



**HAL**  
open science

## On the multiplicative order of $a^n$ modulo $n$

Jonathan Chappelon

► **To cite this version:**

Jonathan Chappelon. On the multiplicative order of  $a^n$  modulo  $n$ . Journal of Integer Sequences, 2010, 13, pp.Article 10.2.1. hal-00371233

**HAL Id: hal-00371233**

**<https://hal.science/hal-00371233>**

Submitted on 22 Mar 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the multiplicative order of $a^n$ modulo $n$

Jonathan Chappelon<sup>a,b,c</sup>

<sup>a</sup>Univ Lille Nord de France, F-59000 Lille, France

<sup>b</sup>ULCO, LMPA J. Liouville, B.P. 699, F-62228 Calais, France

<sup>c</sup>CNRS, FR 2956, France

January 20, 2010

## Abstract

Let  $n$  be a positive integer and  $\alpha_n$  be the arithmetic function which assigns the multiplicative order of  $a^n$  modulo  $n$  to every integer  $a$  coprime to  $n$  and vanishes elsewhere. Similarly, let  $\beta_n$  assign the projective multiplicative order of  $a^n$  modulo  $n$  to every integer  $a$  coprime to  $n$  and vanishes elsewhere. In this paper, we present a study of these two arithmetic functions. In particular, we prove that for positive integers  $n_1$  and  $n_2$  with the same square-free part, there exists an exact relationship between the functions  $\alpha_{n_1}$  and  $\alpha_{n_2}$  and between the functions  $\beta_{n_1}$  and  $\beta_{n_2}$ . This allows us to reduce the determination of  $\alpha_n$  and  $\beta_n$  to the case where  $n$  is square-free. These arithmetic functions recently appeared in the context of an old problem of Molluzzo, and more precisely in the study of which arithmetic progressions yield a balanced Steinhaus triangle in  $\mathbb{Z}/n\mathbb{Z}$  for  $n$  odd.

**2000 Mathematics Subject Classifications:** 11A05, 11A07, 11A25.

**Keywords:** multiplicative order, projective multiplicative order, balanced Steinhaus triangles, Steinhaus triangles, Molluzzo's Problem.

## 1 Introduction

We start by introducing some notation relating to the order of certain elements modulo  $n$ . For every positive integer  $n$  and every prime number  $p$ , we denote by  $v_p(n)$  the *p-adic valuation* of  $n$ , i.e., the greatest exponent  $e \geq 0$  for which  $p^e$  divides  $n$ . The prime factorization of  $n$  may then be written as

$$n = \prod_{p \in \mathbb{P}} p^{v_p(n)},$$

where  $\mathbb{P}$  denotes the set of all prime numbers. We denote by  $\text{rad}(n)$  the *radical* of  $n$ , i.e., the largest square-free divisor of  $n$ , namely

$$\text{rad}(n) = \prod_{\substack{p \in \mathbb{P} \\ p|n}} p.$$

For every positive integer  $n$  and every integer  $a$  coprime to  $n$ , we denote by  $\mathcal{O}_n(a)$  the *multiplicative order of  $a$  modulo  $n$* , i.e., the smallest positive integer  $e$  such that  $a^e \equiv 1 \pmod{n}$ , namely

$$\mathcal{O}_n(a) = \min \{e \in \mathbb{N}^* \mid a^e \equiv 1 \pmod{n}\}$$

and we denote by  $\mathcal{R}_n(a)$  the *multiplicative remainder of  $a$  modulo  $n$* , i.e., the multiple of  $n$  defined by

$$\mathcal{R}_n(a) = a^{\mathcal{O}_n(a)} - 1.$$

The multiplicative order of  $a$  modulo  $n$  also corresponds with the order of the element  $\pi_n(a)$ , where  $\pi_n : \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  is the canonical surjective morphism, in the multiplicative group  $(\mathbb{Z}/n\mathbb{Z})^*$ , the group of units of  $\mathbb{Z}/n\mathbb{Z}$ . Note that  $\mathcal{O}_n(a)$  divides  $\varphi(n)$ ,  $\varphi$  being the Euler's totient function.

For every positive integer  $n$ , we define and denote by  $\alpha_n$  the arithmetic function

$$\alpha_n : \mathbb{Z} \rightarrow \mathbb{N} \\ a \mapsto \begin{cases} \mathcal{O}_n(a^n), & \text{for all } \gcd(a, n) = 1; \\ 0, & \text{otherwise,} \end{cases}$$

where  $\gcd(a, n)$  denotes the greatest common divisor of  $a$  and  $n$ , with the convention that  $\gcd(0, n) = n$ . Observe that, for every  $a$  coprime to  $n$ , the integer  $\alpha_n(a)$  divides  $\varphi(n)/\gcd(\varphi(n), n)$ . This follows from the previous remark on  $\mathcal{O}_n(a)$  and the equality  $\alpha_n(a) = \mathcal{O}_n(a^n) = \mathcal{O}_n(a)/\gcd(\mathcal{O}_n(a), n)$ .

For every positive integer  $n$  and every integer  $a$  coprime to  $n$ , we denote by  $\mathcal{PO}_n(a)$  the *projective multiplicative order of  $a$  modulo  $n$* , i.e., the smallest positive integer  $e$  such that  $a^e \equiv \pm 1 \pmod{n}$ , namely

$$\mathcal{PO}_n(a) = \min \{e \in \mathbb{N}^* \mid a^e \equiv \pm 1 \pmod{n}\}.$$

The projective multiplicative order of  $a$  modulo  $n$  also corresponds with the order of the element  $\pi_n(a)$  in the multiplicative quotient group  $(\mathbb{Z}/n\mathbb{Z})^*/\{-1, 1\}$ .

