary independent increments. Two particular types of such processes are considered:

• a subordinator is a Lévy process taking values in $[0, \infty)$, which implies that its sample paths are increasing,

• a Lévy process is called completely asymmetric when all its jumps have the same sign. We will consider here Lévy processes without positive jumps.

The Laplace exponent of a subordinator is characterized by[2]

$$\mathbf{E} \exp -\lambda\sigma_t = \exp -t\Phi(\lambda) \tag{11}$$

and the process

$$(\exp -p\sigma_t + t\Phi(p))_{t\geq 0} \tag{12}$$

is a martingale. We define the probability measure $\mathbf{P}^{(p)}$ as the h-transform of $\mathbf{P}$ by means of this martingale:

$$d\mathbf{P}^{(p)}|_{\mathcal{E}_t} = \exp\{-p\sigma_t + t\Phi(p)\}\ d\mathbf{P}|_{\mathcal{E}_t}\,. \tag{13}$$

Under $\mathbf{P}^{(p)}$, σ_t is a subordinator with Laplace exponent $q \mapsto \Phi(p+q) - \Phi(p)$. The change of probability shifts the drift.

We will need some technical notions about completely asymmetric Lévy processes, mostly taken from [4] and [5]. Let $Y = (Y_t)_{t\geq 0}$ be a Lévy process with no positive jumps and $(\mathcal{E}_t)_{t\geq 0}$ the natural filtration associated to $(Y_t)_{t\geq 0}$. The case where Y is the negative of a subordinator is degenerate for our purpose and therefore is implicitly excluded in the rest of the paper. The law of the Lévy process started at $x \in \mathbb{R}$ will be denoted by $\mathbf{P}_x$, its Laplace transform is given by

$$\mathbf{E}_0(\mathrm{e}^{\lambda Y_t}) = \mathrm{e}^{t\psi(\lambda)},\ \ \lambda,\ t \geq 0,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}$ is called the Laplace exponent.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be the right inverse of ψ (which exists because $\psi : \mathbb{R}_+ \to \mathbb{R}$ is convex with $\lim_{t\to\infty} \psi(\lambda) = \infty$), i.e. $\psi(\phi(\lambda)) = \lambda$ for every $\lambda \geq 0$. Let $W : \mathbb{R}_+ \to \mathbb{R}_+$ be the scale function, that is the unique continuous function with Laplace transform:

$$\int_0^\infty e^{-\lambda x} W(x)dx = \frac{1}{\psi(\lambda)} \quad ,\ \lambda > \phi(0).$$

For $q \in \mathbb{R}$, let $W^{(q)} : \mathbb{R}_+ \to \mathbb{R}_+$ be the continuous function such that for every $x \in \mathbb{R}_+$

$$W^{(q)}(x) := \sum_{k=0}^{\infty} q^k W^{*k+1}(x),$$

where $W^{*n} = W * ... * W$ denotes the nth convolution power of the function W, so that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\psi(\lambda) - q} \quad ,\ \lambda > \phi(q).$$

The functions $W^{(q)}$ are useful to investigate the two-sided exit problem for Lévy processes. Their properties are well exposed in the book of Kyprianou [20] and in [13], examples are in [17].

The following theorems taken from [21] and [5] yield another important martingale and its corresponding change of probability.

[2]Bold symbols $\mathbf{P}$ and $\mathbf{E}$ will refer to Lévy processes while $\mathbb{P}$ and $\mathbb{E}$ refer to fragmentations.

Theorem 2.1. *Let us define the critical value*

$$\rho_\beta := \inf\{q \geq 0\ ;\ W^{(-q)}(\beta) = 0\}. \tag{14}$$

Suppose that the one-dimensional distributions of the Lévy process are absolutely continuous. Then the following holds:

1. $\rho_\beta \in (0, \infty)$ *and the function* $W^{(-\rho_\beta)}$ *is strictly positive and continuous on* $(0, \beta)$.
2. *The mapping* $\beta \mapsto \rho_\beta = \inf\{q > 0 : W^{(-q)}(\beta) = 0\}$ *is strictly decreasing and of class* $\mathcal{C}^1$ *on* $(0, \infty)$.
3. *Let for* $\beta > 0$,

$$T_\beta = \inf\{t :\ Y_t \notin (0, \beta)\}\ ; \tag{15}$$

then the process

$$D_t := e^{\rho_\beta t}\ \mathbf{1}_{\{t<T_\beta\}}\ \frac{W^{(-\rho_\beta)}(Y_t)}{W^{(-\rho_\beta)}(x)} \tag{16}$$

is a $(\mathbf{P}_x, (\mathcal{E}_t))$*-martingale, for every* $x \in (0, \beta)$.

Remark 2.2. *The definition of* ρ_β *is complicated, but some examples are given in [18].*

Let the probability measure $\mathbf{P}^\updownarrow$ be the h-transform of $\mathbf{P}$ based on the martingale D_t :

$$d\mathbf{P}_x^\updownarrow|_{\mathcal{E}_t} = D_t d\mathbf{P}_x|_{\mathcal{E}_t}. \tag{17}$$

Theorem 2.3. *With the same assumption as in Theorem 2.1, under* $\mathbf{P}_x^\updownarrow$, (Y_t) *is a homogeneous strong Markov process on* $(0, \beta)$, *positive-recurrent and as* $t \to \infty$, Y_t *converges in distribution to its stationary probability, which has a density.*

It is essentially Theorem 3.1 in [21], the convergence in distribution is a consequence of Theorem 2 (v) of [5].

