

The symmetric property (τ) for the Gaussian measure

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Abstract

We give a proof, based on the Poincaré inequality, of the symmetric property (τ) for the Gaussian measure. If $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous, bounded from below and even, we define $Hf(x) = \inf_y f(x+y) + \frac{1}{2}|y|^2$ and we have

$$\int e^{-f} d\gamma_d \int e^{Hf} d\gamma_d \leq 1.$$

This property is equivalent to a certain functional form of the Blaschke-Santaló inequality, as explained in a paper by Artstein, Klartag and Milman.

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1 Introduction

The Blaschke-Santaló inequality states that if K is a symmetric convex body of \mathbb{R}^d then

$$|K|_d |K^\circ|_d \leq |D|_d |D^\circ|_d = v_d^2, \quad (1)$$

where $|\cdot|_d$ stands for the volume, K° is the polar body of K , D the Euclidean ball and v_d its volume. It was first proved by Blaschke for dimension 2 and 3 and Santaló [15] extended the result to any dimension. K. Ball in [2] was the first to prove a functional version of this inequality. Since then, several authors found either improvements or other versions of Ball's inequality, for example Lutwak and Zhang [10], Artstein, Klartag and Milman [1], or Fradelizi and Meyer [7].

Ball's inequality implies in particular the following: if F is a non negative even measurable function on \mathbb{R}^d then

$$\int F(x) dx \int F^\circ(x) dx \leq (2\pi)^d, \quad (2)$$

where F° is the polar function of F :

$$F^\circ(x) = \inf_{y \in \mathbb{R}^d} \frac{e^{-x \cdot y}}{F(y)}.$$

Let K be a symmetric convex body and N_K the associated norm. Defining $F_K = e^{-\frac{1}{2}N_K^2}$, it is easy to see that $(F_K)^\circ = F_{K^\circ}$. Besides, a standard computation shows that $v_d \int F_K =$

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$(2\pi)^{d/2}|K|_d$ and similarly for K° . Therefore, applying (2) to F_K , we get (1). So (2) is a functional form of Santaló's inequality.

In [1], Artstein, Klartag and Milman extend this inequality to the non even setting: they prove that (2) holds as soon as the barycenter of F° is at 0 (which is true when F is even). At the end of their paper is an interesting remark on the property (τ) . This property was introduced by Talagrand and named after him by Maurey in [11]. Here is the definition: let μ be a probability measure on \mathbb{R}^d and w a weight (which is a non negative function equal to 0 at 0). We say that (μ, w) satisfies property (τ) if for any non negative continuous function f , the following inequality holds:

$$\int e^{-f} d\mu \int e^{f \square w} d\mu \leq 1, \quad (3)$$

where $f \square w$ is the infimum convolution of f and w :

$$f \square w(x) = \inf_{y \in \mathbb{R}^d} (f(x+y) + w(y)).$$

Clearly, we may replace " $f \geq 0$ " by " f bounded from below". Of course $f \square w(x) \leq f(x)$ (take $y = 0$). In order to find $f \square w(x)$, one is allowed to move from x to some $x+y$ such that $f(x+y)$ is smaller than $f(x)$, but the displacement costs $w(y)$.

Let γ_d be the standard Gaussian measure on \mathbb{R}^d . For $x \in \mathbb{R}^d$, we let $|x|$ be its Euclidean norm and we define the quadratic cost $c: x \mapsto |x|^2$. Maurey remarked that $(\gamma_d, \frac{c}{4})$ satisfies property (τ) . It follows from the famous Prékopa-Leindler inequality: let u, v and w be measurable functions satisfying $w(\frac{x+y}{2}) \leq \frac{1}{2}(u(x) + v(y))$ for all x and y in \mathbb{R}^d , then

$$\left(\int e^{-u} dx \right)^{\frac{1}{2}} \left(\int e^{-v} dx \right)^{\frac{1}{2}} \leq \int e^{-w} dx. \quad (4)$$

This is a reverse form of Hölder's inequality, we refer to [3] for a proof and selected applications. Applying this inequality to $u = f + \frac{c}{2}$, $v = -(f \square \frac{c}{4}) + \frac{c}{2}$ and $w = \frac{c}{2}$ yields property (τ) for the Gaussian measure.