For every positive integer  $n$ , we define and denote by  $\beta_n$  the arithmetic function

$$\beta_n : \mathbb{Z} \rightarrow \mathbb{N} \\ a \mapsto \begin{cases} \mathcal{PO}_n(a^n), & \text{for all } \gcd(a, n) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that we have the alternative  $\alpha_n = \beta_n$  or  $\alpha_n = 2\beta_n$ .

In this paper, we study in detail these two arithmetic functions. In particular, we prove that, for every positive integers  $n_1$  and  $n_2$  such that

$$\begin{cases} \text{rad}(n_1) \mid n_2 \text{ and } n_2 \mid n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1) \mid n_2 \text{ and } n_2 \mid n_1, & \text{if } v_2(n_1) \geq 2, \end{cases}$$

the integer  $\alpha_{n_1}(a)$  (respectively  $\beta_{n_1}(a)$ ) divides  $\alpha_{n_2}(a)$  (resp.  $\beta_{n_2}(a)$ ), for every integer  $a$ . More precisely, we determine the exact relationship between the functions  $\alpha_{n_1}$  and  $\alpha_{n_2}$  and between  $\beta_{n_1}$  and  $\beta_{n_2}$ . We prove that we have

$$\alpha_{n_1}(a) = \frac{\alpha_{n_2}(a)}{\gcd\left(\alpha_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right)} \text{ for all } \gcd(a, n_1) = 1$$

in Theorem 2.5 of Section 2 and that we have

$$\beta_{n_1}(a) = \frac{\beta_{n_2}(a)}{\gcd\left(\beta_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right)} \text{ for all } \gcd(a, n_1) = 1$$

in Theorem 3.3 of Section 3. Thus, for every integer  $a$  coprime to  $n$ , the determination of  $\alpha_n(a)$  is reduced to the computation of  $\alpha_{\text{rad}(n)}(a)$  and  $\mathcal{R}_{\text{rad}(n)}(a)$  if  $v_2(n) \leq 1$  and of  $\alpha_{2\text{rad}(n)}(a)$  and  $\mathcal{R}_{2\text{rad}(n)}(a)$  if  $v_2(n) \geq 2$ . These theorems on the functions  $\alpha_n$  and  $\beta_n$  are derived from Theorem 2.6 of Section 2, which states that

$$\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))},$$

for all integers  $a$  coprime to  $n_1$  and  $n_2$ . This result generalizes the following theorem of Nathanson which, in the above notation, states that for every odd prime number  $p$  and for every positive integer  $k$ , we have the equality

$$\mathcal{O}_{p^k}(a) = \mathcal{O}_p(a) \cdot \frac{p^k}{\gcd(p^k, \mathcal{R}_p(a))}$$

for all integers  $a$  not divisible by  $p$ .

**Theorem 3.6 of [3].** *Let  $p$  be an odd prime, and let  $a \neq \pm 1$  be an integer not divisible by  $p$ . Let  $d$  be the order of  $a$  modulo  $p$ . Let  $k_0$  be the largest integer such that  $a^d \equiv 1 \pmod{p^{k_0}}$ . Then the order of  $a$  modulo  $p^k$  is  $d$  for  $k = 1, \dots, k_0$  and  $dp^{k-k_0}$  for  $k \geq k_0$ .*

For every finite sequence  $S = (a_1, \dots, a_m)$  of length  $m \geq 1$  in  $\mathbb{Z}/n\mathbb{Z}$ , we denote by  $\Delta S$  the *Steinhaus triangle* of  $S$ , that is the finite multiset of cardinality  $\binom{m+1}{2}$  in  $\mathbb{Z}/n\mathbb{Z}$  defined by

$$\Delta S = \left\{ \sum_{k=0}^i \binom{i}{k} a_{j+k} \mid 0 \leq i \leq m-1, 1 \leq j \leq m-i \right\}.$$

A finite sequence  $S$  in  $\mathbb{Z}/n\mathbb{Z}$  is said to be *balanced* if each element of  $\mathbb{Z}/n\mathbb{Z}$  occurs in its Steinhaus triangle  $\Delta S$  with the same multiplicity. For instance, the sequence  $(2, 2, 3, 3)$  of length 4 is balanced in  $\mathbb{Z}/5\mathbb{Z}$ . Indeed, as depicted in Figure 1, its Steinhaus triangle is composed by each element of  $\mathbb{Z}/5\mathbb{Z}$  occurring twice.

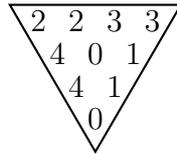


Figure 1: The Steinhaus triangle of a balanced sequence in  $\mathbb{Z}/5\mathbb{Z}$

Note that, for a sequence  $S$  of length  $m \geq 1$  in  $\mathbb{Z}/n\mathbb{Z}$ , a necessary condition on  $m$  for  $S$  to be balanced is that the integer  $n$  divides the binomial coefficient  $\binom{m+1}{2}$ . In 1976, John C. Molluzzo [2] posed the problem to determine whether this necessary condition on  $m$  is also sufficient to guarantee the existence of a balanced sequence. In [1], it was proved that, for each odd number  $n$ , *there exists a balanced sequence of length  $m$  for every  $m \equiv 0$  or  $-1 \pmod{\alpha_n(2) \cdot n}$  and for every  $m \equiv 0$  or  $-1 \pmod{\beta_n(2) \cdot n}$* . This was achieved by analyzing the Steinhaus triangles generated by arithmetic progressions. In particular, since  $\beta_{3^k}(2) = 1$  for all  $k \geq 1$ , the above result implies a complete and positive solution of Molluzzo's Problem in  $\mathbb{Z}/n\mathbb{Z}$  for all  $n = 3^k$ .

