Stability of Static Walls for a three dimensional Model of Ferromagnetic Material

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Abstract. In this paper we consider a three dimensional model of ferromagnetic material. We deal with the static domain wall configuration calculated by Walker. We prove the stability of this configuration for the Landau-Lifschitz equation with a simplified expression of the demagnetizing field.

Résumé. Dans cet article, on considère un modèle tridimensionnel de matériau ferromagnétique. On étudie les profils de murs statiques calculés initialement par Walker. On démontre la stabilité de ces profils pour l'équation de Landau-Lifschitz avec un modèle simplifié pour le champ démagnétisant.

1 Introduction and main results

The formation and the dynamics of domain walls are among the most studied topics in micromagnetism. In his pioneering works [27], Walker performed the exact integration of the equations of motion for a planar wall (see [24].) In this paper, we tackle the problem of the stability of these exact solutions for the Landau-Lifschitz equation in a simplified 3-dimensional model.

Let us recall the general framework of the ferromagnetism (see [5], [15] , [25] and [28]). We consider an infinite homogeneous ferromagnetic medium. We denote by m the magnetization:

$$m : \quad \begin{aligned} I\!\!R^+ \times I\!\!R^3 &\rightarrow I\!\!R^3 \\ (t,x,y,z) &\mapsto m(t,x,y,z). \end{aligned}$$

The magnetic moment m links the magnetic induction B and the magnetic field H by the relation $B = m + H$. In addition, we assume that the material is saturated so that the norm of m is constant. After renormalization we assume that

$$|m| = 1 \text{ at any point.} \tag{1.1}$$

The variations of m are described by the Landau-Lifschitz equation:

$$\partial_t m = -m \times H_{eff} - m \times (m \times H_{eff}), \tag{1.2}$$

where the effective field $H_{eff} = -\nabla \mathcal{E}$. The ferromagnetism energy $\mathcal{E}$ in supposed to be given by

$$\mathcal{E} = \mathcal{E}_{exch} + \mathcal{E}_{dem} + \mathcal{E}_{anis},$$

where

- the exchange energy $\mathcal{E}_{exch}$ writes

$$\mathcal{E}_{exch} = \frac{1}{2} \int_{I\!\!R^3} |\nabla m|^2,$$

- the anisotropy energy reflects the existence of a preferential axis of magnetization:

$$\mathcal{E}_{anis} = \int_{\mathbb{R}^3} (1 - |m_3|^2),$$

where $m = (m_1, m_2, m_3)$,

- $\mathcal{E}_{dem}$ is the demagnetizing energy:

$$\mathcal{E}_{dem} = \int_{\mathbb{R}^3} |h_d(m)|^2,$$

where the demagnetizing field $h_d(m)$ is characterized by

$$\begin{cases} \text{curl } h_d(m) = 0, \\ \\ \text{div } (h_d(m) + m) = 0. \end{cases} \qquad (1.3)$$

We then obtain that

$$H_{eff} = \Delta m + m_3 e_3 + h_d(m),$$

where e_3 is the third vector of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$.

Existence results for the Landau-Lifschitz equation can be found in [26], [2], [6] and [18] for the weak solutions, and in [7], [8] and [9] for the strong solutions. Numerical simulations are performed in [3], [4], [19], [20] and [21].

In case of a magnetic moment only depending on the x variable, the demagnetizing field given integrating (1.3) reads $h_d(m) = -m_1 e_1$. With this expression of the demagnetizing field, Walker calculated in [24] the following wall profile, which is a static solution to Landau-Lifschitz equation:

$$M_0(x, y, z) = M_0(x) = \begin{pmatrix} 0 \\ 1/\text{ch } x \\ -\text{th } x \end{pmatrix}. \qquad (1.4)$$

In our paper we simplify the model assimilating h_d to $-m_1 e_1$ even for perturbations of M_0. So we deal with the following system:

$$\partial_t m = -m \times H_{eff} - m \times (m \times H_{eff}),$$
$$\qquad (1.5)$$
$$H_{eff} = \Delta m + m_3 e_3 - m_1 e_1,$$

and we adress the stability of the static solution M_0 for the system (1.5). Our main result is the following:

Theorem 1 *Let $\varepsilon > 0$. There exists $\delta > 0$ such that for all $m_0 \in H^2(\mathbb{R}^3; \mathbb{R}^3)$, if m_0 satisfies the saturation constraint $|m_0| = 1$ and verifies $\|m_0 - M_0\|_{H^2(\mathbb{R}^3)} \leq \delta$, then the solution m of the Landau-Lifschitz equation (1.5) together with the initial data $m(0, x, y, z) = m_0(x, y, z)$ satisfies*

$$\forall\, t \geq 0, \|m(t, .) - M_0\|_{H^2(\mathbb{R}^3)} \leq \varepsilon.$$

In [10], we proved the same kind of stability result for a one dimensional model of ferromagnetic nanowire. We extended this result in [11] by proving the controlability of the wall position for this 1-d model. In the present paper, we deal with the 3-d model (1.5). The proof of the stability result somewhat follows that presented in [10]. The first two steps are formally similar.

At the begining we must consider perturbations m of the profile M_0 satisfying the physical constraint $|m| = 1$. In order to do that, we describe m in the mobile frame $(M_0(x), M_1(x), M_2)$ where

$$M_1(x) = \begin{pmatrix} 0 \\ \text{th } x \\ \dfrac{1}{\text{ch } x} \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

writting

$$m(t, x, y, z) = r_1(t, x, y, z)M_1(x) + r_2(t, x, y, z)M_2 + \left(1 - (r_1(t, x, y, z))^2 - (r_2(t, x, y, z))^2\right)^{\frac{1}{2}} M_0(x).$$

The new unknown $r = (r_1, r_2)$ now takes its values in the flat space $\mathbb{R}^2$. We then rewrite the Landau-Lifschitz equation with the unknown r, and we obtain in Section 2 that the Landau-Lifschitz equation is equivalent to a non linear equation on r, and the stability of M_0 is equivalent to the stability of 0 for this new equation.

Now the problem is that the linearized around zero admits 0 as a simple eigenvalue. This is due to the invariance of the Landau-Lifschitz equation (1.5) by translation in the x- variable (see Section 3). Following the method developped in [29], [14], [16] and [17] (for travelling waves solutions to semilinear parabolic equations), we decompose the perturbations into a spacial translation component (the "front") and a normal component. The front satisfies a quasilinear parabolic equation which linear part looks likes the heat equation in $\mathbb{R}^2$, so that we don't have dissipation for the front L^2 norm. The normal component is shown to satisfy a very dissipative quasilinear parabolic equation (see Section 4).

The last section is devoted to variational estimates to prove the stability. The situation in the present paper is much more complicated than the one dimensional case, because in 1-d, the front part satisfies an ordinary differential equation. In addition, here the equations are quasilinear, and Kapitula's method with semigroup estimates for the heat flow cannot be applied (see [16] for example).

The lack of dissipation for the front is compensated by a carrefull study of the nonlinear part. The key point is that we can control this non linear part by the gradient of the front for which we do have dissipation.

Remark 1 *When a constant magnetic field is applied in the x-direction on the ferromagnetic material, it is observed that the domain wall is translated in the x-direction. In [24] such solutions are calculated. They are described as traveling waves of a profile obtained from M_0 by rotation and dilation. The stability of these moving walls remains an open problem and our method does not work in that case. In the same way, the stability of walls with the non simplified demagnetizing field remains unproved (see Remark 4 below.)*

Remark 2 *In the static case, the formation of domain walls is explained by asymptotic methods. We refer the interested reader to [1], [12], [13] and [23].*

2   Mobile frame

We consider the mobile frame $(M_0(x), M_1(x), M_2)$ given by:

$$\forall x \in \mathbb{R}, \quad M_0(x) = \begin{pmatrix} 0 \\ 1/\operatorname{ch} x \\ -\operatorname{th} x \end{pmatrix}, \quad M_1(x) = \begin{pmatrix} 0 \\ \operatorname{th} x \\ 1/\operatorname{ch} x \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let us introduce the smooth map $\nu : B(0, 1) \to \mathbb{R}$ defined for $\xi = (\xi_1, \xi_2)$ by

$$\nu(\xi) = \sqrt{1 - (\xi_1)^2 - (\xi_2)^2} - 1,$$

where $B(0, 1) = \{(\xi_1, \xi_2, (\xi_1)^2 + (\xi_2)^2 < 1\}$ is the unit ball of $\mathbb{R}^2$.

We write the perturbations of M_0 as:

$$m(t, x, y, z) = M_0(x) + r_1(t, x, y, z)M_1(x) + r_2(t, x, y, z)M_2(x) + \nu(r(t, x, y, z))M_0(x),$$

so that the constraint $|m| = 1$ is satisfied.

We will work with the unknown $r(t, x, y, z) = \begin{pmatrix} r_1(t, x, y, z) \\ r_2(t, x, y, z) \end{pmatrix}$.

We remark that we have $r_1(t, x, y, z) = m(t, x, y, z) \cdot M_1(x)$ and $r_2(t, x, y, z) = m(t, x, y, z) \cdot M_2$.