It is pointed out in [1] that there is a strong connection between property (τ) and the functional version of Santaló's inequality. Indeed, applying (2) to $F = e^{-f - \frac{c}{2}}$ we obtain easily

$$\int e^{-f} d\gamma_d \int e^{f \square \frac{c}{2}} d\gamma_d \leq 1.$$

Hence, if we restrict to even functions we have property (τ) for $(\gamma_d, \frac{c}{2})$ (we gain a factor 2 in the cost). This is what we call the *symmetric* property (τ) , and our purpose is to prove it directly. We will show that it is related to the eigenvalues of the Laplacian in Gauss space. This will provide a new proof of the functional Santaló inequality, and this proof avoids using the usual Santaló inequality (for convex sets), which was not the case of Ball's proof.

From now on we denote by Hf the function $f \square \frac{c}{2}$:

$$Hf(x) = \inf_{y \in \mathbb{R}^d} (f(x+y) + \frac{1}{2}|y|^2).$$

We restate the theorem we want to prove:

Theorem 1. *Let f be an even, bounded from below and continuous function on \mathbb{R}^d . Then*

$$\int e^{-f} d\gamma_d \int e^{Hf} d\gamma_d \leq 1.$$

2 A “small ϵ ” inequality

Definition 2. Let $C > 0$ and $\epsilon \in (0, 1]$. Let $\mathcal{F}(C, \epsilon)$ be the class of functions on \mathbb{R}^d satisfying the following properties:

- (i) f is lipschitz with constant $C\epsilon$
- (ii) for any x and h in \mathbb{R}^d :

$$f(x+h) + f(x-h) - 2f(x) \leq C\epsilon^2|h|^2.$$

This class $\mathcal{F}(C, \epsilon)$ is stable under various operations. For example it is clearly a convex set, and, if f is in $\mathcal{F}(C, \epsilon)$ and $u \in \mathbb{R}^d$ then $x \mapsto f(x+u)$ and $x \mapsto f(-x+u)$ are also in $\mathcal{F}(C, \epsilon)$. Besides, if $(f_i)_{i \in I}$ is a family of functions in $\mathcal{F}(C, \epsilon)$ and if for some x_0 , $\inf_{i \in I} f_i(x_0) > -\infty$, then $f = \inf_i f_i$ is still in $\mathcal{F}(C, \epsilon)$. Indeed, it is then clear that f is nowhere equal to $-\infty$ and that it is $C\epsilon$ -lipschitz. Let $x, h \in \mathbb{R}^d$ and let i_0 satisfy $f_{i_0}(x) \leq f(x) + \eta$; we have

$$f(x+h) + f(x-h) - 2f(x) \leq f_{i_0}(x+h) + f_{i_0}(x-h) - 2f_{i_0}(x) + 2\eta.$$

Applying property (ii) for f_{i_0} and letting η go to 0, we get the result.

Let X be a standard Gaussian vector on \mathbb{R}^d , the expectation of a random variable will be denoted by \mathbb{E} .

Lemma 3. Let $C > 0$, for every $\epsilon \in (0, 1]$, for every even function $f \in \mathcal{F}(C, \epsilon)$ we have

$$\mathbb{E} e^{-f(X)} \mathbb{E} e^{Hf(X)} \leq 1 + K\epsilon^3,$$

with a constant K depending on C solely.

The proof of this lemma is based on a Taylor expansion and on a symmetric Poincaré inequality.

Lemma 4. Let f be a smooth function in $L_2(\mathbb{R}^d, \gamma_d)$. Assume that f is orthogonal to constants and to linear functions:

$$\mathbb{E} f(X) = 0 \quad \text{and} \quad \mathbb{E} Xf(X) = 0. \tag{5}$$

Then

$$\mathbb{E} f(X)^2 \leq \frac{1}{2} \mathbb{E} |\nabla f(X)|^2. \tag{6}$$

It is a well-known result, we recall its proof for completeness, the reader can also have a look at [6] where a more general statement is proved.