The change of probability forces the process to be confined in $(0, \beta)$.

3 Two additive martingales and their asymptotic behavior

As seen in [8] (p.133 and in Lemmas 3.9 and 3.10), the process $\xi_t = -\ln|\Pi_1(t)|$ is a $\mathcal{G}(t)$ subordinator, which means in particular that $\xi_{t+s} - \xi_t$ is independent of $\mathcal{G}(t)$. In this section, we will adapt the above statements to the subordinator ξ_t (instead of σ_t) and to the spectrally negative process $Y_t = vt - \xi_t - \log a$, starting at $x = -\log a$. It should be stressed that there is a slight change since $\mathcal{G}(t)$ is not the proper filtration of these processes, but the martingale properties remain true, as well as Markov property. We then perform a projection on the filtration $\mathcal{F}(t)$ of the ranked fragmentation.

3.1 The classical additive martingale $M_t^{(p)}$

As in (13), we define for $p > \underline{p}$ the $\mathcal{G}(t)$-martingale

$$D_t^{(p)} = \mathrm{e}^{-p\xi(t)+t\kappa(p)} ,$$

and the probability measure $\mathbb{P}^{(p)}$ as the h-transform of $\mathbb{P}$ based on $D_t^{(p)}$:

$$d\mathbb{P}^{(p)}|_{\mathcal{G}(t)} \ = D_t^{(p)} d\mathbb{P}|_{\mathcal{G}(t)}. \tag{18}$$

When the martingale $D_t^{(p)}$ is projected on the sub-filtration $(\mathcal{F}(t))_{t\geq 0}$, we obtain the well-known additive $\mathcal{F}(t)$-martingale

$$M_t^{(p)} = \sum_{j=1}^{\infty} |\Pi_j(t)|^{p+1} e^{\kappa(p)t} = \sum_{i=1}^{\infty} |J_i(t)|^{p+1} e^{\kappa(p)t} , \tag{19}$$

and the projection of (18) allows to build the new probability :

$$d\mathbb{P}^{(p)}|_{\mathcal{F}(t)} \ = M_t^{(p)} d\mathbb{P}|_{\mathcal{F}(t)} . \tag{20}$$

In [10], there is a complete description of the behavior of the process $(\Pi(t)\ ,\ t \geq 0)$ under $\mathbb{P}^{(p)}$, but since it is not used here, it is omitted.

3.2 The martingale $M_t^{(v,a,b)}$ associated to the set $G_{v,a,b}(t)$.

Since we are interested in the set of the "good" intervals at time t as

$$G_{v,a,b}(t) = \{I_x(t) :\ x \in (0,1) \ \text{ and } \ ae^{-vs} < |I_x(s)| < be^{-vs} \ \ \forall\ s \leq t\}. \tag{21}$$

it is natural to study the process

$$Y_t := vt - \xi_t - \log a\ ,\ t \geq 0$$

and its time of exit from $(0, \log b/a)$. It is a Lévy process with no positive jump. Its Laplace exponent is

$$\psi(\lambda) = v\lambda - \kappa(\lambda)$$

with κ defined in (2).

Assumption (A) guarantees that the marginals of the Lévy process Y_t are absolutely continuous and Theorem 2.1 can thus be applied.

For this Lévy process Y let

$$T := T_{\log(b/a)}$$

where T_β is defined in (15) and

$$\rho(v, a, b) := \rho_{\log(b/a)},$$

where ρ_β is defined in (14). To put it shortly, we will use frequently the notation ρ instead of $\rho(v,a,b)$.

To simplify the notation, let also

$$h(y) := W^{(-\rho)}(y - \log a)\mathbf{1}_{\{y\in(\log a, \log b)\}} \tag{22}$$

for all $y \in \mathbb{R}$, and $h(-\infty) = 0$. This function is well defined thanks to Theorem 2.1 and $h(0) \neq 0$.

By rewriting (16) with the new notation we get a $(\mathcal{G}(t))$-martingale

$$D_t = \mathrm{e}^{\rho t}\ \mathbf{1}_{\{t<T\}}\ \frac{h(vt-\xi_t)}{h(0)}, \qquad t \geq 0\,, \tag{23}$$

and then a new probability defined by

$$d\mathbb{P}^{\updownarrow}|_{\mathcal{G}(t)}\ = D_t d\mathbb{P}|_{\mathcal{G}(t)}. \tag{24}$$

For $i \geq 1$, let $P_i(t)$ be the block of $\Pi(t)$ which contains i at time t. We define the killed partition as follows

$$\Pi_j^\dagger(t) = \Pi_j(t)\mathbf{1}_{\{\exists i\in\mathbb{N}^* |\ \Pi_j(t)=P_i(t);\ \forall s\leq t\quad |P_i(s)|\in(ae^{-vs}, be^{-vs})\}}.$$