After a rather long algebraic calculation, we obtain that if m satisfies (1.5) then r verifies:

$$\partial_t r = \Lambda r + F(x, r, \nabla r, \Delta r), \tag{2.6}$$

where

$$\Lambda r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Lr_1 \\ Lr_2 + r_2 \end{pmatrix},$$

with $L = -\Delta + f$, $f(x) = 2\mathrm{th}^2 x - 1$, and where the non linear part $F : \mathbb{R} \times B(0, 1) \times \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by:

$$F(x, r, \nabla r, \Delta r) = A(r)\Delta r + \sum_{i=1}^{3} B(r)(\partial_i r, \partial_i r) + C(x, r)(\partial_x r) + D(x, r),$$

where

- $A \in \mathcal{C}^\infty(B(0, 1); \mathcal{M}_2(\mathbb{R}))$ ($\mathcal{M}_2(\mathbb{R})$ is the set of the real 2×2 matrices):

$$A(r) = \begin{pmatrix} 2\nu(r) + (\nu(r))^2 + (r_2)^2 & \nu(r) - r_1 r_2 \\ \\ \nu(r) - r_1 r_2 & 2\nu(r) + (\nu(r))^2 + (r_1)^2 \end{pmatrix} + (r_1 - r_2 - r_2\nu(r))\nu'(r),$$

- $B \in \mathcal{C}^\infty(B(0, 1); \mathcal{L}_2(\mathbb{R}^2))$ ($\mathcal{L}_2(\mathbb{R}^2; \mathbb{R}^2)$ is the set of the bilinear functions defined on $\mathbb{R}^2 \times \mathbb{R}^2$ with values in $\mathbb{R}^2$):

$$B(r)(\xi, \xi) = \begin{pmatrix} -r_2 - r_1 - r_1\nu(r) \\ \\ r_1 - r_2 - r_2\nu(r) \end{pmatrix} \nu''(r)(\xi, \xi),$$

- $\partial_1 r = \dfrac{\partial r}{\partial x}$, $\partial_2 r = \dfrac{\partial r}{\partial y}$, $\partial_3 r = \dfrac{\partial r}{\partial z}$,

- $C \in \mathcal{C}^\infty(\mathbb{R} \times B(0, 1); \mathcal{M}_2(\mathbb{R}))$:

$$C(x, r)(\xi) = \frac{2}{\mathrm{ch}\, x} \begin{pmatrix} -r_2 - r_1 - r_1\nu(r) \\ \\ r_1 - r_2 - r_2\nu(r) \end{pmatrix} \xi_1 + \frac{2}{\mathrm{ch}\, x} \begin{pmatrix} -(1 + \nu(r))^2 - (r_2)^2 \\ \\ 1 + \nu(r) + r_1 r_2 \end{pmatrix} \nu'(r)(\xi),$$

- $D \in \mathcal{C}^\infty(\mathbb{R} \times B(0, 1); \mathbb{R}^2)$: $D(x, r) = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ with

$$D_1 = -\left(r_2 + r_2 f + 2fr_1 + fr_1\nu(r)\right)\nu(r) + (r_2)^2 r_1 + \frac{2\mathrm{sh}\, x}{\mathrm{ch}^2 x} r_1 (r_2 + r_1 + r_1\nu(r)),$$

$$D_2 = \left(fr_1 - 2fr_2 - fr_2\nu(r) - 2r_2 - r_2\nu(r)\right)\nu(r) - r_2(r_1)^2 - \frac{2\mathrm{sh}\, x}{\mathrm{ch}^2 x} r_1 (r_1 - r_2 - r_2\nu(r)).$$

In fact, both forms of the Landau-Lifschitz equation are equivalent as it is stated in the following proposition:

Proposition 1 *Let $m \in \mathcal{C}^1(0, T; H^2(\mathbb{R}^3; \mathbb{R}^3))$ such that $|m| = 1$ and satisfying*

$$\forall t \in [0, T[, \ \forall (x, y, z) \in \mathbb{R}^3, \ |m(t, x, y, z) - M_0(x)| < \sqrt{2}. \tag{2.7}$$

We introduce $r = (r_1, r_2) \in \mathcal{C}^1(0, T; H^2(\mathbb{R}^2; \mathbb{R}^2))$ defined by

$$m(t, x, y, z) = M_0(x) + r_1(t, x, y, z)M_1(x) + r_2(t, x, y, z)M_2(x) + \nu(r(t, x, y, z))M_0(x)$$

(Assumption (2.7) implies that $r(t, x, y, z) \in B(0, 1)$ for all (t, x, y, z).)
Then m is solution to the Landau-Lifschitz equation (1.5) if and only if r is solution to (2.6) and M_0 is stable for (1.5) if and only if 0 is stable for (2.6).

4

Sketch of the proof. By projection on M_1 and M_2, it is clear that if m satisfies the Landau-Lifschitz equation (1.5) then m satisfies (2.6). The converse is proved in [10] using the fact that if $|m| = 1$ and if m satisfies the projection of (1.5) onto $\mathbb{R}M_1$ and $\mathbb{R}M_2$, then it satisfies (1.5).

Let us estimate the non linear functions appearing in (2.6). Since $\nu(\xi) = \mathcal{O}(|\xi|^2)$, by straightforward calculations, we obtain the following proposition:

Proposition 2 *There exists a constant K such that for $r \in B(0, 1/2)$ and for $x \in \mathbb{R}$,*

- $|A(r)| \leq K|r|^2$ *and* $|A'(r)| \leq K|r|$,

- $|B(r)| \leq K|r|$ *and* $|B'(r)| \leq K$,

- $|C(x,r)| \leq \dfrac{K}{\operatorname{ch} x}|r|$ *and* $|\partial_r C(x,r)| \leq \dfrac{K}{\operatorname{ch} x}$,

- $|D(x,r)| \leq K|r|^3 + \dfrac{K}{\operatorname{ch} x}|r|^2$ *and* $|\partial_r D(x,r)| \leq K|r|^2 + \dfrac{K}{\operatorname{ch} x}|r|$.

3   Linear properties

We denote by L the linear operator acting on $H^2(\mathbb{R}^3)$ defined by

$$Lu = -\Delta u + fu,$$

with $f(x,y,z) = 2\operatorname{th}^2 x - 1$.
We denote by L_1 the reduced operator acting on $H^2(\mathbb{R})$ given by

$$L_1 = -\partial_{xx} + f.$$

Proposition 3 *The operator L_1 is positive symmetric. Its spectrum is $\{0\} \cup [1, +\infty[$, where 0 is the unique eigenvalue, and $[1, +\infty[$ is the essential spectrum. In addition, 0 is simple.*

Proof. On one hand, since $f(x) = 2\operatorname{th}^2 x - 1$, the essential spectrum is $[1, +\infty[$ (see the Weyl Theorem in [22].)
On the other hand, $L_1 = l^* \circ l$ where $l = \partial_x + \operatorname{th} x$. So L_1 is positive. The kernel of L_1 is directed by $\frac{1}{\operatorname{ch} x}$:

$$\operatorname{Ker} L_1 = \operatorname{Ker} l = \mathbb{R}\frac{1}{\operatorname{ch} x}.$$

Finally we have $l \circ l^* = -\partial_{xx} + 1$, so if v is an eignevalue of L_1, then

$$l \circ l^* \circ lv = \lambda lv,$$

that is, if $v \notin \operatorname{Ker} l$, then λ is an eignevalue of $-\partial_{xx} + 1$, which leads to a contradiction.

Remark 3 *As we remarked in [10] and [11], a direct consequence of Proposition 3 is the following. Let $\mathcal{E}_1$ defined by*

$$\mathcal{E}_1 = (\operatorname{Ker} L_1)^{\perp} = \left\{ v \in H^2(\mathbb{R}), \int_{\mathbb{R}} v(x)\frac{1}{\operatorname{ch} x}dx = 0 \right\}.$$

Then on $\mathcal{E}_1$, the H^2-norm is equivalent to $\|L_1 u\|_{L^2(\mathbb{R})}$ and the H^3-norm is equivalent to $\|L_1^{\frac{3}{2}} u\|_{L^2(\mathbb{R})}$.

Proposition 4 *The operator $L = -\Delta + f$ is a positive self-adjoint operator defined on $H^2(\mathbb{R}^3)$. Let us consider $\mathcal{E}$ defined by*

$$\mathcal{E} = \left\{ v \in H^2(\mathbb{R}^3), \forall (y,z) \in \mathbb{R}^2, \int_{x \in \mathbb{R}} v(x,y,z)\frac{1}{\operatorname{ch} x}dx = 0 \right\}.$$

There exists K such that

$$\forall v \in \mathcal{E}, \|v\|_{H^2} \leq K\|Lv\|_{L^2},$$

$$\forall v \in H^3 \cap \mathcal{E}, \|v\|_{H^3} \leq K\|L^{\frac{3}{2}}v\|_{L^2}.$$

Proof. From Proposition 3, there exists a constant K such that for $u \in H^2(\mathbb{R})$, if $\int_{\mathbb{R}} u(x)\frac{1}{\operatorname{ch} x}dx = 0$, then

$$\|u\|^2_{L^2(\mathbb{R})} + \|\partial_{xx}u\|^2_{L^2(\mathbb{R})} \leq K\|L_1 u\|^2_{L^2(\mathbb{R})}.$$

Now for $v \in \mathcal{E}$, we have for almost every $(y, z) \in \mathbb{R}^2$:

$$\int_{x \in \mathbb{R}} \left(|v(x, y, z)|^2 + |\partial_{xx}v(x, y, z)|^2\right) dx \leq K \int_{\mathbb{R}} |L_1 v(x, y, z)|^2 dx.$$

So integrating for $(y, z) \in \mathbb{R}^2$ we obtain:

$$\|v\|^2_{L^2[\mathbb{R}^3]} + \|\partial_{xx}v\|^2_{L^2(\mathbb{R}^3)} \leq K\|L_1 v\|^2_{L^2(\mathbb{R}^3)}.$$

On the other hand,

$$\int_{\mathbb{R}^3} |Lv|^2 = \int_{\mathbb{R}^3} |L_1 v|^2 + \int_{\mathbb{R}^3} |\Delta_Y v|^2 - 2 \int_{\mathbb{R}^3} L_1 v \Delta_Y v,$$

where $\Delta_Y = \partial_{yy} + \partial_{zz}$. The last term is positive:

$$-2 \int_{\mathbb{R}^3} L_1 v \Delta_Y v = -2 \int_{\mathbb{R}^3} l^* \circ lv \cdot \Delta_Y v = 2 \int_{\mathbb{R}^3} |\nabla lv|^2,$$

by integrations by parts. So

$$\int_{\mathbb{R}^3} |Lv|^2 \geq \int_{\mathbb{R}^3} |L_1 v|^2 + \int_{\mathbb{R}^3} |\Delta_Y v|^2,$$

that is

$$\|v\|^2_{L^2(\mathbb{R}^3)} + \|\Delta v\|^2_{L^2(\mathbb{R}^3)} \leq K\|Lv\|^2_{L^2(\mathbb{R}^3)}.$$

The H^3 estimate can be proved with the same kind of arguments using Remark 3.

