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Edge-partitions of sparse graphs and their applications to game coloring

Mickael Montassier, Arnaud Pêcher, André Raspaud,
Université Bordeaux 1, LaBRI UMR CNRS 5800
33405 Talence Cedex, France
{montassi, pecher, raspaud@labri.fr }

Xuding Zhu∗
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan
National Center for Theoretical Sciences
{zhu@math.nsysu.edu.tw }

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Abstract
In this note, we prove that every graph with maximum average degree less than \( \frac{32}{13} \) (resp. \( \frac{30}{11}, \frac{32}{11}, \frac{70}{23} \)) admits an edge-partition into a forest and a subgraph of maximum degree 1 (resp. 2, 3, 4). This implies that these graphs have game coloring number at most 5, 6, 7, 8, respectively.

1 Introduction
Let \( G \) be a simple graph. The game coloring number of \( G \) is defined through a two-person graph ordering game. Alice and Bob take turns choosing vertices from the set of unchosen vertices of \( G \). This defines a linear order \( L \) of the vertices of \( G \) with \( x < y \), if and only if, \( x \) is chosen before \( y \). The back degree of a vertex \( x \) with respect to \( L \) is the number of its neighbors \( y \) in \( G \) such that \( y < x \). The back degree of \( L \) is the maximum back degree of a vertex of \( G \) with respect to \( L \). Alice’s goal is to minimize the back degree of \( L \) and Bob’s goal is to maximize it. The game coloring number \( \text{col}_g(G) \) of \( G \) is defined to be \( k + 1 \), where \( k \) is the minimum integer such that Alice has a strategy for the graph ordering game to ensure that the back degree of \( L \) is at most \( k \). Equivalently, \( k \) is the maximum integer such that Bob has a strategy for the graph ordering game to ensure that the back degree of \( L \) is at least \( k \). This notion was first formally defined in [5] as a tool to find bounds to the game chromatic number [1].

Recently, Zhu [6] proved that the game coloring number of every planar graph is at most 17. This result was improved in the case of planar graphs with large girth, by Borodin et al. [2] and He et al. [4]. These results are based on some structural properties of planar graphs with large girth:

Theorem 1 (Borodin et. al. [2] + He et. al. [4]) Let \( G \) be a planar graph with girth at least \( g \).

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1. If \( g \geq 9 \), then \( G \) admits an edge-partition into a forest and a matching [2].

2. If \( g \geq 7 \), then \( G \) admits an edge-partition into a forest and a graph with maximum degree 2 [4].

3. If \( g \geq 5 \), then \( G \) admits an edge-partition into a forest and a graph with maximum degree 4 [4].

Zhu established this upper bound of the game coloring number:

**Lemma 1 (Zhu [5])** Suppose that the graph \( G \) has an edge-partition into two subgraphs \( G_1 \) and \( G_2 \), then \( \text{col}_g(G) \leq \text{col}_g(G_1) + \Delta(G_2) \).

Faigle et al. studied the game coloring number of a forest:

**Lemma 2 (Faigle et al. [3])** Let \( T \) be a forest. Then \( \text{col}_g(T) \leq 4 \).

Hence combining these two lemmas with Theorem 1, we have

**Corollary 1 ([2] + [4])** Every planar graph with girth at least 9 (resp. 7, 5) has game coloring number at most 5 (resp. 6, 8).

In this note, we study edge-partitions of sparse graphs, in the meaning of small maximum average degree, and derive bounds on the game coloring number.

The maximum average degree of \( G \), denoted by \( \text{Mad}(G) \) is:

\[
\text{Mad}(G) = \max\{\frac{|E(H)|}{|V(H)|}, H \subseteq G\}
\]

Our main result is:

**Theorem 2** Let \( G \) be a simple graph.

1. If \( \text{Mad}(G) < \frac{32}{13} \), then \( G \) admits an edge-partition into a forest and a matching.

2. If \( \text{Mad}(G) < \frac{30}{11} \), then \( G \) admits an edge-partition into a forest and graph with maximum degree at most 2.