Proof. We consider the sequence $(H_\alpha)_{\alpha \in \mathbb{N}^d}$ of the Hermite polynomials on \mathbb{R}^d . It is an orthonormal basis of $L_2(\gamma_d)$ in which the (unbounded) operator

$$L: f \mapsto \Delta f - x \cdot \nabla f$$

is diagonal: for all α we have $-LH_\alpha = |\alpha|H_\alpha$, where $|\alpha| = \sum \alpha_i$. We refer to [13, chapter 2] for details. Moreover, we have the following integration by parts formula:

$$-\mathbb{E} Lf(X)g(X) = \mathbb{E} \nabla f(X) \cdot \nabla g(X).$$

Let us write $f = \sum f_\alpha H_\alpha$. We have $-Lf = \sum |\alpha| f_\alpha H_\alpha$ and (5) implies that $f_\alpha = 0$ when $|\alpha| = 0$ or 1. Therefore

$$\mathbb{E} |\nabla f(X)|^2 = -\mathbb{E} Lf(X)f(X) = \sum_{|\alpha| \geq 2} |\alpha| f_\alpha^2 \geq 2\mathbb{E} f(X)^2,$$

which concludes the proof. \square

Proof of Lemma 3. In what follows, K_1, K_2, \dots are constants that depend only on C . Let $\epsilon \in (0, 1]$ and $f \in \mathcal{F}(C, \epsilon)$, even. Replace f by its convolution by some even regular Dirac approximation (ρ_n) (Gaussian for example). Clearly $\rho_n * f$ is even and belongs to $\mathcal{F}(C, \epsilon)$. On the other hand, since f is lipschitz, $\rho_n * f \rightarrow f$ uniformly, which implies $H(\rho_n * f) \rightarrow Hf$. Moreover, being lipschitz, f is bounded by a multiple of $1 + |x|$, and the same holds for Hf . Then $e^{-\rho_n * f}$ and $e^{H(\rho_n * f)}$ are both dominated by some function $x \rightarrow Ae^{B|x|}$. By the dominated convergence theorem, we obtain

$$\mathbb{E} e^{-(\rho_n * f)(X)} \mathbb{E} e^{H(\rho_n * f)(X)} \rightarrow \mathbb{E} e^{-f(X)} \mathbb{E} e^{Hf(X)},$$

when n goes to infinity. Hence it is enough to prove the inequality when f is \mathcal{C}^2 . Moreover, the inequality does not change if we add a constant to f . Hence we can also assume that $\mathbb{E} f(X) = 0$. When f is \mathcal{C}^2 , property (ii) implies that $\text{Hess}f(x) \leq C\epsilon^2 \text{Id}$ for any x . Hence by Taylor's formula

$$(ii') \text{ for any } x \text{ and } h, f(x+h) \leq f(x) + \nabla f(x) \cdot h + \frac{C}{2} \epsilon^2 |h|^2.$$

First we estimate $\mathbb{E} e^{-f(X)}$. We write

$$\mathbb{E} e^{-f(X)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mathbb{E} f(X)^k.$$

To estimate the moments of $f(X)$, we use the concentration property of the Gaussian measure: if ϕ is a real valued, mean 0 and 1-lipschitz function on \mathbb{R}^d , then $\phi(X)$ is closed to 0 with high probability. More precisely, for any $t > 0$, we have

$$\text{Prob} \{ |\phi(X)| \geq t \} \leq 2e^{-t^2/2}.$$

We refer to [9, theorem 2.5] for a proof of this statement. This yields that there exists a universal constant M such that for any $p \geq 1$

$$\left(\mathbb{E} |\phi(X)|^p \right)^{1/p} \leq M \sqrt{p}.$$

Since f is $C\epsilon$ -lipschitz, $\frac{f}{C\epsilon}$ is 1-lipschitz, and it has mean 0 so the preceding inequality applies. Thus, for $k \geq 3$, we can bound $|(-1)^k \mathbb{E} f(X)^k|$ from above by $(MC\sqrt{k})^k \epsilon^3$. We obtain

$$\mathbb{E} e^{-f(X)} \leq 1 + \frac{1}{2} \mathbb{E} f(X)^2 + K_1 \epsilon^3. \quad (7)$$

In the same way, we have

$$\mathbb{E} e^{f(X)} \leq 1 + \frac{1}{2} \mathbb{E} f(X)^2 + K_1 \epsilon^3 \quad \text{and} \quad \mathbb{E} |e^{f(X)} - 1| \leq K_2 \epsilon^2. \quad (8)$$

We now deal with Hf . Trying $y = -\nabla f(x)$ in the definition of Hf , we get

$$Hf(x) \leq f(x - \nabla f(x)) + \frac{1}{2}|\nabla f(x)|^2.$$

Applying (ii') to x and $h = -\nabla f(x)$, we obtain

$$Hf(x) \leq f(x) - |\nabla f(x)|^2 + \frac{1}{2}|\nabla f(x)|^2 + \frac{C}{2}\epsilon^2|\nabla f(x)|^2.$$