4 New coordinates

In the one dimensional case, i.e. for solutions depending only on the x-variable, we can construct a one parameter family of static solutions to the Landau-Lifschitz equation (1.5) using the invariance by translation. Indeed, for $s \in \mathbb{R}$, $x \mapsto M_0(x - s)$ satisfies (1.5). On the mobile frame, we then consider the one parameter family $(R(s))_{s \in \mathbb{R}}$ of static solutions to (2.6) obtained from $M_0(x - s)$:

$$R(s)(x) = \begin{pmatrix} M_0(x - s) \cdot M_1(x) \\ M_0(x - s) \cdot M_2 \end{pmatrix} = \begin{pmatrix} \rho(s)(x) \\ 0 \end{pmatrix},$$

where $\rho(s)(x) = \dfrac{\operatorname{th} x}{\operatorname{ch}(x - s)} - \dfrac{\operatorname{th}(x - s)}{\operatorname{ch} x}$.

Following Kapitula [16], for r in a neighbourhood of 0, it would be desirable to use the coordinate system given by (σ, φ, W) with perturbations of zero being given by:

$$r(t, x, y, z) = R(\sigma(t, y, z))(x) + \begin{pmatrix} 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix} \varphi(t, y, z) + W(t, x, y, z), \tag{4.8}$$

where both coordinates of W take their values in $\mathcal{E}$. We prove that this system of coordinates is relevant in the following proposition.

Proposition 5 *There exists $\delta_0 > 0$, such that if $r \in H^2(\mathbb{R}^3; \mathbb{R}^2)$ satisfies $\|r\|_{H^2(\mathbb{R}^3)} \leq \delta_0$, there exists $(\sigma, \varphi, W) \in H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2) \times \mathcal{E}^2$ such that*

$$r(x, y, z) = R(\sigma(y, z))(x) + \begin{pmatrix} 0 \\ 1 \\ \operatorname{ch} x \end{pmatrix} \varphi(y, z) + W(x, y, z).$$

In addition, in a neighbourhood of zero, we have the equivalence for the following couple of norms:

- $\|r\|_{H^2(\mathbb{R}^3)} \sim \left(\|\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^2(\mathbb{R}^2)} + \|W\|_{H^2(\mathbb{R}^3)} \right)$,

- $\|r\|_{H^3(\mathbb{R}^3)} \sim \left(\|\sigma\|_{H^3(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)} \right)$.

Proof. Let us introduce l^1 and l^2 defined for $r = (r_1, r_2) \in H^2(\mathbb{R}^3; \mathbb{R}^2)$ by:

$$l^1(r)(y, z) = \frac{1}{2} \int_{x \in \mathbb{R}} r_1(x, y, z) \cdot \frac{1}{\operatorname{ch} x} dx, \quad l^2(r)(y, z) = \frac{1}{2} \int_{x \in \mathbb{R}} r_2(x, y, z) \frac{1}{\operatorname{ch} x} dx.$$

The operators l^1 and l^2 are linear and map $H^2(\mathbb{R}^3; \mathbb{R}^2)$ and $H^3(\mathbb{R}^3; \mathbb{R}^2)$ respectively into $H^2(\mathbb{R}^2)$ and $H^3(\mathbb{R}^2)$.

In addition we remark that $\mathcal{E}^2 = \left\{ W \in H^2(\mathbb{R}^3; \mathbb{R}^2), l^1(W) = l^2(W) = 0 \right\}$.

For a fixed r in a neigbourhood of 0, the desired (σ, φ, W) can be found in the following manner:

- applying l^2 on (4.8) we obtain:
$$l^2(r)(y, z) = \varphi(y, z),$$

- applying l^1 on (4.8) yields:

$$l^1(r) = \frac{1}{2} \int_{x \in \mathbb{R}} \rho(\sigma(y, z))(x) \frac{1}{\operatorname{ch} x} dx.$$

Let us consider the map $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(s) = \frac{1}{2} \int_{x \in \mathbb{R}} \rho(s)(x) \frac{1}{\operatorname{ch} x} dx.$$

This map is smooth, $\psi(0) = 0$ and $\psi'(0) = 1$. So there exists $\delta_0 > 0$ such that ψ is a $\mathcal{C}^\infty$-diffeomorphism from $]-\delta_0, \delta_0[$ to a neighbourhood of zero. We obtained

$$l^1(r)(y, z) = \psi(\sigma(y, z)),$$

so σ is obtained by

$$\sigma(y, z) = \psi^{-1}(l^1(r)(y, z)).$$

- By subtraction, we set

$$W(x, y, z) = r(x, y, z) - R(\sigma(y, z))(x) - \begin{pmatrix} 0 \\ 1 \\ \operatorname{ch} x \end{pmatrix} \varphi(y, z),$$

and by construction, we obtain that $l^1(W) = l^2(W) = 0$, that is $W \in \mathcal{E}^2$.

Concerning the equivalence of norms, with straighforward estimates, using that $\rho(0)(x) = 1$ and $\partial_s \rho(0)(x) = \frac{1}{\operatorname{ch} x}$ we obtain for example that for $\sigma \in H^2(\mathbb{R}^3)$ is sufficiently small

$$\|(x, y, z) \mapsto R(\sigma(y, z))(x)\|_{H^2(\mathbb{R}^3)} \leq K \|\sigma\|_{H^2(\mathbb{R}^2)},$$

so

$$\|r\|_{H^2(\mathbb{R}^3)} \leq K \left(\|\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^2(\mathbb{R}^2)} + \|W\|_{H^2(\mathbb{R}^3)} \right).$$

By the continuity of the linear operators l^1 and l^2 for the H^2 norm, since ψ^{-1} is smooth in a neighbourhood of 0 and satisfies $\psi^{-1}(s) = s + \mathcal{O}(s^2)$, we obtain that

$$\|\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^2(\mathbb{R}^2)} \leq K\|r\|_{H^2(\mathbb{R}^3)},$$

and by difference we obtain the desired estimate on W. This concludes the proof of Proposition 5.

So in a neighbourhood of zero, we describe r in the coordinates (σ, φ, W) given by (4.8). Let us rewrite (2.6) in these coordinates. We assume tha δ_0 is small enough to ensure that $\|r\|_{L^\infty} < 1$, so that r can satisfy (2.6).

We first remark that in the one dimensional case, for a fixed s, the map $x \mapsto R(s)(x)$ is a static solution to (2.6). So we have

$$\Lambda_1 R(\sigma) + A(R(\sigma))\partial_{xx}R(\sigma) + B(R(\sigma))(\partial_x R(\sigma), \partial_x R(\sigma)) + C(R(\sigma))(\partial_x R(\sigma)) + D(R(\sigma)) = 0, \quad (4.9)$$

where we denote by Λ_1 the reduced operator:

$$\Lambda_1 w = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} L_1 w_1 \\ L_1 w_2 + w_2 \end{pmatrix}.$$

Furthermore, we have:

$$\partial_t (R(\sigma(t, y, z))(x)) = \partial_s R(\sigma(t, y, z))\partial_t \sigma(t, y, z),$$

and

$$\Delta(R(\sigma(t, y, z))(x)) = \partial_{xx}R(\sigma(t, y, z)) + \partial_s R(\sigma(t, y, z))(\Delta_Y \sigma) + \partial_{ss}R(\sigma(t, y, z))|\nabla_Y \sigma|^2,$$

where $\Delta_Y = \partial_{yy} + \partial_{zz}$ and where $|\nabla_Y \sigma|^2 = |\partial_y \sigma|^2 + |\partial_z \sigma|^2$. So, we have:

$$\Lambda R(\sigma) = \Lambda_1 R(\sigma) + \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} (-\partial_s R(\sigma)\Delta_Y \sigma - \partial_{\sigma\sigma}R(\sigma)|\nabla_Y \sigma|^2).$$

Plugging (4.8) in (2.6) and using (4.9) yield:

$$
\begin{aligned}
\partial_s R(\sigma)\partial_t \sigma + \begin{pmatrix} 0 \\ 1 \\ \operatorname{ch} x \end{pmatrix} \partial_t \varphi + \partial_t W = {} & \left(\partial_s \rho(\sigma)\Delta_Y \sigma - \partial_{ss}\rho(\sigma)|\nabla_Y \sigma|^2\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
& + \frac{1}{\operatorname{ch} x}(-\Delta_Y \varphi + \varphi)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \Lambda W + G,
\end{aligned}
\tag{4.10}
$$

where the non linear term G is defined by

$$G = G_1 + G_2 + \ldots + G_5, \tag{4.11}$$

where

- $G_1 = A(R(\sigma))\Delta_Y R(\sigma) + \tilde{A}(R(\sigma), w)(w)(\Delta r) + A(r)\Delta w,$

- $G_2 = 2B(R(\sigma))(\partial_x R(\sigma), \partial_x w) + B(R(\sigma))(\partial_x w, \partial_x w) + \tilde{B}(R(\sigma), w)(w)(\partial_x r, \partial_x r),$

- $G_3 = \displaystyle\sum_{i=2}^{3} B(r)(\partial_i r, \partial_i r),$

- $G_4 = C(x, R(\sigma))(\partial_x w) + \tilde{C}(x, R(\sigma), w)(w)(\partial_x r),$