3. If \( \text{Mad}(G) < \frac{32}{17} \), then \( G \) admits an edge-partition into a forest and graph with maximum degree at most 3.

4. If \( \text{Mad}(G) < \frac{70}{23} \), then \( G \) admits an edge-partition into a forest and graph with maximum degree at most 4.

In contrary to Theorem 1, Theorem 2 is not restricted to planar graphs. We note however that we can not infer Theorem 1 from Theorem 2, by using the usual inequality \( \text{Mad}(G) \leq \frac{2g}{(g - 2)} \) for every planar graph \( G \) of girth at least \( g \).

Combining with Lemmas 1 and 2, we get:

**Corollary 2** Let \( G \) be a simple graph.

1. If \( \text{Mad}(G) < \frac{32}{13} \), then \( \text{col}_g(G) \leq 5 \).

2. If \( \text{Mad}(G) < \frac{30}{11} \), then \( \text{col}_g(G) \leq 6 \).

3. If \( \text{Mad}(G) < \frac{32}{17} \), then \( \text{col}_g(G) \leq 7 \).

4. If \( \text{Mad}(G) < \frac{70}{23} \), then \( \text{col}_g(G) \leq 8 \).

Section 2 is dedicated to the proof of Theorem 2. Section 3 contains some final remarks.
2 Proof of Theorem 2

Let \( G \) be a simple graph. Let \( d(x) \) denote the degree of \( x \) in \( G \). A vertex of degree \( k \) (resp. at least \( k \), at most \( k \)) is called a \( k \)-vertex (resp. \( \geq k \)-vertex, \( \leq k \)-vertex). An \((a, b)\)-alternating cycle is an even cycle \( x_1x_2x_3 \ldots x_{2k}x_1 \) such that \( d(x_i) = a \) if \( i \) is even and \( d(x_i) = b \) otherwise. An \( k \)-vertex is a vertex of degree \( k \) adjacent to exactly \( k \) vertices.

Let \( G \) be a counterexample of Theorem 2, i.e. a graph that does not admit an edge-partition into a forest and a subgraph with maximum degree \( k \) \((k = 1, 2, 3, 4)\), minimizing \( \sigma(G) = |V(G)| + |E(G)| \).

2.1 Structural properties of \( G \)

Claim 1 The counterexample \( G \) does not contain:

1. 1-vertices,
2. two adjacent \( \leq k + 1 \)-vertices,
3. \((k + 2, 2)\)-alternating cycles.

Proof

1. By contradiction, assume that \( G \) contains an 1-vertex \( v \) adjacent to \( u \). By minimality, of \( G \), the graph \( H = G \setminus u \) admits an edge-partition into a forest \( F \) and a subgraph \( D \) with maximum degree \( k \). We can extend this edge-partition to \( G \) by adding the edge \( uv \) into \( F \), a contradiction.

2. Assume that \( G \) contains two adjacent \( \leq k + 1 \)-vertices, say \( u \) and \( v \). By minimality, of \( G \), the graph \( H = G \setminus uv \) admits an edge-partition into a forest \( F \) and a subgraph \( D \) with maximum degree \( k \). If at least one of \( u \) and \( v \) is incident to \( k \) edges in \( D \), then add \( uv \) in \( F \); otherwise, add \( uv \) into \( D \). This extends the edge-partition to \( G \), a contradiction.