Similarly, if I is an interval component of $F(t)$, we define the "killed" interval $I^\dagger$ by $I^\dagger = I$ if I is good (i.e. $I \in G_{v,a,b}(t)$ with $G_{v,a,b}(t)$ defined in (21)), else by $I^\dagger = \emptyset$. Projecting the martingale D_t on the sub-filtration $(\mathcal{F}(t))_{t\geq 0}$, we obtain an additive martingale

$$\begin{aligned} M_t^{(v,a,b)} &= \frac{e^{\rho t}}{h(0)}\sum_{j\in\mathbb{N}} h\left(vt + \log|\Pi_j^\dagger(t)|\right)|\Pi_j^\dagger(t)| \\ &= \frac{e^{\rho t}}{h(0)}\sum_{i\in\mathbb{N}} h\left(vt + \log|J_i^\dagger(t)|\right)|J_i^\dagger(t)|. \end{aligned} \tag{25}$$

Finally, let the absorption time of M_t at 0 be

$$\zeta := \inf\{t : M_t = 0\} = \inf\{t : G_{v,a,b}(t) = \emptyset\},$$

with the convention $\inf\emptyset = \infty$.

The projection of (24) on the sub-filtration $\mathcal{F}(t)$ gives the identity:

$$d\mathbb{P}^{\updownarrow}|_{\mathcal{F}(t)}\ = M_t^{(v,a,b)} d\mathbb{P}|_{\mathcal{F}(t)}\,. \tag{26}$$

The interesting fact is that the change of probability $\mathbb{P}^{\updownarrow}$ only affects the behavior of the block which contains 1. More precisely, like in lemma 8 (ii) [10], we obtain

Lemma 3.1. *Suppose Assumption (A) holds. Under $\mathbb{P}^{\updownarrow}$, the restriction of w to $\mathbb{R}_+ \times \mathcal{P} \times \{2, 3, ...\}$ has the same distribution as under $\mathbb{P}$ and is independent of the restriction to the fiber $\mathbb{R}_+ \times \mathcal{P} \times \{1\}$.*

3.3 Growth of martingales

The above theorems rule the asymptotic behavior of our martingales. It should be noted that assertion 1 of Theorem 3.2 was claimed in [18] Theorem 2, but unfortunately there was a mistake in the proof. Indeed it is not true in general that the function h is Lipschitz.

Theorem 3.2. *In the previous notation, with the assumption A, then:*

1. *If $v > \rho(v,a,b)$, the martingale $M_t^{(v,a,b)}$ is bounded in $\mathrm{L}^2(\mathbb{P})$.*
2. *If $v < \rho(a,b,v)$,*

   a) $\lim_{t\to\infty} M_t^{(v,a,b)} = 0$, $\mathbb{P}$-a.s.,

   b) there exists $C_1, C_2 > 0$ such that for every t

$$C_1 \leq \mathrm{e}^{(v-\rho(v,a,b))t}\, \mathbb{E}\left[M_t^{(v,a,b)}\right]^2 \leq C_2\,. \tag{27}$$

Theorem 3.3. *1. If $p \in (\underline{p}, \overline{p})$, there exists $\alpha > 0$ such that the martingale $M_t^{(p)}$ is bounded in $\mathrm{L}^{1+\alpha}(\mathbb{P})$.*

2. *If $p \geq \overline{p}$,*

   a) $\lim_{t\to\infty} M_t^{(p)} = 0$, $\mathbb{P}$-a.s.

   b) There exists $\alpha_0 > 0$ such that for $\alpha \in (0,\alpha_0)$,

$$d(p,\alpha) := (1+\alpha)\kappa(p) - \kappa\left((1+\alpha)(p+1)-1\right) > 0 \tag{28}$$

   and then for those α, we have for every $t > 0$

$$\mathrm{e}^{d(p,\alpha)t}\, \mathbb{E}\, |M_t^{(p)}|^{1+\alpha} \leq C_{\alpha,p}\,, \tag{29}$$

   where $C_{\alpha,p}$ depends on α and p.

4 Proofs

4.1 Proof of Theorem 1.2

Proof: • We first show the upper bound of (9), i.e.

$$\limsup_{t\to\infty}\ t^{-1}\log \mathbb{P}(G_{v,a,b}(t) \neq \emptyset) \leq v - \rho\,. \tag{30}$$

Let $0 < \bar{a} < a < 1 < b < \bar{b}$. As in Section 3.2, we associate to $\bar{a}, \bar{b}$ and v, the parameter $\bar{\rho}$, as well as the set of 'good" intervals

$$G_{v,\bar{a},\bar{b}}(t) := \{I_x(t) :\quad x \in (0,1) \quad\text{and}\quad |I_x(s)| \ \in\ (\bar{a}e^{-vs}, \bar{b}e^{-vs}) \quad \forall\ s \leq t\}\,,$$

and the martingale $\bar{M}_t$.