## 2 The arithmetic function $\alpha_n$

The table depicted in Figure 2 gives us the first values of  $\alpha_n(a)$  for every positive integer  $n$ ,  $1 \leq n \leq 20$ , and for every integer  $a$ ,  $-20 \leq a \leq 20$ .

$n \backslash a$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
3	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
4	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
5	1	4	4	2	0	1	4	4	2	0	1	4	4	2	0	1	4	4	2	0
6	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
7	1	3	6	3	6	2	0	1	3	6	3	6	2	0	1	3	6	3	6	2
8	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
9	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
10	1	0	2	0	0	0	2	0	1	0	1	0	2	0	0	0	2	0	1	0
11	1	10	5	5	5	10	10	10	5	2	0	1	10	5	5	5	10	10	10	5
12	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
13	1	12	3	6	4	12	12	4	3	6	12	2	0	1	12	3	6	4	12	12
14	1	0	3	0	3	0	0	0	3	0	3	0	1	0	1	0	3	0	3	0
15	1	4	0	2	0	0	4	4	0	0	2	0	4	2	0	1	4	0	2	0
16	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
17	1	8	16	4	16	16	16	8	8	16	16	16	4	16	8	2	0	1	8	16
18	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
19	1	18	18	9	9	9	3	6	9	18	3	6	18	18	18	9	9	2	0	1
20	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0

Figure 2: The first values of  $\alpha_n(a)$

The positive integer  $\alpha_n(a)$  seems to be difficult to determine. Indeed, there is no general formula known to compute the multiplicative order of an integer modulo  $n$  but, however, we get the following helpful propositions.

**Lemma 2.1.** *Let  $n_1$  and  $n_2$  be two positive integers such that  $\text{rad}(n_1) = \text{rad}(n_2)$ . Then, an integer  $a$  is coprime to  $n_1$  if, and only if, it is also coprime to  $n_2$ .*

*Proof.* This follows from the definition of the greatest common divisor of two integers and from the definition of the radical of an integer.  $\square$

**Proposition 2.2.** *Let  $n_1$  and  $n_2$  be two positive integers such that  $\text{rad}(n_1) | n_2$  and  $n_2 | n_1$ . Then, for every integer  $a$ , the integer  $\alpha_{n_1}(a)$  divides  $\alpha_{n_2}(a)$ .*

*Proof.* If  $a$  is not coprime to  $n_1$  and  $n_2$ , then, by definition of the functions  $\alpha_{n_1}$  and  $\alpha_{n_2}$  and by Lemma 2.1, we have

$$\alpha_{n_1}(a) = \alpha_{n_2}(a) = 0.$$

Suppose that  $a$  is coprime to  $n_1$  and  $n_2$ . If  $v_p(n_1) = 1$  for all prime factors  $p$  of  $n_1$ , then  $n_2 = n_1$ . Otherwise, let  $p$  be a prime factor of  $n_1$  such that  $v_p(n_1) \geq 2$ . We shall show that  $\alpha_{n_1}(a)$  divides  $\alpha_{n_1/p}(a)$ . By definition of  $\alpha_{n_1/p}(a)$ , there exists an integer  $u$  such that

$$a^{\alpha_{n_1/p}(a) \cdot \frac{n_1}{p}} = 1 + u \cdot \frac{n_1}{p}.$$

Therefore, by the binomial theorem, we have

$$a^{\alpha_{n_1/p}(a) \cdot n_1} = \left( a^{\alpha_{n_1/p}(a) \cdot \frac{n_1}{p}} \right)^p = \left( 1 + u \cdot \frac{n_1}{p} \right)^p = 1 + u \cdot n_1 + \sum_{k=2}^p \binom{p}{k} \cdot u^k \cdot \left( \frac{n_1}{p} \right)^k.$$

Since  $v_p(n_1) \geq 2$ , it follows that  $(n_1/p)^k$  is divisible by  $n_1$  for every integer  $k \geq 2$  and so

$$a^{\alpha_{n_1/p}(a) \cdot n_1} \equiv 1 \pmod{n_1}.$$

Hence  $\alpha_{n_1}(a)$  divides  $\alpha_{n_1/p}(a)$ . This completes the proof.  $\square$

An exact relationship between  $\alpha_{n_1}(a)$  and  $\alpha_{n_2}(a)$ , for every integer  $a$  coprime to  $n_1$  and  $n_2$ , is determined at the end of this section. We first settle the easy prime power case.

**Proposition 2.3.** *Let  $p$  be a prime number and let  $a$  be an integer. Then we have*

$$\alpha_{p^k}(a) = \mathcal{O}_p(a)$$

for every positive integer  $k$ .