- $G_5 = \tilde{D}(x, R(\sigma), w)(w),$

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where we denote by $w = \varphi(x) \begin{pmatrix} 0 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix} + W$, by $r = R(\sigma) + w$, and where:

- $\tilde{A} \in \mathcal{C}^\infty(B(0, 1/2) \times B(0, 1/2); \mathcal{L}(I\!\!R^2; \mathcal{M}_2(I\!\!R)))$:

$$\tilde{A}(u, v) = \int_0^1 A'(u + sv)ds,$$

- $\tilde{B} \in \mathcal{C}^\infty(B(0, 1/2) \times B(0, 1/2); \mathcal{L}(I\!\!R^2; \mathcal{L}_2(I\!\!R^2; I\!\!R^2)))$:

$$\tilde{B}(u, v) = \int_0^1 B'(u + sv)ds,$$

- $\tilde{C} \in \mathcal{C}^\infty(B(0, 1/2) \times B(0, 1/2); \mathcal{L}(I\!\!R^2; \mathcal{M}_2(I\!\!R)))$:

$$\tilde{C}(x, u, v) = \int_0^1 \partial_r C(x, u + sv)ds,$$

- $\tilde{D} \in \mathcal{C}^\infty(B(0, 1/2) \times B(0, 1/2); \mathcal{L}(I\!\!R^2; I\!\!R^2))$:

$$\tilde{D}(x, u, v) = \int_0^1 \partial_\xi D(x, u + sv)ds.$$

The tilda terms come from the fundamental theorem of the analysis applied between $R(\sigma)$ and $R(\sigma) + w$.

In order to separate the unknown, we will use the projectors l^1 and l^2.

We multiply (4.10) by $\begin{pmatrix} \frac{1}{2\operatorname{ch} x} \\ 0 \end{pmatrix}$ and we integrate in the x variable. We obtain:

$$\tilde{g}(\sigma)\partial_t\sigma = \tilde{g}(\sigma)\Delta_Y\sigma + \Delta_Y\varphi - \varphi + \tilde{K}(\sigma)|\nabla_Y\sigma|^2 + l^1(G),$$

where

$$\tilde{g}(s) = \frac{1}{2}\int_{x \in I\!\!R} \partial_s\rho(s)(x)\frac{1}{\operatorname{ch} x}dx = \frac{1}{2}\int_{I\!\!R}\left[\frac{\operatorname{sh}(x-s)\operatorname{th} x}{\operatorname{ch}^2(x-s)} + \frac{1}{\operatorname{ch}^2(x-s)\operatorname{ch} x}\right]\frac{1}{\operatorname{ch} x}dx,$$

and

$$\tilde{K}(s) = \frac{1}{2}\int_{x \in I\!\!R} \partial_{ss}\rho(s)(x)\frac{1}{\operatorname{ch} x}dx$$

$$= \int_{I\!\!R}\left[-\frac{\operatorname{th} x}{\operatorname{ch}(x-s)} + 2\frac{\operatorname{sh}^2(x-s)\operatorname{th} x}{\operatorname{ch}^3(x-s)} + 2\frac{\operatorname{sh}(x-s)}{\operatorname{ch}^3(x-s)\operatorname{ch} x}\right]\frac{1}{\operatorname{ch} x}dx.$$

We remark that $\tilde{g}$ and $\tilde{K}$ are in $\mathcal{C}^\infty(I\!\!R; I\!\!R)$ and that $\tilde{g}(0) = 1$ and $\tilde{K}(0) = 0$.

Then we write $\frac{1}{\tilde{g}(s)} = 1 + \gamma(s)$ where $\gamma(s) = \mathcal{O}(|s|)$ in a neighbourhood of zero. So we obtain that

$$\partial_t\sigma = \Delta_Y\sigma + \Delta_Y\varphi - \varphi + T_1(\sigma, \varphi, W), \tag{4.12}$$

where:

$$T_1(\sigma, \varphi, W) = \gamma(\sigma)(\Delta_Y\varphi - \varphi) + \frac{\tilde{K}(\sigma)}{\tilde{g}(\sigma)}|\nabla_Y\sigma|^2 + \frac{1}{\tilde{g}(\sigma)}l^1(G).$$

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Now we multiply (4.10) by $\begin{pmatrix} 0 \\ \frac{1}{2\mathrm{ch}\,x} \end{pmatrix}$ and we integrate in the x variable. We get:

$$\partial_t \varphi = -\Delta\sigma + \Delta\varphi - \varphi + T_2(\sigma, \varphi, W), \qquad (4.13)$$

where

$$T_2(\sigma, \varphi, W) = (1 - \tilde{g}(\sigma))\Delta_Y\sigma + \tilde{K}(\sigma)|\nabla_Y\sigma|^2 + l^2(G).$$

Multiplying (4.12) by $\partial_s R(\sigma)$, (4.13) by $\begin{pmatrix} 0 \\ \frac{1}{\mathrm{ch}\,x} \end{pmatrix}$ and subtracting from (4.10) yield:

$$\partial_t W = \Lambda W + T_3(x, \sigma, \varphi, W), \qquad (4.14)$$

where

$$T_3(x, \sigma, \varphi, W) = G + \begin{pmatrix} -|\nabla_Y\sigma|^2\partial_{ss}\rho(\sigma) + (\Delta_Y\varphi - \varphi)\left(\frac{1}{\mathrm{ch}\,x} - \partial_s\rho(\sigma)\right) - \rho(\sigma)T_1(\sigma, \varphi, W) \\ |\nabla_Y\sigma|^2\partial_{ss}\rho(\sigma) + \Delta_Y\sigma\left(\frac{1}{\mathrm{ch}\,x} - \partial_s\rho(\sigma)\right) - \frac{1}{\mathrm{ch}\,x}T_2(\sigma\varphi, W) \end{pmatrix}.$$

We have proved the following proposition:

Proposition 6 *Let $r \in \mathcal{C}^1(0, T; H^2(I\!\!R^3; I\!\!R^2))$ such that for all $t \geq 0$, $\|r(t, .)\|_{H^2(I\!\!R^3)} \leq \delta_0$. Let $(\sigma, \varphi, W) \in \mathcal{C}^1(0, T[; H^2(I\!\!R^2) \times H^2(I\!\!R^2) \times \mathcal{E}^2)$ given by proposition (5). Then r satisfies (2.6) if and only if (σ, φ, W) satisfies the system (4.12)-(4.13)-(4.14), and 0 is stable for (2.6) is and only if $(0, 0, 0)$ is stable for (4.12)-(4.13)-(4.14).*

Remark 4 *The key point of this step is that with l^1 and l^2, we can separate the variables σ, φ and W in order to obtain the system (4.12)-(4.13)-(4.14) in which the linear parts are almost independant. When we deal with the complete model for the demagnetizing field or with the travelling waves solutions when a magnetic field is applied, this splitting is not possible and the variational estimates cannot be successfull.*

5 Non linear Terms Estimates

On $\Sigma = H^2(I\!\!R^2) \times H^2(I\!\!R^2) \times \mathcal{E}^2$ we define the followings quantities for $k \in \{1, 2, 3\}$:

$$\|(\sigma, \varphi, W)\|_{\mathcal{H}^k} = \|\sigma\|_{H^k(I\!\!R^2)} + \|\varphi\|_{H^k(I\!\!R^2)} + \|L^{\frac{k}{2}}W_1\|_{L^2(I\!\!R^3)} + \|L^{\frac{k}{2}}W_2\|_{L^2(I\!\!R^3)}.$$

We recall that from Proposition 4, on $\mathcal{E}$, $\|L^{\frac{k}{2}}u\|_{L^2(I\!\!R^3)}$ is equivalent to $\|u\|_{H^k(I\!\!R^3)}$. In addition from Proposition 5, there exists a constant K such that if

$$r(x, y, z) = R(\sigma(y, z))(x) + \varphi(y, z)\begin{pmatrix} 0 \\ \frac{1}{\mathrm{ch}\,x} \end{pmatrix} + W(x, y, z)$$

in a neighbourhood of 0, then

$$\frac{1}{K}\|(\sigma, \varphi, W)\|_{\mathcal{H}^k} \leq \|r\|_{H^k(I\!\!R^3)} \leq K\|(\sigma, \varphi, W)\|_{\mathcal{H}^k}.$$

We introduce $\gamma_1 > 0$ such that if $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \gamma_1$, then $\|r\|_{L^\infty} \leq \delta_0$, so that we are in the framework of Proposition 6.

We first recall Gagliardo-Nirenberg type inequalities in 2-d.

Lemma 1 *There exists a constant K such that for all $u \in H^2(\mathbb{R}^2)$,*

$$\|u\|_{L^4(\mathbb{R}^2)} \le K\|u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\nabla_Y u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}},$$

$$\|\nabla_Y u\|_{L^{2p}(\mathbb{R}^2)}^2 \le K\|u\|_{L^\infty(\mathbb{R}^2)}\|\Delta_Y u\|_{L^p(\mathbb{R}^2)} \ \textit{for } p = 1, 2, 4.$$

Proof: in the 2-dimensional case, from Sobolev imbeddings, $W^{1,1}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$ and there exists K such that

$$\|v\|_{L^2(\mathbb{R}^2)} \le K\|\nabla_Y v\|_{L^1(\mathbb{R}^2)}.$$

We apply the previous inequality to u^2 to obtain the first desired estimate.
Concerning the second estimate, we have, for $i \in \{2, 3\}$ and for $p = 1, 2, 4$:

$$\begin{aligned}
\int_{\mathbb{R}^2} (\partial_i u)^{2p} &= \int_{\mathbb{R}^2} \partial_i u (\partial_i u)^{2p-1} \\
&= -(2p-1) \int_{\mathbb{R}^2} u \partial_{ii} u (\partial_i u)^{2p-2} \\
&\le K\|u\|_{L^\infty(\mathbb{R}^2)} \|\partial_{ii} u\|_{L^p(\mathbb{R}^2)} \|\partial_i u\|_{L^{2p}(\mathbb{R}^2)}^{2p-2},
\end{aligned}$$

which concludes the proof of Lemma 1

In the following proposition, we estimate the non linear part G of (4.10) defined in (4.11).