Let for every $t \in \mathbb{R}$

$$\bar{h}(t) := W^{(-\bar{\rho})}(t + \log(1/\bar{a}))\mathbf{1}_{\{t \in (\log \bar{a}, \log \bar{b})\}}.$$

For $t \geq 0$ fixed, we have:

$$\begin{aligned} 1 = \mathbb{E}\bar{M}_t &= \frac{e^{\bar{\rho}t}}{\bar{h}(0)}\mathbb{E}\left(\sum_{i\in\mathbb{N}} \bar{h}(vt + \log|J_i(t)|)|J_i(t)| \ \mathbf{1}_{\{J_i(t)\in \bar{G}_{v,a,b}(t)\}}\right) \\ &\geq \frac{\bar{a}e^{(\bar{\rho}-v)t}}{\bar{h}(0)}\mathbb{E}\left(\sum_{i\in\mathbb{N}} \bar{h}(vt + \log|J_i(t)|) \ \ \mathbf{1}_{\{J_i(t)\in G_{v,a,b}(t)\}}\right) . \end{aligned}$$

Since $(a,b) \subsetneq (\bar{a},\bar{b})$, $\bar{h}$ is continuous and stricly positive on $[\log a, \log b]$ so that, if

$$C_5 := \bar{h}(0)/\left(\bar{a} \inf_{x\in[\log a,\log b]} \bar{h}(x)\right) < \infty,$$

then, for all $t \geq 0$:

$$C_5 \geq e^{(\bar{\rho}-v)t}\mathbb{E}\left(\sum_{i\in\mathbb{N}} \mathbf{1}_{\{J_i(t)\in G_{v,a,b}(t)\}}\right) \geq e^{(\bar{\rho}-v)t}\mathbb{P}(G_{v,a,b}(t) \neq \emptyset)$$

Hence for all $\bar{a},\bar{b}$ such that $0 < \bar{a} < a < 1 < b < \bar{b}$:

$$\limsup_{t\to\infty} \ t^{-1}\log\mathbb{P}(G_{v,a,b}(t) \neq 0) \leq v - \bar{\rho}\,.$$

For $\bar{a} \to a$ and $\bar{b} \to b$, by the continuity of $\rho_{\cdot}$ (see Theorem 2.1.2) the inequality (30) is obtained.

- Let us prove the lower bound of (9), i.e.

$$\liminf_{t\to\infty} \ t^{-1}\log\mathbb{P}(G_{v,a,b}(t) \neq \emptyset) \geq v - \rho\,. \tag{31}$$

Since M_t is a positive martingale and $\{G_{v,a,b}(t) \neq \emptyset\} = \{M_t \neq 0\}$, we have

$$1 = \mathbb{E}(M_t) = \mathbb{E}(M_t \mathbb{1}_{\{M_t\neq 0\}}) = \mathbb{E}(M_t \mathbb{1}_{\{G_{v,a,b}(t)\neq\emptyset\}})\,.$$

Now, thanks to the Cauchy-Schwarz inequality:

$$\mathbb{E}(M_t \mathbb{1}_{\{G_{v,a,b}(t)\neq\emptyset\}}) \leq \left(\mathbb{E}(M_t^2)\mathbb{P}(G_{v,a,b}(t) \neq \emptyset)\right)^{1/2}$$

and applying (27), we get

$$\mathbb{P}(G_{v,a,b}(t) \neq \emptyset) \geq L^{-1}e^{(v-\rho)t}$$

which yields (31). ■

4.2 Proof of Theorem 1.1

The upper bound is straightforward. We have for all $p \geq \underline{p}$,

$$1 = \mathbb{E}M_t^{(p)} = \mathbb{E}\left(\sum_{i=1}^{\infty} |J_i(t)|^{p+1} \mathrm{e}^{\kappa(p)t}\right) \geq a^{p+1} \mathrm{e}^{\kappa(p)t-(p+1)vt} \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) .$$

Hence

$$\mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \leq a^{-(p+1)} \mathrm{e}^{[(p+1)v-\kappa(p)]t}$$

and

$$\limsup_{t\to\infty} \; t^{-1} \log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \leq (p+1)v - \kappa(p).$$

In particular, for $p = \Upsilon_v$, we get from (4)

$$\limsup_{t\to\infty} \; t^{-1} \log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \leq C(v) . \tag{32}$$

To prove the lower bound:

$$\liminf_{t\to\infty} \; t^{-1} \log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \geq C(v) , \tag{33}$$

we use again the change of probability (20) with $p = \Upsilon_v$. We have,

$$\begin{aligned} \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) &= \mathbb{E}^{(p)} \left((M_t^{(p)})^{-1} ; \widetilde{G}_{v,a,b}(t) \neq \emptyset \right) \geq \\ \geq \;\; & \mathrm{e}^{tC(v)-t\varepsilon} \, \mathbb{P}^{(p)} \left(\sup_{0<s\leq t} M_s^{(p)} < \mathrm{e}^{-tC(v)+t\varepsilon} ; vt - \xi_t \in [\log a, \log b] \right) \end{aligned} \tag{34}$$

and

$$\begin{aligned} & \mathbb{P}^{(p)} \left(\sup_{0<s\leq t} M_s^{(p)} < \mathrm{e}^{-tC(v)+t\varepsilon} ; vt - \xi_t \in [\log a, \log b] \right) \geq \\ \geq \;\; & \mathbb{P}^{(p)} \left(vt - \xi_t \in [\log a, \log b] \right) - \mathbb{P}^{(p)} \left(\sup_{0<s\leq t} M_s^{(p)} \geq \mathrm{e}^{-tC(v)+t\varepsilon} \right) \end{aligned} \tag{35}$$

From Sections 2.2 and 3.1 we see that under $\mathbb{P}^{(p)}$, the Lévy process $(vt - \xi_t)_{t\geq 0}$ has mean $-\kappa'(p) + v = 0$ and variance $\sigma_p^2 := -\kappa''(p)$. From Proposition 2 of Bertoin and Doney [9] it satisfies the local central limit theorem, if it is not lattice. We get

$$\sigma_p \sqrt{2\pi t} \; \mathbb{P}^{(p)}(vt - \xi_t \in [\log a, \log b]) \to \log \frac{b}{a} . \tag{36}$$

and then

$$\liminf_t \; t^{-1} \log \mathbb{P}^{(p)}(vt - \xi_t \in [\log a, \log b]) = 0 . \tag{37}$$

In the case of a r-lattice fragmentation, under assumption B, there is at least an integer multiple of r in the interval $[vt-\log b, vt-\log a]$. We can use the lattice version of the local central limit theorem (see for instance [14] Theorem 2 iii)), and we get (37) again.