3. Assume that \( G \) contains a \((k + 2, 2)\)-alternating cycle \( C = x_1x_2x_3 \ldots x_{2l}x_1 \) with \( d(x_i) = k + 2 \) if \( i \) is even and \( d(x_i) = 2 \) otherwise. By minimality of \( G \), the graph \( H = G \setminus \{x_1x_2, x_2x_3, \ldots, x_{2l-1}x_{2l}, x_{2l}x_1\} \) admits an edge-partition into a forest \( F \) and a subgraph \( D \) with maximum degree \( k \). We may assume that \( x_{2j} \) is incident to at least one edge of \( F \), for otherwise we can add an arbitrary edge incident to \( x_{2j} \) into \( F \). Now by adding the edges \( x_{2j}x_{2j+1} \) into \( D \), adding \( x_{2j}x_{2j-1} \) into \( F \), we obtain a required edge-partition of \( G \), a contradiction.

\( \square \)

2.2 Discharging procedures

In what follows, we will define an additional structure, called bank, which is a subgraph of \( G \) composed of maximal connected components, called agencies. In fact, we will show that each bank is a forest and each agency a tree. These structures will be used, during the discharging procedure, to transfer charges. Usually, the discharging rules operate locally; agencies will allow us to transfer charges non locally. In our discharging procedures, the vertices adjacent to an agency \( C \) will give their excess charge to \( C \) which will redistribute this excess charge to the vertices of \( C \) which does not have enough charges.

First we assign to each vertex \( v \) a charge \( \omega(v) \) equal to its degree, i.e. \( \forall v \in V(G), \omega(v) = d(v) \). Moreover we assign to each agency \( C \) (that will be defined later) a charge \( \omega(C) = 0 \). We define then discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge function \( \omega^* \) is produced. However, the total sum of charges is kept
fixed when the discharging is in process. Nevertheless, we can show that \( \omega^*(v) \geq \frac{32}{13} \) (resp. \( \frac{32}{17}, \frac{32}{11}, \frac{70}{23} \)) for all \( v \in V(G) \) and \( \omega^*(C) \geq 0 \) for all agency \( C \) of \( G \). Hence the following equation follows:

\[
\frac{32}{13}|V(G)| \leq \sum_{v \in V(G)} \omega^*(v) + \sum_{C \text{ agency of } G} \omega^*(C) = \sum_{v \in V(G)} \omega(v) + \sum_{C \text{ agency of } G} \omega(C) = \sum_{v \in V(G)|d(v)} = 2|E(G)|
\]

This leads to the following obvious contradiction:

\[
\frac{32}{13} = \frac{32}{|V(G)|} \leq \frac{2|E(G)|}{|V(G)|} \leq Mad(G) < \frac{32}{13}
\]

and hence demonstrates that no such counterexample can exist (as well for \( Mad(G) < \frac{32}{17}, \frac{32}{11}, \frac{70}{23} \)).

2.2.1 Graphs with \( Mad < \frac{32}{13} \)

Here, the bank of \( G \) is the subgraph of \( G \) defined as follows: its set of vertices contains all the \( 3_2 \)-vertices, \( 3_3 \)-vertices and the 2-vertices adjacent to \( 3_2 \)-vertices, or \( 3_3 \)-vertices ; its set of edges is the set of edges between the 2-vertices and the \( 3_2 \)-vertices, \( 3_3 \)-vertices. By Claim 1.3, an agency is a tree whose each leaf is a 2-vertex.

We say that a vertex, which does not belong to an agency, is adjacent to an agency if it is adjacent to a 2-vertex belonging to an agency.

The discharging rules are defined as follows:

**R1.** Every \( \geq 3 \)-vertex gives \( \frac{4}{13} \) to each adjacent 2-vertex.

**R2.** Every \( \geq 3 \)-vertex not belonging to an agency gives \( \frac{2}{13} \) to each adjacent agency.

**R3.** Each agency gives \( \frac{2}{17} \) to each of its own \( 3 \)-vertices.

Let us check first that for each vertex \( v, \omega^*(v) \geq \frac{32}{13} \). Let \( v \) be a \( k \)-vertex \((k \geq 2 \text{ by Claim } 1.1)\).