Proposition 7 *There exists K such that for all $(\sigma, \varphi, W) \in \Sigma$, if $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \le \gamma_1$, then*

$$\|G\|_{L^2(\mathbb{R}^3)} + \|\nabla G\|_{L^2(\mathbb{R}^3)} \le K\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \left(\|\Delta_Y\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right).$$

First we prove preliminary estimates.

Lemma 2 *There exists K such that for all $(\sigma, \varphi, W) \in \Sigma$, if $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \le \gamma_1$, then*

$$\|R(\sigma)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla \partial_x R(\sigma)\|_{L^4(\mathbb{R}^3)} \le K\|(\sigma, \varphi, W)\|_{\mathcal{H}^2},$$

and

$$\begin{aligned}
\|\Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} &+ \|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla \Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} \\
&\le K \left(\|\Delta_Y\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right).
\end{aligned}$$

Proof. We recall that there exists K such that for s in the neighbourhood of 0, we have

- $|R(s)(x) + |\partial_x R(s)(x)| + |\partial_{xx} R(s)(x)| \le K\dfrac{|s|}{\operatorname{ch} x}$,

- $|\partial_s R(s)(x)| + |\partial_x \partial_s R(s)| \le \dfrac{K}{\operatorname{ch} x}$,

- $|\partial_{ss} R(s)(x)| + |\partial_x \partial_{ss} R(s)(x)| \le \dfrac{K}{\operatorname{ch} x}$,

- $|\partial_{sss} R(s)(x)| \le \dfrac{K}{\operatorname{ch} x}$.

On one hand, the first waited estimate is a straightforward consequence of the previous remarks and the Sobolev injections of $H^2(\mathbb{R}^2)$ into $L^\infty(\mathbb{R}^2)$ and $W^{1,4}(\mathbb{R}^2)$.

On the other hand,

$$\Delta_Y(R(\sigma)) = \partial_s R(\sigma)\Delta_Y \sigma + \partial_{ss} R(\sigma)|\nabla_Y \sigma|^2,$$

so

$$|\Delta_Y(R(\sigma))| \le K(|\Delta_Y \sigma| + |\nabla_Y \sigma|^2)\frac{1}{\operatorname{ch} x}.$$

With Lemma 1,
$$\|\Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} \le K \|\Delta_Y \sigma\|_{L^2(\mathbb{R}^2)}.$$

In addition,
$$\|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} \le K \|\Delta_Y \sigma\|_{L^4(\mathbb{R}^2)} + K \|\nabla_Y \sigma\|^2_{L^8(\mathbb{R}^2)}$$
$$\le K \|\Delta_Y \sigma\|_{L^4(\mathbb{R}^2)} \text{ by the last estimate in Lemma 1,}$$
$$\le K \|\nabla_Y \sigma\|_{H^1(\mathbb{R}^2)} \text{ by Sobolev injection.}$$

To conclude, we have
$$\partial_x \Delta_Y R(\sigma) = \partial_x \partial_s R(\sigma) \Delta_Y \sigma + \partial_x \partial_{ss} R(\sigma) |\nabla_Y \sigma|^2,$$

so the estimate on $\partial_x \Delta_Y R(\sigma)$ is straightforward.
Concerning the derivatives in y and z, we have
$$\nabla_Y \Delta_Y R(\sigma) = \partial_{ss} R(\sigma)(\nabla_Y \sigma) \Delta_Y \sigma + \partial_s R(\sigma) \nabla_Y \Delta_Y \sigma + \partial_{sss} R(\sigma)(\nabla_Y \sigma)|\nabla_Y \sigma|^2$$
$$+ 2\partial_{ss} R(\sigma) \nabla^2_Y \sigma \cdot \nabla_Y \sigma,$$

so
$$\|\nabla_Y \Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} \le K\|\nabla_Y\sigma\|_{L^4(\mathbb{R}^2)}\|\Delta_Y\sigma\|_{L^4(\mathbb{R}^2)} + K\|\nabla_Y\Delta_Y\sigma\|_{L^2(\mathbb{R}^2)} + K\|\nabla_Y\sigma\|^3_{L^6(\mathbb{R}^2)}$$
$$+ K\|\nabla^2_Y\sigma\|_{L^4(\mathbb{R}^2)}\|\nabla_Y\sigma\|_{L^4(\mathbb{R}^2)}$$
$$\le K\left(\|\nabla^2_Y\sigma\|_{L^2(\mathbb{R}^2)} + \|\nabla^3_Y\sigma\|_{L^2(\mathbb{R}^2)}\right)$$
$$\le \|\nabla_Y\sigma\|_{H^1(\mathbb{R}^2)}.$$

So we conclude the proof of Lemma 2.

We recall that we denote by w the quantity
$$w(t,x,y,z) = \varphi(t,x,y,z)\begin{pmatrix} 0 \\ 1 \\ \frac{1}{\operatorname{ch} x} \end{pmatrix} + W(t,x,y,z).$$

Lemma 3 *There exists a constant K such that*
$$\|w\|_{L^\infty(\mathbb{R}^3)} + \|w\|_{H^2(\mathbb{R}^3)} + \|\nabla w\|_{L^4(\mathbb{R}^3)} \le K\|(\sigma, \varphi, W)\|_{\mathcal{H}^2},$$

and
$$\|w\|_{H^2(\mathbb{R}^3)} + \|\Delta w\|_{L^4(\mathbb{R}^3)} + \|\nabla \Delta w\|_{L^2(\mathbb{R}^3)} \le K\left(\|\Delta_Y\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right).$$

Proof. This lemma is a direct consequence of the Sobolev inequalities.

Proof of Proposition 7. We estimate each term of G separately (see (4.11).)

- We recall that
$$G_1 = A(R(\sigma))\Delta_Y R(\sigma) + \tilde{A}(R(\sigma),w)(w)(\partial_{xx}R(\sigma)) + \tilde{A}(R(\sigma,w)(w)\Delta_Y R(\sigma) + A(R(\sigma)+w)\Delta w.$$

In addition from proposition 2, there exists K such that for $|\xi| \le \frac{1}{2}$,
$$|A(\xi)| \le K|\xi|, \quad |A'(\xi)| \le K,$$

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$$\tilde{A}(u,v) \le K(|u| + |v|) \text{ and } |\partial_u \tilde{A}(u,v)| + |\partial_v \tilde{A}(u,v)| \le K.$$

Therefore

$$|G_1| \le K|R(\sigma)||\Delta_Y R(\sigma)| + K|w||\partial_{xx} R(\sigma)| + K|w||\Delta_Y R(\sigma)| + (|R(\sigma)| + |w|)|\Delta w|,$$

so that

$$
\begin{aligned}
\|G_1\|_{L^2(\mathbb{R}^3)} \le \ & K \left(\|R(\sigma)\|_{L^\infty(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)} \right) \left(\|\Delta_Y R(\sigma)\| + \|\Delta w\|_{L^2(\mathbb{R}^3)} \right) \\
& + K\|\partial_{xx} R(\sigma)\|_{L^\infty(\mathbb{R}^3)} \|w\|_{L^2(\mathbb{R}^3)} \\
\le \ & K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \left(\|\Delta_Y\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)} \right),
\end{aligned}
$$

from Lemma 2 and Lemma 3.

Concerning the gradient we have

$$
\begin{aligned}
|\nabla G_1| \le \ & K|\nabla R(\sigma)||\Delta_Y R(\sigma)| + K|R(\sigma)||\nabla \Delta_Y R(\sigma)| + K\left(|\nabla R(\sigma)| + |\nabla w|\right)|w||\partial_{xx} R(\sigma)| \\
& + K|\nabla w||\partial_{xx} R(\sigma)| + K\left(|\nabla R(\sigma)| + |\nabla w|\right)|w||\Delta_Y R(\sigma)| \\
& + K|\nabla w||\Delta_Y R(\sigma)| + (|\nabla R(\sigma)| + |\nabla w|)|\Delta w| + K\left(|R(\sigma)| + |w|\right)|\nabla \Delta w|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\nabla G_1\|_{L^2(\mathbb{R}^3)} \le \ & K\|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)}\|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} + K\|R(\sigma)\|_{L^\infty(\mathbb{R}^3)}\|\nabla \Delta_Y R(\sigma)\|_{L^2(\mathbb{R}^3)} \\
& + K\left(\|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla w\|_{L^4(\mathbb{R}^3)}\right)\|w\|_{L^4(\mathbb{R}^3)}\|\partial_{xx} R(\sigma)\|_{L^\infty(\mathbb{R}^3)} \\
& + K\|\nabla w\|_{L^2(\mathbb{R}^3)}\|\partial_{xx} R(\sigma)\|_{L^\infty(\mathbb{R}^3)} \\
& + K\left(\|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla w\|_{L^4(\mathbb{R}^3)}\right)\|w\|_{L^\infty(\mathbb{R}^3)}\|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} \\
& + K\|\nabla w\|_{L^4(\mathbb{R}^3)}\|\Delta_Y R(\sigma)\|_{L^4(\mathbb{R}^3)} \\
& + \left(\|\nabla R(\sigma)\|_{L^4(\mathbb{R}^3)} + \|\nabla w\|_{L^4(\mathbb{R}^3)}\right)\|\Delta w\|_{L^4(\mathbb{R}^3)} \\
& + K\left(\|R(\sigma)\|_{L^\infty(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)}\right)\|\nabla \Delta w\|_{L^2(\mathbb{R}^3)} \\
\le \ & K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \left(\|\Delta_Y\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)} \right),
\end{aligned}
$$

using lemmas 2 and 3.