Let us tackle now the second term of the RHS of (35). By convexity $(M_t^{(p)})^{1+\alpha}$ is a $\mathbb{P}$-submartingale, so $(M_t^{(p)})^{\alpha}$ is a $\mathbb{P}^{(p)}$-submartingale[3]. Hence, by Doob's inequality,

$$\begin{aligned}
\mathbb{P}^{(p)}\left(\sup_{0<s\le t} |\, M_s^{(p)} \,| \ge \mathrm{e}^{-tC(v)+t\varepsilon}\right) &\le \mathrm{e}^{t\alpha C(v)-\alpha t\varepsilon}\mathbb{E}^{(p)} \,|\, M_t^{(p)} \,|^{\alpha} \\
&= \mathrm{e}^{t\alpha C(v)-\alpha t\varepsilon}\mathbb{E} \,|\, M_t^{(p)} \,|^{1+\alpha}\,, \qquad (38)
\end{aligned}$$

and by (29) we have for $\alpha \in (0, \alpha_1]$ for some $\alpha_1 > 0$

$$\mathbb{P}^{(p)}\left(\sup_{0<s\le t} |\, M_s^{(p)} \,| \ge \mathrm{e}^{-tC(v)+t\varepsilon}\right) \le K'_{\alpha,p}\mathrm{e}^{tH(\alpha)}\,, \qquad (39)$$

where

$$H(\alpha) = \alpha C(v) - \alpha\varepsilon + d(p,\alpha)\,,$$

and $K'_{\alpha,p}$ is some constant. Now, a second order development of κ around p gives

$$H(\alpha) = -\alpha\varepsilon - \frac{\alpha^2(p+1)^2}{2}\kappa''(p)(1+o(\alpha))$$

and, since $\kappa'' < 0$ (κ is concave), we may choose α being small enough such that $H(\alpha) < 0$. This yields

$$\limsup_t \; t^{-1}\log \mathbb{P}^{(p)}\left(\sup_{0<s\le t} |\, M_s^{(p)} \,| \ge \mathrm{e}^{-tC(v)+t\varepsilon}\right) < 0. \qquad (40)$$

So, gathering (40) and (36), we get

$$\liminf_{t\to\infty} t^{-1}\log \mathbb{P}^{(p)}\left(\sup_{0<s\le t} M_s^{(p)} < \mathrm{e}^{-tC(v)+t\varepsilon}; vt-\xi_t \in [\log a, \log b]\right) = 0\,,$$

which with (34) yields

$$\liminf_{t\to\infty} t^{-1}\log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \ge C(v) - \varepsilon$$

for every $\varepsilon > 0$. Letting $\varepsilon \to 0$ proves (33), and ends the proof of Theorem 1.1 . ■

[3] This the same argument as in [16]

4.3 Proof of Theorem 3.2 :

We use the change of probability (26), with $p = \Upsilon_v$:

$$\mathbb{E}(M_t^2) = \mathbb{E}^{\updownarrow}(M_t)\,, \tag{41}$$

and the spine decomposition:

$$M_t = c_t + d_t,$$

where

$$c_t := \frac{e^{\rho t}}{h(0)} h\left(vt + \log(|\Pi_1^{\dagger}(t)|)\right)\ |\Pi_1^{\dagger}(t)| \tag{42}$$

and

$$d_t := \frac{e^{\rho t}}{h(0)} \sum_{i=2}^{\infty} h\left(vt + \log\left(|\Pi_i^{\dagger}(t)|\right)\right)\ |\Pi_i^{\dagger}(t)|\,. \tag{43}$$

The asymptotic behavior of c_t and d_t is ruled by the two following lemmas.

Lemma 4.1. *Suppose Assumption (A) holds. Under $\mathbb{P}^{\updownarrow}$, $\mathrm{e}^{-(\rho-v)t}c_t$ converges in distribution as $t \to \infty$, to a random variable η with no mass en 0. Moreover there exists $C > 0$ such that*

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(c_t)\mathrm{e}^{-(\rho-v)t} = C\,. \tag{44}$$

Lemma 4.2. *Suppose Assumption (A) holds, if $\rho \neq v$, there exists $C > 0$ such that*

$$\mathbb{E}^{\updownarrow} d_t \leq C \max\{\mathrm{e}^{(\rho-v)t}, 1\}\,. \tag{45}$$

4.3.1 Proof of Theorem 3.2 1)

From (41), it is enough to prove that

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(M_t) < \infty\ .$$

By (44), we have

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(c_t) = 0\,. \tag{46}$$

By (45), we have $\sup_t \mathbb{E}^{\updownarrow}(d_t) < \infty$. ∎

4.3.2 Proof of Theorem 3.2 2) a)

The method is now classic, (see for instance [19]) and uses a decomposition which may be found in Durrett [15] p. 241. It will be used also in the proof of Theorem 3.3 below. We have only to prove that $\mathbb{P}^{\updownarrow}(\limsup M_t = \infty) = 1$.