Remark that since f is $C\epsilon$ -lipschitz, we have $|\nabla f(x)| \leq C\epsilon$ for any x . We obtain

$$Hf(x) \leq f(x) - \frac{1}{2}|\nabla f(x)|^2 + \frac{1}{2}C^3\epsilon^4.$$

Taking the exponential, and using $|\nabla f| \leq C\epsilon$ again, we get

$$\begin{aligned} e^{Hf(x)} &\leq e^{f(x)} \left(1 - \frac{1}{2}|\nabla f(x)|^2 + K_3\epsilon^4\right) (1 + K_4\epsilon^4) \\ &\leq e^{f(x) - \frac{1}{2}|\nabla f(x)|^2} + \frac{1}{2}(1 - e^{f(x)})|\nabla f(x)|^2 + K_5e^{f(x)}\epsilon^4. \end{aligned}$$

We take the expectation and we use (8), we obtain

$$\mathbb{E} e^{Hf(X)} \leq 1 + \frac{1}{2}\mathbb{E} f(X)^2 - \frac{1}{2}\mathbb{E} |\nabla f(X)|^2 + K_6\epsilon^3. \quad (9)$$

Multiplying (7) and (9) we get

$$\mathbb{E} e^{-f(X)} \mathbb{E} e^{Hf(X)} \leq 1 + \mathbb{E} f(X)^2 - \frac{1}{2}\mathbb{E} |\nabla f(X)|^2 + K\epsilon^3. \quad (10)$$

Now we use Lemma 4: since f has mean 0 and is even, it satisfies (5), hence $\mathbb{E} f(X)^2 \leq \frac{1}{2}\mathbb{E} |\nabla f(X)|^2$, which concludes the proof. \square

Remark. Without the assumption “ f even” the Poincaré inequality would only say

$$\mathbb{E} f(X)^2 \leq \mathbb{E} |\nabla f(X)|^2,$$

and we would be able to prove

$$\mathbb{E} e^{-f(X)} \mathbb{E} e^{(f \square \frac{C}{4})(X)} \leq 1 + K\epsilon^3.$$

In order to prove the symmetric property (τ) , we need to tensorize Lemma 3. This requires various technical tools, most of which are well known to specialists. We begin with a very simple property of the class $\mathcal{F}(C, \epsilon)$.

Lemma 5. *Let E_1 and E_2 be Euclidean spaces, and μ a probability measure on E_1 . Let $f: E_1 \times E_2 \rightarrow \mathbb{R}$ be such that for any $x \in E_1$ the function $y \mapsto f(x, y)$ is in $\mathcal{F}(C, \epsilon)$. We define ϕ by*

$$e^{-\phi(y)} = \int_{E_1} e^{-f(x,y)} d\mu(x).$$

Then, unless $\phi = -\infty$, ϕ belongs to $\mathcal{F}(C, \epsilon)$.

Proof. Property (i) is simple. We check (ii): multiplying by $-\frac{1}{2}$ the inequality $f(x, y+h) + f(x, y-h) \leq 2f(x, y) + C\epsilon^2|h|^2$, taking the exponential and integrating with respect to μ , we obtain

$$\int_{E_1} e^{-f(x,y)} d\mu(x) e^{-\frac{C\epsilon^2}{2}|h|^2} \leq \int_{E_1} e^{-\frac{1}{2}f(x,y+h)} e^{-\frac{1}{2}f(x,y-h)} d\mu(x).$$

Then we use Cauchy-Schwarz to bound the right hand side, we take the log and we end up with the desired inequality for ϕ . \square

3 Symmetrisation

An important tool in geometry is Steiner's symmetrisation, see [4] for definition, properties and applications. In particular, it is the main idea behind Meyer and Pajor's proof of the Blaschke-Santaló inequality. It is thus natural, and as far as we know this idea dates back to [1] to introduce functional analogues of this symmetrisation.

Definition 6. Let E be a Euclidean space and $f: E \rightarrow \mathbb{R}$ be continuous. We define

$$Sf: x \mapsto \inf_{u \in E} \left(\frac{1}{2}(f(u+x) + f(u-x)) + \frac{1}{2}|u|^2 \right).$$

The function Sf is even.