- We have

$$
\begin{aligned}
G_2 = \ & 2B(R(\sigma))(\partial_x R(\sigma), \partial_x w) + B(R(\sigma))(\partial_x w, \partial_x w) + \tilde{B}(R(\sigma,w)(w)(\partial_x R(\sigma), \partial_x R(\sigma)) \\
& + 2\tilde{B}(R(\sigma,w)(w)(\partial_x R(\sigma), \partial_x w) + \tilde{B}(R(\sigma,w)(w)(\partial_x w, \partial_x w).
\end{aligned}
$$

In addition, we recall that from Proposition 2, there exists K such that for $|\xi| \le \frac{1}{2}$ one has

$$|B(\xi)| \le K|\xi|, \ |B'(\xi)| \le K,$$

and for $|u| \le 1/2$ and $|v| \le 1/2$,

$$|\tilde{B}(u,v)| + |\partial_u \tilde{B}(u,v)| + |\partial_v \tilde{B}(u,v)| \le K.$$

A straightforward calculation, lemma 2 and lemma 3 yield the desired estimates on G_2 and ∇G_2.

- The term G_3 is given by

$$G_3 = B(r)(\partial_s R(\sigma), \partial_s R(\sigma))|\nabla\sigma|^2 + 2\sum_{i=2}^{3} B(r)(\partial_s R(\sigma), \partial_i w)\partial_i\sigma + \sum_{i=2}^{3} B(r)(\partial_i w, \partial_i w).$$

Using that $|B(\xi)| \leq K|\xi|$ and that $|B'(\xi)| \leq K$ for $\xi \in B(0, 1/2)$, since $\|\nabla_Y\sigma\|_{L^4(\mathbb{R}^2)} \leq K\|\sigma\|_{L^\infty(\mathbb{R}^2)}\|\Delta_Y\sigma\|_{L^2(\mathbb{R}^2)}$ we obtain the waited estimate on G_3.

- To estimate G_4, we remark that

$$G_4 = C(x, R(\sigma))(\partial_x w) + \tilde{C}(x, R(\sigma), w)(w)(\partial_x R(\sigma)) + \tilde{C}(x, R(\sigma), w)(w)(\partial_x w),$$

and we recall that for $|\xi| \leq 1/2$,

$$|C(x, \xi)| + |\partial_x C(x, \xi)| \leq \frac{K}{\operatorname{ch} x}|\xi|,$$

and

$$|\partial_\xi C(x, r)| + |\partial_x \partial_\xi C(x, \xi)| + |\partial_{\xi\xi} C(x, \xi)| \leq \frac{K}{\operatorname{ch} x},$$

so that

$$|\tilde{C}(x, u, v)| + |\partial_u \tilde{C}(x, u, v)| + |\partial_v \tilde{C}(x, u, v)| \leq \frac{K}{\operatorname{ch} x}.$$

The waited estimate of G_4 is then a straightforward consequence of these remarks.

- The last term G_5 is estimated with the same kind of arguments, using that

$$|\tilde{D}(x, u, v)| + |\partial_x \tilde{D}(x, u, v)| \leq K(|u| + |v|),$$

and that

$$|\partial_u \tilde{D}(x, u, v)| + |\partial_v \tilde{D}(x, u, v)| \leq K$$

for u and v in $B(0, 1/2)$.

With all these estimates, we conclude the proof of Proposition 7.

As a corollary of Proposition 7, we obtain the following estimates of the T_i's:

Proposition 8 *There exists K such that for all $(\sigma, \varphi, W) \in \Sigma$, if $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \gamma_1$, then*

$$\|T_1\|_{H^1(\mathbb{R}^2)} + \|T_2\|_{H^1(\mathbb{R}^2)} + \|T_3\|_{H^1(\mathbb{R}^3)}$$

$$\leq K\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \left(\|\Delta_Y\sigma\|_{H^1(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right).$$

Proof. We remark that for $s \in \mathbb{N}$, there exists C such that if $u \in H^s(\mathbb{R}^3; \mathbb{R}^3)$, then $l^i(u) \in H^s(\mathbb{R}^2; \mathbb{R})$ and

$$\|l^i(u)\|_{H^s(\mathbb{R}^2)} \leq C\|u\|_{H^s(\mathbb{R}^3)}.$$

This estimate toghether with Proposition 7 yield the desired estimates on T_1 and T_2. By difference we obtain the waited result on T_3.

For the L^2 estimate in the last Section, we need a more precise control on $T_1 - T_2$. It is explained in the following proposition.

Proposition 9 *We can split $T_1 - T_2$ on the form : $T_1 - T_2 = \tilde{T}_a + \tilde{T}_b$, where $\tilde{T}_a$ and $\tilde{T}_b$ satisfy the following estimates. There exists K such that for all $(\sigma, \varphi, W) \in \Sigma$, if $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \gamma_1$, then*

$$\|\tilde{T}_a\|_{L^1(\mathbb{R}^2)} \leq K \left(\|\nabla_Y\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)^2$$

and

$$\|\tilde{T}_b\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \leq K \left(\|\nabla_Y\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)\|\sigma\|_{L^4(\mathbb{R}^2)}.$$

Proof. The spirit of the proof is the following: each term of $T_1 - T_2$ is at least quadratic. Either it contains a product of two "dissipated" components (that is $\nabla_Y \sigma$, $\Delta_Y \sigma$, or φ, W and their derivatives), and we put this term in $\tilde{T}_a$, or it contains σ multiplicated by a dissipated component, and we put it in $\tilde{T}_b$ (the terms quadratic in σ are removed by using (4.9) in Section 4.) Let us precise this splitting.

We recall that

$$T_1(\sigma, \varphi, W) = \gamma(\sigma)(\Delta_Y \varphi - \varphi) + \frac{\tilde{K}(\sigma)}{\tilde{g}(\sigma)}|\nabla_Y \sigma|^2 + \frac{1}{\tilde{g}(\sigma)}l^1(G),$$

$$T_2(\sigma, \varphi, W) = (1 - \tilde{g}(\sigma))\Delta_Y \sigma + \tilde{K}(\sigma)|\nabla_Y \sigma|^2 + l^2(G),$$

where $\gamma(s) = \mathcal{O}(s)$, $\tilde{g}(s) = 1 + \mathcal{O}(s)$ and $\tilde{K}(s) = \mathcal{O}(s)$.

We denote by

$$\tilde{T}_a^1 = (\frac{\tilde{K}(\sigma)}{\tilde{g}(\sigma)} - \tilde{K}(\sigma))|\nabla_Y \sigma|^2,$$

$$\tilde{T}_b^1 = \gamma(\sigma)(\Delta_Y \varphi - \varphi) - (1 - \tilde{g}(\sigma))\Delta_Y \sigma.$$

On one hand we have

$$\|\tilde{T}_a^1\|_{L^1(\mathbb{R}^2)} \leq K\|\sigma\|_{L^\infty}\|\nabla_Y \sigma\|_{L^2(\mathbb{R}^2)}^2 \leq K \left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)^2.$$

On the other hand,

$$|\tilde{T}_b^1| \leq K|\sigma||\Delta_Y \varphi - \varphi| + K|\sigma||\Delta_Y \sigma|,$$

so,

$$\|\tilde{T}_b^1\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \leq K\|\sigma\|_{L^4(\mathbb{R}^2)} \left(\|\Delta_Y \varphi - \varphi\|_{L^2(\mathbb{R}^2)} + \|\Delta_Y \sigma\|_{L^2(\mathbb{R}^2)}\right)$$

$$\leq K \left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right) \|\sigma\|_{L^4(\mathbb{R}^2)}.$$

Concerning the other two terms, we will split G on the form $G = G_a + G_b$ with the corresponding estimates on G_a and G_b. Let us describe this splitting for each term G_i defining G (see (4.11).)

- Concerning G_1, we recall that

$$\Delta_Y R(\sigma) = \partial_s R(\sigma)\Delta_Y \sigma + \partial_{ss} R(\sigma|\nabla_Y \sigma|^2,$$

and that

$$A(r) = A(R(\sigma + w) = A(R(\sigma)) + \tilde{A}(R(\sigma, w)(w),$$

with

$$\tilde{A}(u, v) = \int_0^1 A'(u + sv)ds.$$

Then we set $G_1 = G_1^a + G_1^b$ with

$$G_1^a = A(R(\sigma))(\partial_{ss} R(\sigma)|\nabla_Y \sigma|^2) + \tilde{A}(R(\sigma), w)(w)(\partial_s R(\sigma)\Delta_Y \sigma)$$

$$+ \tilde{A}(R(\sigma), w)(w)(\partial_{ss} R(\sigma)|\nabla_Y \sigma|^2) + 2\tilde{A}(R(\sigma), w)(w)(\Delta w),$$

$$G_1^b = A(R(\sigma))(\partial_s R(\sigma)\Delta_Y \sigma) + A(R(\sigma))(\Delta w).$$

If (σ, φ, W) is bounded as it is assumed, we have:

$$|G_1^a| \leq \frac{K}{\text{ch } x}|\nabla_Y \sigma|^2 + \frac{K}{\text{ch } x}|w||\Delta_Y \sigma| + K|w||\Delta w|,$$

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so,
$$\|G_1^a\|_{L^1(\mathbb{R}^3)} \leq K \left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)} \right)^2.$$

On the other hand,
$$|G_1^g| \leq \frac{K}{\operatorname{ch} x} |\sigma|(|\Delta_Y \sigma| + |\Delta w|),$$

so
$$\|G_1^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq K \left(\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)} \right) \|\sigma\|_{L^4(\mathbb{R}^2)}.$$