We have the obvious lower bound

$$M_t \geq c_t$$

For $v < \rho$, Lemma (4.1) yields $\lim c_t = \infty$ in $\mathbb{P}^{\updownarrow}$ probability, or in other words $\mathbb{P}^{\updownarrow}(\limsup_t c_t = \infty) = 1$ which implies $\mathbb{P}^{\updownarrow}(\limsup_t M_t = \infty) = 1$, hence $\mathbb{P}(\lim_t M_t = 0) = 1$. ∎

4.3.3 Proof of Theorem 3.2 2 b)

This result is a straightforward consequence of (41) and the two lemmas. ■

4.3.4 Proof of Lemma 4.1 :

From the definition of c_t (42), we see that the distribution of $\left(\mathrm{e}^{-(\rho-v)t}c_t, t \geq 0\right)$ under $\mathbb{P}^{\updownarrow}$ is the distribution of $\left(\frac{1}{h(0)}W^{(-\rho)}(Y_t)\mathrm{e}^{Y_t}\,\mathbf{1}_{\{t<T\}}, t \geq 0\right)$ under $\mathbf{P}^{\updownarrow}_{\log(1/a)}$. Under $\mathbf{P}^{\updownarrow}_{\log(1/a)}$, the stopping time T is a.s. infinite and from Theorem 2.3 the process Y_t is positive-recurrent, converges in distribution and the limit has no mass in 0. Since the function $y \mapsto W^{(-\rho)}(y)\mathrm{e}^y$ is continuous, it is bounded on the compact support of the distribution of Y_t, and $y_t = \mathbb{E}^{\updownarrow}\left(c_t\mathrm{e}^{-(\rho-v)t}\right)$ has a positive limit. ■

4.3.5 Proof of Lemma 4.2 :

We start from the definition of d_t decomposing the time interval $[0, t]$ into pieces $[k-1, k[$ and splitting the sum (43) according to the time where the fragment separates from 1:

$$h(0)\mathrm{e}^{-\rho t}d_t = \sum_k \sum_{i\in\mathcal{I}_k} h(vt + \log|\Pi^{\dagger}_i(t)|)|\Pi^{\dagger}_i(t)| \tag{47}$$

where $\mathcal{I}_k$ is the set of $i \geq 2$ such that the block $\Pi_i(t)$ separates at some instant $r \in \mathcal{D}_1 \cap [k-1, k[$. The block after the split which contains 1 is $\Pi_1(r)$. Thus, there is some index $\ell \geq 2$ such that $\Pi_i(t) \subseteq \pi_\ell(r) \cap \Pi_1(r-)$. Then, at time k, $\pi_\ell(r) \cap \Pi_1(r-)$ is partitioned into $\Pi_j(k), j \in \mathcal{J}_{\ell,r}$ where $\mathcal{J}_{\ell,r}$ is some set of indices measurable with respect to

$$\mathcal{G}_{1,k}(t) := \mathcal{G}_1(t) \vee \mathcal{G}(k)\,.$$

Conditionally upon $\mathcal{G}_{1,k}(t)$, the partition $(\Pi_i(t), i \in \mathcal{I}_k)$ can be written in the form $\tilde{\Pi}^{(j)}(t-k)_{|\Pi_j(k)}, j \in \mathcal{J}_k$, where $\mathcal{J}_k$ is some set of indices $\mathcal{G}_{1,k}(t)$-measurable and where $(\tilde{\Pi}^{(j)})_{j\in\mathbb{N}}$ is a family of i.i.d. homogeneous fragmentations distributed as Π under $\mathbb{P}$ and independent of the sigma-field $\mathcal{G}_{1,k}(t)$.

As a consequence:

$$\underset{i\in\mathcal{I}_k}{\cup}\Pi_i(t) = \cup_{j\in\mathcal{J}_k}\tilde{\Pi}^{(j)}(t-k)_{|\Pi_j(k)}\,, \tag{48}$$

and for all $m \in \mathbb{N}$

$$|\tilde{\Pi}^{(j)}_m(t-k)_{|\Pi_j(k)}| = |\tilde{\Pi}^{(j)}_m(t-k)||\Pi_j(k)|. \tag{49}$$

Now, we have to take into account the killings.