Case \( k = 2 \) Initially, \( \omega(v) = 2 \). The vertex \( v \) receives \( \frac{4}{13} \) from each of its neighbors (which are \( \geq 3 \)-vertices by Claim 1.2). Hence, \( \omega^*(v) = 2 + 2 \cdot \frac{4}{13} = \frac{32}{13} \).

Case \( k = 3 \) Initially, \( \omega(v) = 3 \). If \( v \) is adjacent to at most one 2-vertex, then \( \omega^*(v) \geq 3 - \frac{4}{13} \geq \frac{34}{13} \). If \( v \) is a \( 3 \)-vertex, then \( v \) belongs to an agency and gives two times \( \frac{4}{13} \) by R1 and nothing by R2. Hence \( \omega^*(v) = 3 - 2 \cdot \frac{4}{13} = \frac{5}{13} \). Finally assume that \( v \) is a \( 3 \)-vertex. The vertex \( v \) gives three times \( \frac{4}{13} \) by R1 and receives \( \frac{2}{13} \) by R3. Hence \( \omega^*(v) = 3 - 3 \cdot \frac{4}{13} + \frac{2}{13} = \frac{22}{13} \).

Case \( k \geq 4 \) Initially, \( \omega(v) = k \). The vertex \( v \) is adjacent to at most \( k \) 2-vertices and at most \( k \) agencies. Hence by R1 and R2, \( \omega^*(v) \geq k - k \cdot \frac{4}{13} \geq \frac{16}{13} \geq \frac{22}{13} \) if \( k \geq 4 \).

It remains to prove that the charge remaining on each agency is non-negative. Let \( C \) be an agency. Let \( n_{3_2}(C) \), and \( n_{3_3}(C) \) be the number of \( 3_2 \)-vertices, and leaves of \( C \) respectively. Observe that:

\[
n_{l}(C) \geq n_{3_3}(C) \tag{1}
\]

By R2, the agency \( C \) receives \( \frac{2}{13} \) from its adjacent vertices (i.e. it receives \( n_{l}(C) \cdot \frac{2}{13} \)), and gives \( \frac{4}{13} \) to each of its own \( 3_3 \)-vertices by R3 (i.e. it gives \( n_{3_3}(C) \cdot \frac{4}{13} \)). Hence, \( \omega^*(C) = n_{l}(C) \cdot \frac{2}{13} - n_{3_3}(C) \cdot \frac{4}{13} \geq n_{3_3}(C) \cdot \frac{2}{13} - n_{3_3}(C) \cdot \frac{4}{13} \geq 0 \) by Equation (1). This completes the proof of Theorem 2.1.
2.2.2 Graphs with $\text{Mad} < \frac{30}{11}$

Here, the bank of $G$ is the subgraph of $G$ defined as follows: its set of vertices contains all the 4$_4$-vertices, 4$_5$-vertices and the 2-vertices adjacent to 4$_3$-vertices, or 4$_4$-vertices; its set of edges is the set of edges between the 2-vertices and the 4$_3$-vertices, 4$_4$-vertices. By Claim 1.3, an agency is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

R1. Every $\geq 4$-vertex gives $\frac{4}{11}$ to each adjacent 2-vertex.

R2. Every $\geq 4$-vertex not belonging to an agency gives $\frac{1}{11}$ to each adjacent agency.

R3. Each agency gives $\frac{2}{11}$ to each of its own 4$_4$-vertices.

Let us check first that for each vertex $v$, $\omega^*(v) \geq \frac{30}{11}$. Let $v$ be a $k$-vertex.

Case $k = 2$Initially, $\omega(v) = 2$. The vertex $v$ receives $\frac{4}{11}$ from each of its neighbors (which are $\geq 4$-vertices by Claim 1.2). Hence, $\omega^*(v) = 2 + 2 \cdot \frac{1}{11} = \frac{30}{11}$.

Case $k = 3$Initially, $\omega(v) = 3$. The vertex $v$ is not affected by the discharging procedure; hence $\omega^*(v) = \omega(v) = 3 \geq \frac{30}{11}$.