- The splitting for G_2 is the following: $G_2 = G_2^a + G_2^b$ where

$$G_2^a = \; B(R(\sigma))(\partial_x w, \partial_x w) + 2\tilde{B}(R(\sigma), w)(w)(\partial_x R(\sigma), \partial_x w) + \tilde{B}(R(\sigma), w)(w)(\partial_x w, \partial_x w)$$

$$G_2^b = \; 2B(R(\sigma))(\partial_x R(\sigma), \partial_x w) + \tilde{B}(R(\sigma), w)(w)(\partial_x R(\sigma), \partial_x R(\sigma)).$$

Since $|\partial_x R(\sigma)| \leq \dfrac{K}{\operatorname{ch} x} |\sigma|$, we have

$$|G_2^b| \leq \frac{K}{\operatorname{ch} x} |\sigma|(|\partial_x w| + |w|),$$

so
$$\|G_2^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq K\|w\|_{H^1(\mathbb{R}^3)}\|\sigma\|_{L^4(\mathbb{R}^2)}.$$

In addition,
$$|G_2^a| \leq K|\partial_x w|^2 + K|w||\partial_x w|,$$

so
$$\|G_2^a\|_{L^1(\mathbb{R}^3)} \leq K\|w\|_{H^1(\mathbb{R}^3)}^2.$$

- Since $\partial_i r = \partial_s R(\sigma)\partial_i \sigma + \partial_i w$ for $i = 2$ or $i = 3$, we set $G_3^a = G_3$ and $G_3^b = 0$ and we have

$$\|G_3^a\|_{L^1(\mathbb{R}^3)} \leq K(\|\nabla_Y \sigma\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla w\|_{L^2(\mathbb{R}^3)}^2).$$

- We define the decomposition of G_4 setting

$$G_4^a = \; \tilde{C}(x, R(\sigma), w)(w)(\partial_x w),$$

$$G_4^b = \; C(x, R(\sigma))(\partial_x w) + \tilde{C}(x, R(\sigma), w)(w)(\partial_x R(\sigma)).$$

Since $|C(x, R(\sigma))| \leq \dfrac{K}{\operatorname{ch} x} |\sigma|$, we have

$$|G_4^b| \leq \frac{K}{\operatorname{ch} x} |\sigma|(|\partial_x w| + |w|),$$

so
$$\|G_4^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq K\|w\|_{H^1(\mathbb{R}^3)}\|\sigma\|_{L^4(\mathbb{R}^2)}.$$

In addition,
$$\|G_4^a\|_{L^1(\mathbb{R}^3)} \leq K\|w\|_{H^1(\mathbb{R}^3)}^2.$$

- Lastly, for G_5, from the Taylor expansion, we have

$$\tilde{D}(x, R(\sigma), w)(w) = \partial_\xi D(x, R(\sigma))(w) + \tilde{\tilde{D}}(x, R(\sigma), w)(w, w),$$

where
$$\tilde{\tilde{D}}(x, u, v) = \frac{1}{2} \int_0^1 (1 - s)\partial_{\xi\xi} D(x, u + sv)ds.$$

We set
$$G_5^a = \tilde{\tilde{D}}(x, R(\sigma), w)(w, w) \text{ and } G_5^b = \partial_\xi D(x, R(\sigma))(w),$$

and we have
$$|G_5^b| \le \frac{K}{\operatorname{ch} x}|\sigma||w| \text{ so } \|G_5^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \le K\|w\|_{L^2(\mathbb{R}^3)}\|\sigma\|_{L^4(\mathbb{R}^2)}.$$

In addition,
$$\|G_5^a\|_{L^1(\mathbb{R}^3)} \le K\|w\|_{L^2(\mathbb{R}^3)}^2.$$

Denoting $G^a = \sum_i G_i^a$ and $G^b = \sum_i G_i^b$, we have obtained that $G = G^a + G^b$ with
$$\|G^a\|_{L^1(\mathbb{R}^3)} \le K\left(\|\nabla_Y\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)^2, \tag{5.15}$$

and
$$\|G^b\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \le K\left(\|\nabla_Y\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)\|\sigma\|_{L^4(\mathbb{R}^2)}. \tag{5.16}$$

We set
$$\tilde{T}_a^2 = \frac{1}{\tilde{g}(\sigma)}l^1(G^a) - l^2(G^a) \text{ and } \tilde{T}_b^2 = \frac{1}{\tilde{g}(\sigma)}l^1(G^b) - l^2(G^b).$$

By properties of the operators l^1 and l^2, (5.15) and (5.16) yield
$$\|\tilde{T}_a^2\|_{L^1(\mathbb{R}^2)} \le K\left(\|\nabla_Y\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)^2,$$

and
$$\|\tilde{T}_b^2\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \le K\left(\|\nabla_Y\sigma\|_{H^2(\mathbb{R}^2)} + \|\varphi\|_{H^3(\mathbb{R}^2)} + \|W\|_{H^3(\mathbb{R}^3)}\right)\|\sigma\|_{L^4(\mathbb{R}^2)}.$$

Defining $\tilde{T}_a$ and $\tilde{T}_b$ respectively by
$$\tilde{T}_a = \tilde{T}_a^1 + \tilde{T}_a^2 \text{ and } \tilde{T}_b = \tilde{T}_b^1 + \tilde{T}_b^2,$$

we have obtained the desired decomposition. This concludes the proof of Proposition 9.

6 Variational Estimates

We recall that we deal with the following system:
$$\partial_t\sigma = \Delta_Y\sigma + \Delta_Y\varphi - \varphi + T_1(\sigma, \varphi, W), \tag{6.17}$$

$$\partial_t\varphi = -\Delta\sigma + \Delta\varphi - \varphi + T_2(\sigma, \varphi, W), \tag{6.18}$$

$$\partial_t W = \begin{pmatrix} -LW_1 - (L+1)W_2 \\ LW_1 - (L+1)W_2 \end{pmatrix} + T_3(x, \sigma, \varphi, W). \tag{6.19}$$

6.1 H^1 and H^2 estimates

Taking the inner product of (6.17) with $-\Delta_Y\sigma$, we obtain
$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla_Y\sigma\|_{L^2(\mathbb{R}^2)}^2\right) + \|\Delta_Y\sigma\|_{L^2(\mathbb{R}^2)}^2 = -\int_{\mathbb{R}^2}(\Delta_Y\varphi - \varphi)\Delta_Y\sigma - \int_{\mathbb{R}^2}T_1(\sigma, \varphi, W)\Delta_Y\sigma.$$

Taking the inner product of (6.18) with $-\Delta_Y\varphi + \varphi$ we get:
$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla_Y\varphi\|_{L^2(\mathbb{R}^2)}^2 + \|\varphi\|_{L^2(\mathbb{R}^2)}^2\right) + \|\Delta_Y\varphi - \varphi\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2}(\Delta_Y\varphi - \varphi)\Delta_Y\sigma$$

$$-\int_{\mathbb{R}^2}T_2(\sigma, \varphi, W)(\Delta_Y\varphi - \varphi).$$

Adding the previous equations, we obtain:

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla_Y\sigma\|^2_{L^2(\mathbb{R}^2)} + \|\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\nabla_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right) + \left[\|\Delta_Y\sigma\|^2_{L^2(\mathbb{R}^2)} + \|\varphi\|^2_{L^2(\mathbb{R}^2)} + 2\|\nabla_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right.$$

$$\left.+\|\Delta_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right] = -\int_{\mathbb{R}^2} T_1(\sigma,\varphi,W)\Delta_Y\sigma - \int_{\mathbb{R}^2} T_2(\sigma,\varphi,W)(\Delta_Y\varphi - \varphi).$$
(6.20)

Taking the inner product of (6.17) with $\Delta_Y^2\sigma$ and the product of (6.18) with $\Delta_Y(\Delta_Y\varphi - \varphi)$ yield:

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta_Y\sigma\|^2_{L^2(\mathbb{R}^2)} + \|\nabla_Y\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\Delta_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right) + \left[\|\nabla_Y\Delta_Y\sigma\|^2_{L^2(\mathbb{R}^2)} + \|\nabla_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right.$$

$$\left.+2\|\Delta_Y\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\nabla_Y\Delta_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right] = -\int_{\mathbb{R}^2} \nabla_Y(T_1(\sigma,\varphi,W))\cdot\nabla_Y\Delta_Y\sigma$$

$$-\int_{\mathbb{R}^2} \nabla_Y(T_2(\sigma,\varphi,W))\cdot\nabla_Y(\Delta_Y\varphi - \varphi).$$
(6.21)

The estimates in Proposition 8 toghether with (6.20) and (6.21) yield that while $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \le \gamma_1$, then

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla_Y\sigma\|^2_{L^2(\mathbb{R}^2)} + \|\Delta_Y\sigma\|^2_{L^2(\mathbb{R}^2)} + \|\varphi\|^2_{L^2(\mathbb{R}^2)} + 2\|\nabla_Y\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\Delta_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right)$$

$$+\left[\|\Delta_Y\sigma\|^2_{L^2(\mathbb{R}^2)} + \|\nabla_Y\Delta_Y\sigma\|^2_{L^2(\mathbb{R}^2)} + \|\varphi\|^2_{L^2(\mathbb{R}^2)} + 3\|\nabla_Y\varphi\|^2_{L^2(\mathbb{R}^2)} + 3\|\Delta_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right.$$ (6.22)

$$\left.+\|\nabla_Y\Delta_Y\varphi\|^2_{L^2(\mathbb{R}^2)}\right] \le K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}\left(\|\Delta_Y\varphi\|^2_{H^1(\mathbb{R}^2)} + \|\varphi\|^2_{H^3(\mathbb{R}^2)} + \|W\|^2_{H^3(\mathbb{R}^3)}\right).$$

Taking the inner product of (6.19) with $\begin{pmatrix} L^2W_1 \\ L(L+1)W_2 \end{pmatrix}$ yields:

$$\frac{1}{2}\frac{d}{dt}\left(\|LW_1\|^2_{L^2(\mathbb{R}^3)} + \|(L+Id)W_2\|^2_{L^2(\mathbb{R}^3)}\right) + \|L^{\frac{3}{2}}W_1\|^2_{L^2(\mathbb{R}^3)} + \|L^{\frac{1}{2}}(L+Id)W_2\|^2_{L^2(\mathbb{R}^3)}$$

$$\le \|L^{\frac{1}{2}}T_3\|_{L^2(\mathbb{R}^3)}\left(\|L^{\frac{3}{2}}W_1\|_{L^2(\mathbb{R}^3)} + \|L^{\frac{1}{2}}(L+Id)W_2\|_{L^2(\mathbb{R}^3)}\right)$$ (6.23)

$$\le K\|(\sigma,\varphi,W)\|_{\mathcal{H}^2}\left(\|\Delta_Y\varphi\|^2_{H^1(\mathbb{R}^2)} + \|\varphi\|^2_{H^3(\mathbb{R}^2)} + \|W\|^2_{H^3(\mathbb{R}^3)}\right).$$

while $\|(\sigma,\varphi,W)\|_{\mathcal{H}^2} \le \gamma_1$ (by Proposition 8).