Let us call "good fragment" a fragment which satisfies the constraint all along its history up to time t. In the sum

$$\begin{aligned} S_k &:= \sum_{i\in\mathcal{I}_k} h(vt + \log|\Pi^{\dagger}_i(t)|)|\Pi^{\dagger}_i(t)| \\ &= \sum_{j\in\mathcal{J}_k}\sum_m h(vt + \log(|\tilde{\Pi}^{(j)}_m(t-k)||\Pi_j(k)|))|\tilde{\Pi}^{(j)}_m(t-k)||\Pi_j(k)|\mathbb{1}_{j,m,k} \end{aligned}$$

where $\mathbb{1}_{j,m,k} = 1$ if and only if $\tilde{\Pi}^{(j)}_m(t-k)_{|\Pi_j(k)}$ is a good fragment. From the definition of h in (22) we have

$$\sum_m h(vt + \log(|\tilde{\Pi}^{(j)}_m(t-k)||\Pi_j(k)|))|\tilde{\Pi}^{(j)}_m(t-k)|\mathbb{1}_{j,m,k}$$
$$= \mathrm{e}^{\rho(k-t)} h(vk + \log(|\Pi_j(k)|)\mathbb{1}_{j,k}\tilde{M}^j_{t-k}$$

where $\tilde{M}^j_{t-k}$ is a martingale and where $\mathbb{1}_{j,k} = 1$ if and only if $\Pi_j(k)$ is a good fragment. We have then

$$\mathbb{E}^{\updownarrow}\left(S_k|\mathcal{G}_{1,k}(t)\right) = \mathrm{e}^{\rho(k-t)} \sum_{j\in\mathcal{J}_k} |\Pi_j(k)| h(vk + \log(|\Pi_j(k)|)\mathbb{1}_{j,k}$$

Now again by the definition of h and its continuity, there exists $C_3 > 0$ such that

$$h(vk + \log(|\Pi_j(k)|) \leq C_3 \mathbb{1}_{vk+\log(|\Pi_j(k)|\in(\log a, \log b)}$$

and

$$\mathbb{E}^{\updownarrow}\left(S_k|\mathcal{G}_{1,k}(t)\right) \leq C_3 \mathrm{e}^{(\rho-v)k}\mathrm{e}^{-\rho t} \sum_{j\in\mathcal{J}_k} \mathbb{1}_{j,k}$$

It is clear that the only terms that contribute to the sum $\sum_{j\in\mathcal{J}_k} \mathbb{1}_{j,k}$ correspond to good fragments at time k which where dislocated from good $\Pi_1(k-1)$ during $[k-1,k[$. Since the fragmentation is conservative, there were at most be^v/a such dislocations during that time. So we get:

$$\mathbb{E}^{\updownarrow}\left(S_k|\mathcal{G}_{1,k}(t)\right) \leq C_3 b e^v a^{-1} \mathrm{e}^{(\rho-v)k}\mathrm{e}^{-\rho t},$$

and from (47) there exists $C_4 > 0$ such that

$$\mathbb{E}^{\updownarrow}(d_t|\mathcal{G}_1(t)) \leq C_4 \sum_{k=1}^{\lfloor t \rfloor} \mathrm{e}^{(\rho-v)k}.$$

In other words, for all $v \neq \rho$ there exists $L > 0$ such that

$$\mathbb{E}^{\updownarrow}(d_t|\mathcal{G}_1(t)) \leq L \max(\mathrm{e}^{(\rho-v)t}, 1), \tag{50}$$

which proves (45), hence Lemma 4.2. ■

4.4 Proof of Theorem 3.3

1) The first point is in the proof of Theorem 2 of [7] p.406. The core of the argument is the estimate:

$$\mathbb{E} \sup_{0<s\leq t} |M^{(p)}_s|^{1+\alpha} \leq K_\alpha c(p,\alpha) \int_0^t \exp\left(d(p,\alpha)s\right)\, ds \tag{51}$$

where

- K_α is a universal constant depending only on α
- $d(p,\alpha) = (1+\alpha)\kappa(p) - \kappa\big((1+\alpha)(p+1)-1\big)$
- $$c(p,\alpha) = \int_{\mathcal{S}^*} |\sum_{i=1}^\infty (x_i^{p+1} - x_i)\,|^{1+\alpha}\ \nu(dx) < \infty$$

for every $p > \underline{p}$ and $\alpha \in [0,\alpha_1]$ for some α_1.

For $p < \bar{p}$, we know that $\kappa(p) - (p+1)\kappa'(p) < 0$, so that the function $\alpha \mapsto d(p,\alpha)$ is decreasing on some interval $[0,\alpha_2]$ and vanishes at $\alpha = 0$. The integral on the right hand side of (51) is then uniformly bounded in t.

2) The point a) is in ([10]), but we recall the argument for the sake of completeness. The martingale is lower bounded by the contribution of the spine:

$$M_t^{(p)} \geq \mathrm{e}^{t\kappa(p)}|\Pi_1(t)|^{p+1} = \exp\{t\kappa(p) - (p+1)\xi_t\}\,.$$

From Sections 2.2 and 3.1, we know that under $\mathbb{P}^{(p)}$, the Lévy process $\big(\kappa(p)t - (p+1)\xi_t\big)_{t\geq 0}$ has mean $\kappa(p) - (p+1)\kappa'(p)$, which is 0 if $p = \bar{p}$ and positive if $p > \bar{p}$. In both cases $\mathbb{P}^{(p)}(\limsup_{t\to\infty}(\kappa(p)t - (p+1)\xi_t) = \infty) = 1$. This shows that $\mathbb{P}^{(p)}(\limsup M_t^{(p)} = \infty) = 1$, hence $\mathbb{P}(\lim M_t^{(p)} = 0)$.