*Proof.* Let  $k$  be a positive integer. If  $a$  is not coprime to  $p$ , then we have  $\alpha_{p^k}(a) = \alpha_p(a) = 0$ . Suppose now that  $a$  is coprime to  $p$ . By Proposition 2.2, the integer  $\alpha_{p^k}(a)$  divides  $\alpha_p(a)$ . It remains to prove that  $\alpha_p(a)$  divides  $\alpha_{p^k}(a)$ . The congruence

$$a^{\alpha_{p^k}(a) \cdot p^k} \equiv 1 \pmod{p^k}$$

implies that

$$a^{\alpha_{p^k}(a) \cdot p^k} \equiv 1 \pmod{p},$$

and hence, by Fermat's Little Theorem, it follows that

$$a^{\alpha_{p^k}(a) \cdot p} \equiv a^{\alpha_{p^k}(a) \cdot p^k} \equiv 1 \pmod{p}.$$

Therefore  $\alpha_p(a)$  divides  $\alpha_{p^k}(a)$ . Finally, we have

$$\alpha_{p^k}(a) = \alpha_p(a) = \mathcal{O}_p(a^p) = \mathcal{O}_p(a).$$

This completes the proof.  $\square$

*Remark.* If  $p = 2$ , then, for every positive integer  $k$ , we obtain

$$\alpha_{2^k}(a) = \mathcal{O}_2(a) = \begin{cases} 0, & \text{for } a \text{ even;} \\ 1, & \text{for } a \text{ odd.} \end{cases}$$

**Proposition 2.4.** *Let  $n_1$  and  $n_2$  be two coprime numbers and let  $a$  be an integer. Then  $\alpha_{n_1 n_2}(a)$  divides  $\text{lcm}(\alpha_{n_1}(a), \alpha_{n_2}(a))$ , the least common multiple of  $\alpha_{n_1}(a)$  and  $\alpha_{n_2}(a)$ .*

*Proof.* If  $\text{gcd}(a, n_1 n_2) \neq 1$ , then  $\text{gcd}(a, n_1) \neq 1$  or  $\text{gcd}(a, n_2) \neq 1$  and so

$$\alpha_{n_1 n_2}(a) = \text{lcm}(\alpha_{n_1}(a), \alpha_{n_2}(a)) = 0.$$

Suppose now that  $\text{gcd}(a, n_1 n_2) = 1$  and hence that the integers  $a$ ,  $n_1$  and  $n_2$  are coprime pairwise. Let  $i \in \{1, 2\}$ . The congruences

$$a^{\alpha_{n_i}(a) \cdot n_i} \equiv 1 \pmod{n_i}$$

imply that

$$a^{n_1 n_2 \text{lcm}(\alpha_{n_1}(a), \alpha_{n_2}(a))} \equiv 1 \pmod{n_i}.$$

Therefore  $\alpha_{n_1 n_2}(a)$  divides  $\text{lcm}(\alpha_{n_1}(a), \alpha_{n_2}(a))$  by the Chinese remainder theorem.  $\square$

Let  $n_1$  and  $n_2$  be two positive integers such that

$$\begin{cases} \text{rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

By definition, we know that  $\alpha_{n_1}(a) = \alpha_{n_2}(a) = 0$  for every integer  $a$  not coprime to  $n_1$  and  $n_2$ . We end this section by determining the exact relationship between  $\alpha_{n_1}(a)$  and  $\alpha_{n_2}(a)$  for every integer  $a$  coprime to  $n_1$  and  $n_2$ .

**Theorem 2.5.** *Let  $n_1$  and  $n_2$  be two positive integers such that*

$$\begin{cases} \text{rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

*Then, for every integer  $a$  coprime to  $n_1$  and  $n_2$ , we have*

$$\alpha_{n_1}(a) = \frac{\alpha_{n_2}(a)}{\text{gcd}\left(\alpha_{n_2}(a), \frac{\text{gcd}(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right)}$$

This result is a corollary of the following theorem.

**Theorem 2.6.** *Let  $n_1$  and  $n_2$  be two positive integers such that*

$$\begin{cases} \text{rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1) | n_2 \text{ and } n_2 | n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

*Then, for every integer  $a$  coprime to  $n_1$  and  $n_2$ , we have*

$$\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\text{gcd}(n_1, \mathcal{R}_{n_2}(a))}.$$

The proof of this theorem is based on the following lemma.

**Lemma 2.7.** *Let  $n$  be a positive integer and let  $a$  be an integer coprime to  $n$ . Let  $m$  be an integer such that  $\text{rad}(m) | \text{rad } n$ . Then, there exists an integer  $u_m$ , coprime to  $n$  if  $m$  is odd, or coprime to  $n/2$  if  $m$  is even, such that*

$$a^{\mathcal{O}_n(a) \cdot m} = 1 + u_m \cdot \mathcal{R}_n(a) \cdot m.$$

*Proof.* We distinguish different cases based upon the parity of  $m$ . First, we prove the odd case by induction on  $m$ . If  $m = 1$ , then, by definition of the integer  $\mathcal{R}_n(a)$ , we have

$$a^{\mathcal{O}_n(a)} = 1 + \mathcal{R}_n(a).$$

Therefore the assertion is true for  $m = 1$ .