Case $k = 4$Initially, $\omega(v) = 4$. Suppose that $v$ is a 4-vertex adjacent to at most two 2-vertices. Then $v$ gives at most two times $\frac{2}{11}$ to its adjacent 2-vertices and two times $\frac{1}{11}$ to its adjacent agencies. Hence $\omega^*(v) \geq 4 - 2 \cdot \frac{2}{11} - 2 \cdot \frac{1}{11} = \frac{30}{11} \geq \frac{30}{11}$. Assume that $v$ is a 4$_3$-vertex. Then $v$ belongs to an agency; hence $v$ gives three times $\frac{3}{11}$ to its adjacent 2-vertices by R1 and nothing by R2. So $\omega^*(v) = 4 - 3 \cdot \frac{3}{11} = \frac{30}{11} \geq \frac{30}{11}$. Finally, assume that $v$ is a 4$_4$-vertex. The vertex $v$ gives four times $\frac{1}{11}$ to its incident 2-vertices by R1 and receives $\frac{3}{11}$ from its agency by R3. Hence $\omega^*(v) = 4 - 4 \cdot \frac{1}{11} + \frac{3}{11} = \frac{30}{11}$.

Case $k \geq 5$ Initially, $\omega(v) = k$. The vertex $v$ is adjacent to at most $k$ 2-vertices and to at most $k$ agencies. Hence by R1 and R2, $\omega^*(v) \geq k - k \cdot \frac{1}{11} - k \cdot \frac{1}{11} = \frac{30}{11} \geq \frac{30}{11}$ if $k \geq 5$.

It remains to prove that the charge remaining on each agency is non-negative. Let $C$ be an agency. Let $n_{4_4}(C)$, $n_{4_5}(C)$, and $n_l(C)$ be the number of 4$_4$-vertices, 4$_5$-vertices, and leaves of $C$ respectively.

Observe that:

$$n_l(C) \geq n_{4_5}(C) + 2 \cdot n_{4_4}(C) \quad (2)$$

By R2, the agency $C$ receives $\frac{1}{11}$ from its adjacent vertices (i.e. it receives $n_l(C) \cdot \frac{1}{11}$), and gives $\frac{2}{11}$ to each of its own 4$_4$-vertices by R3 (i.e. it gives $n_{4_4}(C) \cdot \frac{2}{11}$). Hence, $\omega^*(C) = n_l(C) \cdot \frac{1}{11} - n_{4_5}(C) \cdot \frac{2}{11} \geq 2 \cdot n_{4_4}(C) \cdot \frac{2}{11} - n_{4_4}(C) \cdot \frac{2}{11} \geq 0$ by Equation (2). This completes the proof of Theorem 2.2.

2.2.3 Graphs with $\text{Mad} < \frac{30}{11}$

Here, the bank of $G$ is the subgraph of $G$ defined as follows: its set of vertices contains all the 5$_4$-vertices, 5$_5$-vertices and the 2-vertices adjacent to 5$_4$-vertices, or 5$_5$-vertices; its set of edges is the set of edges between the 2-vertices and the 5$_4$-vertices, 5$_5$-vertices. By Claim 1.3, an agency is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

R1. Every $\geq 5$-vertex gives $\frac{5}{11}$ to each adjacent 2-vertex.

R2. Every $\geq 5$-vertex not belonging to an agency gives $\frac{2}{11}$ to each adjacent agency.

R3. Each agency gives $\frac{2}{11}$ to each of its own 5$_5$-vertices.
Let us check first that for each vertex \( v, \omega^*(v) \geq \frac{22}{11} \). Let \( v \) be a \( k \)-vertex.

Case \( k = 2 \): Initially, \( \omega(v) = 2 \). The vertex \( v \) receives \( \frac{6}{11} \) from each of its neighbors (which are \( \geq 5 \)-vertices by Claim 1.2). Hence, \( \omega^*(v) = 2 + 2 \cdot \frac{6}{11} = \frac{22}{11} \).