2 b) For $p > \bar{p}$, we know that $\kappa(p) - (p+1)\kappa'(p) > 0$, so that the function $\alpha \mapsto d(p,\alpha)$ is increasing on some interval $[0,\alpha_0]$ and vanishes at $\alpha = 0$, hence is positive on $(0,\alpha_0]$. By integration, we get (29). ■

5 Comparison of limits.

Proof of Proposition 1.3: Let us prove the inequality

$$v - \rho(v,a,b) \leq C(v)\,.$$

directly. In the particular case of $v \in (v_{\min}, v_{\max})$ such that $\rho(v,a,b) \geq v_{\min}$, it was already proved in [18], Remark 4.

Let $\beta = \log b - \log a$, fix v and let $\rho = \rho(v,a,b)$. One has to prove that $C(v) \geq v - \rho$, which by the definition (3) of C is equivalent to prove that $pv - \kappa(p) \geq -\rho$ for all $p > \underline{p}$. As in Section 3.2 let us denote by $\psi(p) = pv - \kappa(p)$ the Laplace exponent of $vt - \xi_t$. Since $\rho = \inf\{q \geq 0 : W^{(-q)}(\beta) = 0\} \geq 0$, there is nothing to prove if $\psi(p) \geq 0$. Let us assume $\psi(p) < 0$, so that we have to prove that $-\rho - \psi(p) \leq 0$. But

$$\begin{aligned} -\rho - \psi(p) &= -\inf\{q \geq 0 : W^{(-q)}(\beta) = 0\} - \psi(p) \\ &= -\inf\{q' \geq \psi(p) : W^{(\psi(p)-q')}(\beta) = 0\}\,. \end{aligned}$$

From Lemma 8.4 p.222 in [20], we have

$$W^{(\psi(p)-q')}(\beta) = e^{px} W_p^{(-q')}(x)$$

where W_p denotes the function W associated to the tilted process (see [20] p.213) of Laplace exponent $\lambda \mapsto \psi(\lambda + p) - \psi(p)$. This yields

$$-\rho - \psi(p) \quad = \quad -\inf\{q' \geq \psi(p) : W_p^{(-q')}(\beta) = 0\}\,.$$

For $q' \in [\psi(p), 0]$ we have $-q' \geq 0$ hence $W_p^{(-q')}(\beta) > 0$ and

$$\inf\{q' \geq \psi(p) : W_p^{(-q')}(\beta) = 0\} = \inf\{q' \geq 0 : W_p^{(-q')}(\beta) = 0\} \geq 0\,.$$

Finally we get $-\rho - \psi(p) \leq 0$. ■

Remark 5.1. *A consequence of this proposition is that when $v < v_{min}$, we have $\rho(v, a, b) > v$.*

References

[1] A-L. Basdevant. Fragmentation of ordered partitions and intervals. *Electron. J. Probab.*, 11:no. 16, 394–417, 2006.

[2] J. Berestycki. Ranked fragmentations. *ESAIM Probab. Statist.*, 6:157–175 (electronic), 2002.

[3] J. Berestycki. Multifractal spectra of fragmentation processes. *J. Statist. Phys.*, 113(3-4):411–430, 2003.

[4] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[5] J. Bertoin. Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab.*, 7(1):156–169, 1997.

[6] J. Bertoin. Homogeneous fragmentation processes. *Probab. Theory Related Fields*, 121(3):301–318, 2001.

[7] J. Bertoin. The asymptotic behavior of fragmentation processes. *J. Europ. Math. Soc.*, 5(4):395–416, 2003.

[8] J. Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[9] J. Bertoin and R. A. Doney. Spitzer's condition for random walks and Lévy processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(2):167–178, 1997.

[10] J. Bertoin and A. Rouault. Additive martingales and probability tilting for homogeneous fragmentations. Report 808 of University Paris 6 available at
http://www.proba.jussieu.fr/mathdoc/textes/PMA-808.pdf, 2003.

[11] J. Bertoin and A. Rouault. Asymptotical behaviour of the presence probability in branching random walks and fragmentations. arXiv:math.PR/04095477, 2004.

[12] J. Bertoin and A. Rouault. Discretization methods for homogeneous fragmentations. *J. London Math. Soc. (2)*, 72(1):91–109, 2005.

[13] T. Chan, A.E. Kyprianou, and V. Savov. Smoothness of scale functions for spectrally negative Lévy processes. submitted, 2009.

[14] S. D. Chatterji. Asymptotic formulae derived from the central limit theorem. *Confer. Sem. Mat. Univ. Bari*, 234:37, 1990.

[15] R. Durrett. *Probability: theory and examples*. Duxbury Press, Belmont, CA, second edition, 1996.

[16] R. Hardy and S. C. Harris. A conceptual approach to a path result for branching Brownian motion. *Stochastic Process. Appl.*, 116(12):1992–2013, 2006.

[17] F. Hubaleck and A.E. Kyprianou. Old and new examples of scale functions for spectrally negative Lévy processes. submitted, 2009.

[18] N. Krell. Multifractal spectra and precise rates of decay in homogeneous fragmentations. *Stoch. Proc. and their Appl.*, 118:897–916, 2008.

[19] A. E. Kyprianou. Travelling wave solutions to the K-P-P equation: alternatives to Simon Harris' probabilistic analysis. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(1):53–72, 2004.

[20] A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, Berlin, 2006.

[21] A. Lambert. Completely asymmetric Lévy processes confined in a finite interval. *Ann. Inst. H. Poincaré Probab. Statist.*, 36(2):251–274, 2000.