6.2 L^2-estimates

Subtracting (6.17) to (6.18) yields

$$\partial_t(\sigma - \varphi) = 2\Delta_Y\sigma + T_1(\sigma,\varphi,W) - T_2(\sigma,\varphi,W).$$

Multiplying by $\sigma - \varphi$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\sigma - \varphi\|^2_{L^2(\mathbb{R}^2)} + 2\|\nabla_Y\sigma\|^2_{L^2(\mathbb{R}^2)} = 2\int_{\mathbb{R}^2}\nabla_Y\sigma\nabla_Y\varphi + \int_{\mathbb{R}^2}(T_1 - T_2)\sigma - \int_{\mathbb{R}^2}(T_1 - T_2)\varphi.$$

By Young inequality and with the splitting of $T_1 - T_2$ (see Proposition 9), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma - \varphi\|^2_{L^2(I\!\!R^2)}\right) + 2\|\nabla_Y\sigma\|^2_{L^2(I\!\!R^2)} \leq \|\nabla_Y\sigma\|^2_{L^2(I\!\!R^2)} + \|\nabla_Y\varphi\|^2_{L^2(I\!\!R^2)} + \|\tilde{T}_a\|_{L^1(I\!\!R^2)}\|\sigma\|_{L^\infty(I\!\!R^2)}$$

$$+\|\tilde{T}_b\|_{L^{\frac{4}{3}}(I\!\!R^2)}\|\sigma\|_{L^4(I\!\!R^2)} + (\|T_1\|_{L^2(I\!\!R^2)} + \|T_2\|_{L^2(I\!\!R^2)})\|\varphi\|_{L^2(I\!\!R^2)}.$$

So, applying the estimates given by Proposition 8 and Proposition 9, while $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \gamma_1$, then

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma - \varphi\|^2_{L^2(I\!\!R^2)}\right) + \|\nabla_Y\sigma\|^2_{L^2(I\!\!R^2)} \leq \|\nabla_Y\varphi\|^2_{L^2(I\!\!R^2)}$$

$$+K\|\sigma\|_{L^\infty(I\!\!R^2)}\left[\|\nabla_Y\sigma\|_{H^2(I\!\!R^2)} + \|\varphi\|_{H^3(I\!\!R^2)} + \|W\|_{H^3(I\!\!R^3)}\right]^2$$

$$+K\left[\|\nabla_Y\sigma\|_{H^2(I\!\!R^2)} + \|\varphi\|_{H^3(I\!\!R^2)} + \|W\|_{H^3(I\!\!R^3)}\right]\|\sigma\|^2_{L^4(I\!\!R^2)}$$

$$+K\|(\sigma, \varphi, W)\|_{\mathcal{H}^2}\left[\|\Delta_Y\sigma\|_{L^2(I\!\!R^2)} + \|\varphi\|_{H^2(I\!\!R^2)} + \|W\|_{H^2(I\!\!R^3)}\right]\|\varphi\|_{L^2(I\!\!R^2)}.$$

By Lemma 1,

$$\|\sigma\|^2_{L^4(I\!\!R^2)} \leq K\|\sigma\|_{L^2(I\!\!R^2)}\|\nabla_Y\sigma\|_{L^2(I\!\!R^2)}.$$

So, we obtain that while $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \gamma_1$,

$$\frac{1}{2}\frac{d}{dt}\left(\|\sigma - \varphi\|^2_{L^2(I\!\!R^2)}\right) + \|\nabla_Y\sigma\|^2_{L^2(I\!\!R^2)} \leq \|\nabla_Y\varphi\|^2_{L^2(I\!\!R^2)}$$

$$\tag{6.24}$$

$$+K\|(\sigma, \varphi, W)\|_{\mathcal{H}^2)}\left[\|\nabla_Y\sigma\|^2_{L^2(I\!\!R^2)} + \|\varphi\|^2_{H^3(I\!\!R^2)} + \|W\|^2_{H^3(I\!\!R^3)}\right].$$

6.3 End of the proof

We define $\mathcal{N}$ and $\mathcal{D}$ by

$$\mathcal{N}(t) = \left(\|\sigma - \varphi\|^2_{L^2(I\!\!R^2)} + \|\nabla_Y\sigma\|^2_{L^2(I\!\!R^2)} + \|\Delta_Y\sigma\|^2_{L^2(I\!\!R^2)} + \|\varphi\|^2_{L^2(I\!\!R^2)} + 2\|\nabla_Y\varphi\|^2_{L^2(I\!\!R^2)}\right.$$

$$\left.+\|\Delta_Y\varphi\|^2_{L^2(I\!\!R^2)} + \|LW_1\|^2_{L^2(I\!\!R^3)} + \|(L + Id)W_2\|^2_{L^2(I\!\!R^3)}\right)(t),$$

and

$$\mathcal{D}(t) = \left[\|\nabla_Y\sigma\|^2_{L^2(I\!\!R^2)} + \|\Delta_Y\sigma\|^2_{L^2(I\!\!R^2)} + \|\nabla_Y\Delta_Y\sigma\|^2_{L^2(I\!\!R^2)} + \|\varphi\|^2_{L^2(I\!\!R^2)} + 2\|\nabla_Y\varphi\|^2_{L^2(I\!\!R^2)}\right.$$

$$\left.+3\|\Delta_Y\varphi\|^2_{L^2(I\!\!R^2)} + \|\nabla_Y\Delta_Y\varphi\|^2_{L^2(I\!\!R^2)} + \|L^{\frac{3}{2}}W_1\|^2_{L^2(I\!\!R^3)} + \|L^{\frac{1}{2}}(L + Id)W_2\|^2_{L^2(I\!\!R^3)}\right](t).$$

Adding up (6.22), (6.23) and (6.24), we obtain that

$$\frac{1}{2}\frac{d\mathcal{N}}{dt} + \mathcal{D}(t) \leq K\|(\sigma, \varphi, W)\|_{\mathcal{H}^2}\left[\|\nabla_Y\sigma\|^2_{H^2(I\!\!R^2)} + \|\varphi\|^2_{H^3(I\!\!R^2)} + \|W\|^2_{H^3(I\!\!R^3)}\right]$$

(the term $\|\nabla_Y\varphi\|^2_{L^2(I\!\!R^2)}$ in the right hand side of (6.24) vanishes with a part of the left hand side of (6.22).)

As remarked in Proposition 4, on $\mathcal{E}$, we have the equivalences of norms: $\|L^{\frac{3}{2}}W_1\|_{L^2(I\!\!R^3)} \sim \|W_1\|_{H^3(I\!\!R^3)}$ and $\|L^{\frac{1}{2}}(L + Id)W_2\|_{L^2(I\!\!R^3)} \sim \|W_2\|_{H^3(I\!\!R^3)}$. So there exists a constant C_1 such that

$$\mathcal{D} \geq C_1\left[\|\nabla_Y\sigma\|^2_{H^2(I\!\!R^2)} + \|\varphi\|^2_{H^3(I\!\!R^2)} + \|W\|^2_{H^3(I\!\!R^3)}\right].$$

In addition, $\|\sigma\|_{L^2(\mathbb{R}^2)} \leq \|\sigma - \varphi\|_{L^2(\mathbb{R}^2)} + \|\varphi\|_{L^2(\mathbb{R}^2)}$, so again with Proposition 4, there exists C_2 such that

$$\frac{1}{C_2}\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \mathcal{N}(t) \leq C_2\|(\sigma, \varphi, W)\|_{\mathcal{H}^2}.$$

So while $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2} \leq \gamma_1$, we have

$$\frac{1}{2}\frac{d\mathcal{N}}{dt} + \left[\|\nabla_Y \sigma\|_{H^2(\mathbb{R}^2)}^2 + \|\varphi\|_{H^3(\mathbb{R}^2)}^2 + \|W\|_{H^3(\mathbb{R}^3)}^2\right](C_1 - KC_2\mathcal{N}(t)) \leq 0. \qquad (6.25)$$

Let us introduce $\eta_0 = \min\left\{\dfrac{\gamma_1}{C_2}, \dfrac{C_1}{KC_2}\right\}$. If $\mathcal{N}(0) \leq \eta_0$, then with (6.25), $\mathcal{N}(t)$ remains smaller than $\dfrac{C_1}{KC_2}$, that is $\mathcal{N}(t)$ decreases and remains smaller than η_0, so that $\|(\sigma, \varphi, W)\|_{\mathcal{H}^2}$ remains smaller than γ_1. So we are always in the validity domain of our estimates.

Therefore we have proved the stability of $(0, 0, 0)$ for (4.12)-(4.13)-(4.14). This concludes the proof of Theorem 1 using Proposition 1 and 6.

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