Case \( k = 3, 4 \): The vertex \( v \) is not affected by the discharging procedure; hence \( \omega^*(v) = \omega(v) \geq \frac{22}{11} \).

Case \( k = 5 \): Initially, \( \omega(v) = 5 \). Assume that \( v \) is a 5-vertex adjacent to at most three 2-vertices. Then by R1 and R2 \( \omega^*(v) \geq 5 - 3 \cdot \frac{7}{11} - 3 \cdot \frac{7}{33} = \frac{50}{11} \). Assume now that \( v \) is a 5-vertex. Then \( v \) belongs to an agency and gives nothing by R2. So \( \omega^*(v) = 5 - \frac{5}{11} + \frac{2}{11} = \frac{32}{11} \).

Case \( k \geq 6 \): Initially, \( \omega(v) = k \). The vertex \( v \) is adjacent to at most \( k \) 2-vertices and to at most \( k \) agencies. Hence by R1 and R2, \( \omega^*(v) \geq k - k \cdot \frac{5}{11} - k \cdot \frac{2}{33} = \frac{16k}{33} \geq \frac{32}{11} \) if \( k \geq 6 \).

It remains to prove that the charge remaining on each agency is non-negative. Let \( C \) be an agency. Let \( n_{5_5}(C) \) and \( n_l(C) \) be the number of \( 5_5 \)-vertices and leaves of \( C \) respectively. Observe that:

\[
n_l(C) \geq 3 \cdot n_{5_5}(C) \tag{3}
\]

By R2, the agency \( C \) receives \( \frac{2}{33} \) from its adjacent vertices (i.e., it receives \( n_l(C) \cdot \frac{2}{33} \)), and gives \( \frac{7}{11} \) to each of its own \( 5_5 \)-vertices by R3 (i.e., it gives \( n_{5_5}(C) \cdot \frac{7}{11} \)). Hence, \( \omega^*(C) = n_l(C) \cdot \frac{2}{33} - n_{5_5}(C) \cdot \frac{7}{11} \geq 3 \cdot n_{5_5}(C) \cdot \frac{2}{33} - n_{5_5}(C) \cdot \frac{7}{11} \geq 0 \) by Equation (3). This completes the proof of Theorem 2.3.

### 2.2.4 Graphs with Mad < \( \frac{70}{23} \)

Here, the bank of \( G \) is the subgraph of \( G \) defined as follows: its set of vertices contains all the \( 6_0 \)-vertices and the 2-vertices adjacent to \( 6_0 \)-vertices; its set of edges is the set of edges between the 2-vertices and the \( 6_0 \)-vertices. By Claim 1.3, an agency is a tree whose each leaf is a 2-vertex.

The discharging rules are defined as follows:

**R1.** Every \( \geq 6 \)-vertex gives \( \frac{12}{23} \) to each adjacent 2-vertex, and \( \frac{1}{66} \) to each adjacent 3-vertex.

**R2.** Every \( \geq 6 \)-vertex not belonging to an agency gives \( \frac{1}{23} \) to each adjacent agency.

**R3.** Each agency gives \( \frac{1}{23} \) to each of its own \( 6_0 \)-vertices.

Let us check first that for each vertex \( v, \omega^*(v) \geq \frac{70}{23} \). Let \( v \) be a \( k \)-vertex.

Case \( k = 2 \): Initially, \( \omega(v) = 2 \). The vertex \( v \) receives \( \frac{12}{23} \) from each of its neighbors (which are \( \geq 6 \)-vertices by Claim 1.2). Hence, \( \omega^*(v) = 2 + 2 \cdot \frac{12}{23} = \frac{70}{23} \).

Case \( k = 3 \): The vertex \( v \) is adjacent to \( \geq 6 \)-vertices by Claim 1.2. Then \( v \) receives \( \frac{1}{66} \) from each of its neighbors; hence \( \omega^*(v) = 3 + 3 \cdot \frac{1}{66} = \frac{70}{23} \).