Now, let  $p$  be a prime factor of  $m$  and suppose that the assertion is true for the odd number  $m/p$ , i.e., there exists an integer  $u_{m/p}$ , coprime to  $n$ , such that

$$a^{\mathcal{O}_n(a) \cdot \frac{m}{p}} = 1 + u_{m/p} \cdot \mathcal{R}_n(a) \cdot \frac{m}{p}.$$

Then, we obtain

$$\begin{aligned} a^{\mathcal{O}_n(a) \cdot m} &= \left( a^{\mathcal{O}_n(a) \cdot \frac{m}{p}} \right)^p = \left( 1 + u_{m/p} \cdot \mathcal{R}_n(a) \cdot \frac{m}{p} \right)^p \\ &= 1 + u_{m/p} \cdot \mathcal{R}_n(a) \cdot m + \sum_{k=2}^{p-1} \binom{p}{k} \left( u_{m/p} \cdot \mathcal{R}_n(a) \cdot \frac{m}{p} \right)^k + \left( u_{m/p} \cdot \mathcal{R}_n(a) \cdot \frac{m}{p} \right)^p \\ &= 1 + \left( u_{m/p} + \sum_{k=2}^{p-1} \frac{\binom{p}{k}}{p} \cdot (u_{m/p})^k \cdot \mathcal{R}_n(a)^{k-1} \cdot \left( \frac{m}{p} \right)^{k-1} + \right. \\ &\quad \left. + (u_{m/p})^p \cdot \frac{\mathcal{R}_n(a)^{p-1}}{p} \cdot \left( \frac{m}{p} \right)^{p-1} \right) \cdot \mathcal{R}_n(a) \cdot m \\ &= 1 + u_m \cdot \mathcal{R}_n(a) \cdot m. \end{aligned}$$

Since  $n$  divides  $\mathcal{R}_n(a)$  which divides

$$u_m - u_{m/p} = \sum_{k=2}^{p-1} \frac{\binom{p}{k}}{p} \cdot (u_{m/p})^k \cdot \mathcal{R}_n(a)^{k-1} \cdot \left( \frac{m}{p} \right)^{k-1} + (u_{m/p})^p \cdot \frac{\mathcal{R}_n(a)^{p-1}}{p} \cdot \left( \frac{m}{p} \right)^{p-1},$$

it follows that  $\gcd(u_m, n) = \gcd(u_{m/p}, n) = 1$ . This completes the proof for the odd case.

Suppose now that  $n$  and  $m$  are even. We proceed by induction on  $v_2(m)$ . If  $v_2(m) = 1$ , then  $m/2$  is odd and by the first part of this proof,

$$a^{\frac{m}{2} \cdot \mathcal{O}_n(a)} = 1 + u_{m/2} \cdot \frac{m}{2} \cdot \mathcal{R}_n(a)$$

where  $u_{m/2}$  is coprime to  $n$  and hence to  $n/2$ . Now assume that  $v_2(m) > 1$  and that

$$a^{\frac{m}{2} \cdot \mathcal{O}_n(a)} = 1 + u_{m/2} \cdot \frac{m}{2} \cdot \mathcal{R}_n(a)$$

with  $u_{m/2}$  coprime to  $n/2$ . Then, we obtain

$$\begin{aligned} a^{\mathcal{O}_n(a) \cdot m} &= \left( a^{\mathcal{O}_n(a) \cdot \frac{m}{2}} \right)^2 = \left( 1 + u_{m/2} \cdot \mathcal{R}_n(a) \cdot \frac{m}{2} \right)^2 \\ &= 1 + u_{m/2} \cdot \mathcal{R}_n(a) \cdot m + \left( u_{m/2} \cdot \mathcal{R}_n(a) \cdot \frac{m}{2} \right)^2 \\ &= 1 + \left( u_{m/2} + (u_{m/2})^2 \cdot \frac{\mathcal{R}_n(a)}{2} \cdot \frac{m}{2} \right) \cdot \mathcal{R}_n(a) m \\ &= 1 + u_m \cdot \mathcal{R}_n(a) \cdot m. \end{aligned}$$

Since  $n/2$  divides  $\mathcal{R}_n(a)/2$  which divides  $u_m - u_{m/2}$ , it follows that  $\gcd(u_m, n/2) = \gcd(u_{m/2}, n/2) = 1$ . This completes the proof.  $\square$

We are now ready to prove Theorem 2.6.

*Proof of Theorem 2.6.* The proof is by induction on the integer  $n_1/n_2$ . If  $n_1 = n_2$ , then we have

$$\frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))} = \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_1}(a))} = \frac{n_1}{n_1} = 1,$$

since  $\mathcal{R}_{n_1}(a)$  is divisible by  $n_1$ , and thus the statement is true. Let  $p$  be a prime factor of  $n_1$  and  $n_2$  such that  $n_2$  divides  $n_1/p$  and suppose that

$$\mathcal{O}_{n_1/p}(a) = \mathcal{O}_{n_2}(a) \cdot \frac{n_1/p}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))}.$$

First, the congruence

$$a^{\mathcal{O}_{n_1}(a)} \equiv 1 \pmod{n_1}$$

implies that

$$a^{\mathcal{O}_{n_1}(a)} \equiv 1 \pmod{\frac{n_1}{p}}$$

and so  $\mathcal{O}_{n_1/p}(a)$  divides  $\mathcal{O}_{n_1}(a)$ . We consider two cases.