Let E_1, E_2 be Euclidean spaces and $g: E_1 \times E_2 \rightarrow \mathbb{R}$. We define S_1g and S_2g to be the symmetrisations of g with respect to the first and the second variable, respectively:

$$S_1g: x, y \mapsto \inf_{u \in E_1} \left(\frac{1}{2}(f(u+x, y) + f(u-x, y)) + \frac{1}{2}|u|^2 \right),$$

and similarly for S_2 .

Lemma 7. Let γ be the normal distribution on E . For any continuous f on E , we have

$$\int_E e^{-f} d\gamma \leq \int_E e^{-Sf} d\gamma. \quad (11)$$

Let γ_1, γ_2 be the normal distributions on E_1 and E_2 , respectively. Any continuous g on $E_1 \times E_2$ satisfies

$$\int_{E_1 \times E_2} e^{Hg} d\gamma_1 \otimes d\gamma_2 \leq \int_{E_1 \times E_2} e^{HS_2g} d\gamma_1 \otimes d\gamma_2. \quad (12)$$

Proof. Set $\tilde{f}: x \rightarrow f(-x)$. By definition of Sf , we have for every z and $u \in \mathbb{R}^d$

$$\begin{aligned} Sf(z) + \frac{1}{2}|z|^2 &\leq \frac{1}{2}f(u+z) + \frac{1}{2}f(-z+u) + \frac{1}{2}|u|^2 + \frac{1}{2}|z|^2 \\ &= \frac{1}{2}(f(u+z) + \frac{1}{2}|u+z|^2) + \frac{1}{2}(\tilde{f}(z-u) + \frac{1}{2}|z-u|^2). \end{aligned}$$

Let $x, y \in \mathbb{R}^d$, we apply this inequality to $z = \frac{x+y}{2}$ and $u = \frac{x-y}{2}$. We obtain

$$Sf\left(\frac{x+y}{2}\right) + \frac{1}{2}\left|\frac{x+y}{2}\right|^2 \leq \frac{1}{2}(f(x) + \frac{1}{2}|x|^2) + \frac{1}{2}(\tilde{f}(y) + \frac{1}{2}|y|^2).$$

Applying the Prékopa-Leindler inequality we get (11).

Inequality (12) is the functional version of the following: if K is a centrally symmetric convex body and K_u its Steiner symmetral with respect to some direction u then $|(K_u)^\circ| \geq |K^\circ|$. This is the key argument in Meyer and Pajor's proof of the symmetric Santaló inequality, see [12]. Actually this idea dates back to Saint-Raymond [14] although he considered a symmetrisation of his own instead of Steiner's. Let us prove (12). It follows from

$$-S_1(-Hg) \leq HS_2g. \quad (13)$$

Indeed, combining it with (11) we get for every $y \in E_2$

$$\begin{aligned} \int e^{Hg(x,y)} d\gamma_1(x) &= \int e^{-(-Hg)(x,y)} d\gamma_1(x) \\ &\leq \int e^{-S_1(-Hg(x,y))} d\gamma_1(x) \leq \int e^{HS_2g(x,y)} d\gamma_1(x), \end{aligned}$$

which implies (12). Proof of (13): notations are heavy but it is straightforward.

$$\begin{aligned} 2HS_2g(x, y) &= \inf_{u,v} \{2S_2g(x+u, y+v) + |u|^2 + |v|^2\} \\ &= \inf_{u,v,w} \{g(x+u, w+y+v) + g(x+u, w-y-v) \\ &\quad + |u|^2 + |v|^2 + |w|^2\}. \end{aligned}$$

Since g is even, the latest is the same as

$$\inf_{u,v,w} \{g(x+u, w+y+v) + g(-x-u, -w+y+v) + |u|^2 + |v|^2 + |w|^2\}.$$

Whereas

$$\begin{aligned} -2S_1(-Hg)(x, y) &= -\inf_t \{(-Hg)(t+x, y) + (-Hg)(t-x, y) + |t|^2\} \\ &= \sup_t \{Hg(t+x, y) + Hg(t-x, y) - |t|^2\}. \end{aligned}$$