Case \( k = 4, 5 \): The vertex \( v \) is not affected by the discharging procedure; hence \( \omega^*(v) = \omega(v) \geq \frac{70}{23} \).

Case \( k = 6 \): Initially, \( \omega(v) = 6 \). Assume that \( v \) is adjacent to at most five 2-vertices. Then by R1 and R2 \( \omega^*(v) \geq 6 - 5 \cdot \frac{12}{23} - 5 \cdot \frac{1}{23} = \frac{70}{23} \). Assume finally that \( v \) is a \( 6_0 \)-vertex. The vertex \( v \) belongs to an agency. By R1 and R3 \( \omega^*(v) = 6 - 6 \cdot \frac{12}{23} + \frac{1}{23} = \frac{70}{23} \).

Case \( k \geq 7 \): Initially, \( \omega(v) = k \). The vertex \( v \) is adjacent to at most \( k \leq 3 \)-vertices and to at most \( k \) agencies. Hence by R1 and R2, \( \omega^*(v) \geq k - k \cdot \frac{12}{23} - k \cdot \frac{1}{23} = \frac{10k}{23} \geq \frac{70}{23} \) if \( k \geq 7 \).
It remains to prove that the charge remaining on each agency is non-negative. Let $C$ be an agency. Let $n_{6_6}(C)$ and $n_1(C)$ be the number of $6_6$-vertices and leaves of $C$ respectively. Observe that:

$$n_1(C) \geq 4 \cdot n_{6_6}(C) \quad \text{(4)}$$

By R2, the agency $C$ receives $\frac{4}{23}$ from its adjacent vertices (i.e. it receives $n_1(C) \cdot \frac{4}{23}$), and gives $\frac{4}{23}$ to each of its own $6_6$-vertices by R3 (i.e. it gives $n_{6_6}(C) \cdot \frac{4}{23}$). Hence, $\omega^*(C) = n_{6_6}(C) \cdot \frac{4}{23} - n_{6_6}(C) \cdot \frac{4}{23} \geq 4 \cdot n_{6_6}(C) \cdot \frac{4}{23} - n_{6_6}(C) \cdot \frac{4}{23} \geq 0$ by Equation (4). This completes the proof of Theorem 2.4.

3 Conclusion

In this note, we established that for every simple graph $G$, if $\text{Mad}(G) \leq \frac{32}{11}$ (resp. $\frac{30}{11}$, $\frac{32}{11}$, $\frac{70}{23}$) then $G$ admits an edge-partition into a forest and a graph with maximum degree at most 1 (resp. 2, 3, 4), hence the game chromatic of $G$ is at most 5 (resp. 6, 7, 8). In order to study the tightness of Theorem 2, we introduce a function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(k) = \inf \{ \text{Mad}(H) \mid H \text{ does not admit an edge-partition into a forest and a subgraph with maximum degree } k \}$. It is easy to observe that the complete bipartite graph $K_{2,2k+2}$ has $\text{Mad}(K_{2,2k+2}) = \frac{4(k+1)}{k+2}$ and does not admit an edge-partition into a forest and a subgraph with maximum degree at most $k$. Hence,

$$2,461... = \frac{32}{13} \leq f(1) \leq \frac{8}{3} = 2,666...$$
$$2,727... = \frac{30}{11} \leq f(2) \leq 3$$
$$2,909... = \frac{32}{11} \leq f(3) \leq \frac{16}{5} = 3,2$$
$$3,043... = \frac{70}{23} \leq f(4) \leq \frac{10}{3} = 3,333...$$

We conclude with the following problem:

**Problem 1** For every $k$, what are (if any) the graphs which do not admit an edge-partition into a forest and a subgraph with maximum degree at most $k$, but such that every graph with smaller maximal average degree does?

**References**