**First Case:**  $v_p(n_1) \leq v_p(\mathcal{R}_{n_2}(a))$ .

Since  $n_2$  divides  $n_1/p$ , it follows that  $\mathcal{O}_{n_2}(a)$  divides  $\mathcal{O}_{n_1/p}(a)$ . Let  $r = \frac{\mathcal{O}_{n_1/p}(a)}{\mathcal{O}_{n_2}(a)}$ . Hence

$$\begin{aligned} \mathcal{R}_{n_1/p}(a) &= a^{\mathcal{O}_{n_1/p}(a)} - 1 = a^{\mathcal{O}_{n_2}(a) \cdot r} - 1 = (a^{\mathcal{O}_{n_2}(a)} - 1) \left( \sum_{k=0}^{r-1} a^{k\mathcal{O}_{n_2}(a)} \right) \\ &= \mathcal{R}_{n_2}(a) \left( \sum_{k=0}^{r-1} a^{k\mathcal{O}_{n_2}(a)} \right) \end{aligned}$$

and so  $\mathcal{R}_{n_1/p}(a)$  is divisible by  $\mathcal{R}_{n_2}(a)$ . This leads to

$$v_p(n_1) \leq v_p(\mathcal{R}_{n_2}(a)) \leq v_p(\mathcal{R}_{n_1/p}(a)).$$

Therefore  $\mathcal{R}_{n_1/p}(a)$  is divisible by  $n_1$  and hence we have

$$a^{\mathcal{O}_{n_1/p}(a)} = 1 + \mathcal{R}_{n_1/p}(a) \equiv 1 \pmod{n_1}.$$

This implies that  $\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_1/p}(a)$ . Moreover, the hypothesis  $v_p(n_1) \leq v_p(\mathcal{R}_{n_2}(a))$  implies that  $\gcd(n_1/p, \mathcal{R}_{n_2}(a)) = \gcd(n_1, \mathcal{R}_{n_2}(a))/p$ . Finally, we obtain

$$\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_1/p}(a) = \mathcal{O}_{n_2}(a) \cdot \frac{n_1/p}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))} = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))}.$$

**Second Case:**  $v_p(n_1) > v_p(\mathcal{R}_{n_2}(a))$ .

If  $v_2(n_1) \leq 1$ , then  $(n_1/p)/\gcd(n_1/p, \mathcal{R}_{n_2}(a))$  is odd. Otherwise, if  $v_2(n_1) \geq 2$ , then  $v_2(n_2) \geq 2$  and every integer coprime to  $n_2/2$  is also coprime to  $n_2$ . In both cases,

$v_2(n_1) \leq 1$  or  $v_2(n_1) \geq 2$ , we know, by Lemma 2.7, that there exists an integer  $u$ , coprime to  $n_2$ , such that

$$\begin{aligned} a^{\mathcal{O}_{n_1/p}(a)} &= a^{\mathcal{O}_{n_2}(a) \cdot \frac{n_1/p}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))}} = 1 + u \cdot \mathcal{R}_{n_2}(a) \cdot \frac{n_1/p}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))} \\ &= 1 + u \cdot \frac{\mathcal{R}_{n_2}(a)}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))} \cdot \frac{n_1}{p}. \end{aligned}$$

As  $v_p(\mathcal{R}_{n_2}(a)) \leq v_p(n_1/p)$ , it follows that  $\mathcal{R}_{n_2}(a)/\gcd(n_1/p, \mathcal{R}_{n_2}(a))$  is coprime to  $p$ , and hence  $\mathcal{O}_{n_1/p}(a)$  is a proper divisor of  $\mathcal{O}_{n_1}(a)$  since

$$a^{\mathcal{O}_{n_1/p}(a)} \not\equiv 1 \pmod{n_1}.$$

Moreover, by Lemma 2.7 again, there exists an integer  $u_p$  such that

$$a^{\mathcal{O}_{n_1/p}(a) \cdot p} = 1 + u_p \cdot \mathcal{R}_{n_1/p}(a) \cdot p \equiv 1 \pmod{n_1}.$$

This leads to

$$\mathcal{O}_{n_1}(a) = \mathcal{O}_{n_1/p}(a) \cdot p = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1/p, \mathcal{R}_{n_2}(a))} = \mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))}.$$

This completes the proof of Theorem 2.6.  $\square$

We may view Theorem 2.6 as a generalization of Theorem 3.6 of [3], where  $n_2 = p$  is an odd prime number and  $n_1 = p^k$  for some positive integer  $k$ . Note that the conclusion of Theorem 2.6 fails in general in the case where  $v_2(n_1) \geq 2$  and  $n_2 = \text{rad}(n_1)$ . For instance, for  $n_1 = 24 = 3 \cdot 2^3$ ,  $n_2 = 6 = 3 \cdot 2$  and  $a = 7$ , we obtain that  $\mathcal{O}_{n_1}(a) = 2$  while  $\mathcal{O}_{n_2}(a)n_1/\gcd(n_1, \mathcal{R}_{n_2}(a)) = 24/\gcd(24, 6) = 4$ .

We now turn to the proof of the main result of this paper.

*Proof of Theorem 2.5.* From Theorem 2.6, we obtain

$$\begin{aligned} \alpha_{n_1}(a) &= \mathcal{O}_{n_1}(a^{n_1}) = \frac{\mathcal{O}_{n_1}(a)}{\gcd(\mathcal{O}_{n_1}(a), n_1)} = \frac{\mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))}}{\gcd\left(\mathcal{O}_{n_2}(a) \cdot \frac{n_1}{\gcd(n_1, \mathcal{R}_{n_2}(a))}, n_1\right)} \\ &= \frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_1, \mathcal{R}_{n_2}(a))}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\alpha_{n_2}(a)}{\alpha_{n_1}(a)} &= \frac{\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_2)}}{\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_1, \mathcal{R}_{n_2}(a))}} = \frac{\gcd(\mathcal{O}_{n_2}(a), n_1, \mathcal{R}_{n_2}(a))}{\gcd(\mathcal{O}_{n_2}(a), n_2)} \\ &= \gcd\left(\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_2)}, \frac{n_2}{\gcd(\mathcal{O}_{n_2}(a), n_2)} \cdot \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right). \end{aligned}$$

Finally, since we have

$$\gcd\left(\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_2)}, \frac{n_2}{\gcd(\mathcal{O}_{n_2}(a), n_2)}\right) = \frac{\gcd(\mathcal{O}_{n_2}(a), n_2)}{\gcd(\mathcal{O}_{n_2}(a), n_2)} = 1,$$

it follows that

$$\frac{\alpha_{n_2}(a)}{\alpha_{n_1}(a)} = \gcd\left(\frac{\mathcal{O}_{n_2}(a)}{\gcd(\mathcal{O}_{n_2}(a), n_2)}, \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right) = \gcd\left(\alpha_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right).$$

□

Thus, the determination of  $\alpha_n$  is reduced to the case where  $n$  is square-free.

**Corollary 2.8.** *Let  $n$  be a positive integer such that  $v_2(n) \leq 1$ . Then, for every integer  $a$ , coprime to  $n$ , we have*

$$\alpha_n(a) = \frac{\alpha_{\text{rad}(n)}(a)}{\gcd\left(\alpha_{\text{rad}(n)}(a), \frac{\gcd(n, \mathcal{R}_{\text{rad}(n)}(a))}{\text{rad}(n)}\right)}.$$

**Corollary 2.9.** *Let  $n$  be a positive integer such that  $v_2(n) \geq 2$ . Then, for every integer  $a$ , coprime to  $n$ , we have*

$$\alpha_n(a) = \frac{\alpha_{2\text{rad}(n)}(a)}{\gcd\left(\alpha_{2\text{rad}(n)}(a), \frac{\gcd(n, \mathcal{R}_{2\text{rad}(n)}(a))}{2\text{rad}(n)}\right)}.$$

### 3 The arithmetic function $\beta_n$

First, we can observe that, by definition of the functions  $\alpha_n$  and  $\beta_n$ , we have

$$\alpha_n(a) = \beta_n(a) = 0$$

for every integer  $a$  not coprime to  $n$  and

$$\frac{\alpha_n(a)}{\beta_n(a)} \in \{1, 2\}$$

for every integer  $a$  coprime to  $n$ . There is no general formula known to compute  $\alpha_n(a)/\beta_n(a)$  but, however, we get the following proposition.

**Proposition 3.1.** *Let  $n_1$  and  $n_2$  be two positive integers such that  $\text{rad}(n_1) = \text{rad}(n_2)$ . Let  $a$  be an integer coprime to  $n_1$  and  $n_2$ . If  $v_2(n_1) \leq 1$ , then we have*

$$\frac{\alpha_{n_1}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{n_2}(a)}{\beta_{n_2}(a)}.$$

If  $v_2(n_1) \geq 2$ , then we have

$$\alpha_{n_1}(a) = \beta_{n_1}(a).$$

*Proof.* Let  $n_1$  be a positive integer such that  $v_2(n_1) \leq 1$  and  $a$  be an integer coprime to  $n_1$ . Let  $p$  be an odd prime factor of  $n_1$  such that  $v_p(n_1) \geq 2$ . We will prove that

$$\frac{\alpha_{n_1}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{n_1/p}(a)}{\beta_{n_1/p}(a)}.$$

If  $\alpha_{n_1}(a) = 2\beta_{n_1}(a)$ , then

$$a^{\beta_{n_1}(a) \cdot n_1} \equiv -1 \pmod{n_1}$$

and thus

$$a^{\beta_{n_1}(a) \cdot p \cdot \frac{n_1}{p}} \equiv -1 \pmod{\frac{n_1}{p}}.$$

This implies that  $\alpha_{n_1/p}(a) = 2\beta_{n_1/p}(a)$ . Conversely, if  $\alpha_{n_1/p}(a) = 2\beta_{n_1/p}(a)$ , then we have

$$a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}} \equiv -1 \pmod{\frac{n_1}{p}}.$$

Since  $v_p(n_1) \geq 2$ , it follows that

$$a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}} \equiv -1 \pmod{p}$$

and thus

$$a^{\beta_{n_1/p}(a) \cdot n_1} + 1 = 1 - \left(-a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}}\right)^p = \left(1 + a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}}\right) \sum_{k=0}^{p-1} \left(-a^{\beta_{n_1/p}(a) \cdot \frac{n_1}{p}}\right)^k \equiv 0 \pmod{n_1}.$$

This implies that  $\alpha_{n_1}(a) = 2\beta_{n_1}(a)$ . Continuing this process we have

$$\frac{\alpha_{n_1}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{\text{rad}(n_1)}(a)}{\beta_{\text{rad}(n_1)}(a)}$$

and since  $\text{rad}(n_1) = \text{rad}(n_2)$ ,

$$\frac{\alpha_{n_1}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{n_2}(a)}{\beta_{n_2}(a)}.$$

Now, let  $n_1$  be a positive integer such that  $v_2(n_1) \geq 2$ , and let  $a$  be a non-zero integer. Suppose that we have  $\alpha_{n_1}(a) = 2\beta_{n_1}(a)$ . Since

$$a^{\beta_{n_1}(a) \cdot n_1} \equiv -1 \pmod{n_1}$$

it follows that

$$\left(a^{\beta_{n_1}(a) \cdot \frac{n_1}{4}}\right)^4 \equiv -1 \pmod{4}$$

in contradiction with

$$\left(a^{\beta_{n_1}(a) \cdot \frac{n_1}{4}}\right)^4 \equiv 1 \pmod{4}.$$

Thus  $\alpha_{n_1}(a) = \beta_{n_1}(a)$ . □

If  $n$  is a prime power, then  $\beta_n = \beta_{\text{rad}(n)}$ , in analogy with Proposition 2.3 for  $\alpha_n$ .

**Proposition 3.2.** *Let  $p$  be a prime number and let  $a$  be an integer. Then we have*

$$\beta_{p^k}(a) = \beta_p(a)$$

for every positive integer  $k$ .

*Proof.* This result is trivial for every integer  $a$  not coprime to  $p$ . Suppose now that  $a$  is coprime to  $p$ . For  $p = 2$ , then, by Proposition 3.1, we have

$$\beta_{2^k}(a) = \alpha_{2^k}(a) = 1$$

for every positive integer  $k$ . For an odd prime number  $p \geq 3$ , Proposition 3.1 and Proposition 2.3 lead to

$$\beta_{p^k}(a) = \frac{\alpha_{p^k}(a)}{\alpha_p(a)} \cdot \beta_p(a) = \beta_p(a)$$

for every positive integer  $k$ . This completes the proof.  $\square$

Let  $n_1$  and  $n_2$  be two positive integers such that

$$\begin{cases} \text{rad}(n_1)|n_2 \text{ and } n_2|n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1)|n_2 \text{ and } n_2|n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

It immediately follows that  $\beta_{n_1}(a) = \beta_{n_2}(a) = 0$  for every integer  $a$  not coprime to  $n_1$  and  $n_2$ . Finally, we determine the relationship between  $\beta_{n_1}(a)$  and  $\beta_{n_2}(a)$  for every integer  $a$  coprime to  $n_1$  and  $n_2$ .

**Theorem 3.3.** *Let  $n_1$  and  $n_2$  be two positive integers such that*

$$\begin{cases} \text{rad}(n_1)|n_2 \text{ and } n_2|n_1, & \text{if } v_2(n_1) \leq 1; \\ 2 \text{ rad}(n_1)|n_2 \text{ and } n_2|n_1, & \text{if } v_2(n_1) \geq 2. \end{cases}$$

*Let  $a$  be an integer coprime to  $n_1$  and  $n_2$ . Then, we have*

$$\beta_{n_1}(a) = \frac{\beta_{n_2}(a)}{\gcd\left(\beta_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right)}.$$

*Proof.* If  $v_2(n_1) \leq 1$ , then Theorem 2.5 and Proposition 3.1 lead to

$$\frac{\beta_{n_2}(a)}{\beta_{n_1}(a)} = \frac{\alpha_{n_2}(a)}{\alpha_{n_1}(a)} = \gcd\left(\alpha_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right).$$

Since  $v_2(n_2) = v_2(n_1) \leq 1$ , it follows that  $\gcd(n_1, \mathcal{R}_{n_2}(a))/n_2$  is odd and hence, we have

$$\frac{\beta_{n_2}(a)}{\beta_{n_1}(a)} = \gcd\left(\alpha_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right) = \gcd\left(\beta_{n_2}(a), \frac{\gcd(n_1, \mathcal{R}_{n_2}(a))}{n_2}\right).$$

If  $v_2(n_1) \geq 2$ , then  $\beta_{n_1}(a) = \alpha_{n_1}(a)$  and  $\beta_{n_2}(a) = \alpha_{n_2}(a)$  by Proposition 3.1 and the result follows from Theorem 2.5.  $\square$

Thus, as for  $\alpha_n$ , the determination of  $\beta_n$  is reduced to the case where  $n$  is square-free.

**Corollary 3.4.** *Let  $n$  be a positive integer such that  $v_2(n) \leq 1$ . Then, for every integer  $a$ , coprime to  $n$ , we have*

$$\beta_n(a) = \frac{\beta_{\text{rad}(n)}(a)}{\gcd\left(\beta_{\text{rad}(n)}(a), \frac{\gcd(n, \mathcal{R}_{\text{rad}(n)}(a))}{\text{rad}(n)}\right)}.$$

**Corollary 3.5.** *Let  $n$  be a positive integer such that  $v_2(n) \geq 2$ . Then, for every integer  $a$ , coprime to  $n$ , we have*

$$\beta_n(a) = \frac{\beta_{2 \text{ rad}(n)}(a)}{\gcd\left(\beta_{2 \text{ rad}(n)}(a), \frac{\gcd(n, \mathcal{R}_{2 \text{ rad}(n)}(a))}{2 \text{ rad}(n)}\right)}.$$

## 4 Acknowledgments

The author would like to thank Shalom Eliahou for his help in preparing this paper. He also thanks the anonymous referee for its useful remarks.

## References

- [1] Jonathan Chappelon. On a problem of Molluzzo concerning Steinhaus triangles in finite cyclic groups. *INTEGERS*, 8:#A37, 2008.
- [2] John C. Molluzzo. Steinhaus graphs. In Springer, editor, *Theory and Applications of Graphs*, volume 642 of *Lecture Notes in Mathematics*, pages 394–402, Berlin / Heidelberg, 1978.
- [3] Melvyn B. Nathanson. *Elementary Methods in Number Theory*, pages 92–93. New York, Springer edition, 2000.