Hence we have to prove that for any t, u, v, w

$$\begin{aligned} Hg(t+x, y) + Hg(t-x, y) &\leq g(x+u, w+y+v) + g(-x-u, -w+y+v) \\ &\quad + |u|^2 + |v|^2 + |w|^2 + |t|^2, \end{aligned}$$

which is clear: use twice $Hg(a) - g(b) \leq \frac{1}{2}|b-a|^2$. \square

4 Tensorisation

Definition 8. Let $n \in \mathbb{N}^*$, we define the class $\mathcal{F}_n(C, \epsilon)$ by induction on n : let f be a function on $(\mathbb{R}^d)^n$, we say that f belongs to $\mathcal{F}_n(C, \epsilon)$ if

- for every $y \in \mathbb{R}^d$, the function $x \in (\mathbb{R}^d)^{n-1} \mapsto f(x, y)$ is in $\mathcal{F}_{n-1}(C, \epsilon)$
- for every $x \in (\mathbb{R}^d)^{n-1}$, the function $y \in \mathbb{R}^d \mapsto f(x, y)$ is in $\mathcal{F}(C, \epsilon)$.

In other words $\mathcal{F}_n(C, \epsilon)$ is the class of functions on $\mathbb{R}^d \times \dots \times \mathbb{R}^d$ that belong to $\mathcal{F}(C, \epsilon)$ with respect to each coordinate separately. The crucial point is that the class $\mathcal{F}_n(C, \epsilon)$ is stable under symmetrisation.

Lemma 9. Let f belong to $\mathcal{F}_n(C, \epsilon)$, set $E_1 = (\mathbb{R}^d)^{n-1}$ and $E_2 = \mathbb{R}^d$. Then S_2f also belongs to $\mathcal{F}_n(C, \epsilon)$.

Proof. It follows from the stability properties of the class $\mathcal{F}(C, \epsilon)$ that we mentioned earlier: for every $x \in E_1$ and $u \in E_2$, the function

$$y \mapsto \frac{1}{2}(f(x, u+y) + f(x, u-y)) + \frac{1}{2}|u|^2$$

is in $\mathcal{F}(C, \epsilon)$ so the same is true for the infimum over u , provided that this infimum is not $-\infty$, which is clear, since $\frac{1}{2}(f(x, u+y) + f(x, u-y))$ is lipschitz in u . Similarly, for any $y, u \in E_2$, the function

$$x \mapsto \frac{1}{2}(f(x, u+y) + f(x, u-y)) + \frac{1}{2}|u|^2$$

is in $\mathcal{F}_{n-1}(C, \epsilon)$ and the same is true for the infimum over u . \square

The next lemma is a tensorisation of Lemma 3, with the same constant K .

Lemma 10. *Let X_1, \dots, X_n be independent copies of X and let f be even and in $\mathcal{F}_n(C, \epsilon)$. Then*

$$\mathbb{E} e^{-f(X_1, \dots, X_n)} \mathbb{E} e^{Hf(X_1, \dots, X_n)} \leq (1 + K\epsilon^3)^n.$$

Proof. It is done by induction. When $n = 1$, it is Lemma 3. Let $n \geq 2$, we assume that the result holds for $n - 1$. Let $f: (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ be even and in $\mathcal{F}_n(C, \epsilon)$. Set $E_1 = (\mathbb{R}^d)^{n-1}$, $E_2 = \mathbb{R}^d$ and $\tilde{X} = (X_1, \dots, X_{n-1})$, it is a normal vector on E_1 . We have to prove

$$\mathbb{E} e^{-f(\tilde{X}, X_n)} \mathbb{E} e^{Hf(\tilde{X}, X_n)} \leq (1 + K\epsilon)^n.$$

Let $g = S_2 f$. We use Lemma 7, for every $x \in E_1$ we have: $\mathbb{E} e^{-f(x, X_n)} \leq \mathbb{E} e^{-S_2 f(x, X_n)}$, which implies that

$$\mathbb{E} e^{-f(\tilde{X}, X_n)} \leq \mathbb{E} e^{-S_2 f(\tilde{X}, X_n)}.$$

On the other hand, the second part of Lemma 7 gives

$$\mathbb{E} e^{Hf(\tilde{X}, X_n)} \leq \mathbb{E} e^{HS_2 f(\tilde{X}, X_n)}.$$

Hence it is enough to prove that

$$\mathbb{E} e^{-g(\tilde{X}, X_n)} \mathbb{E} e^{Hg(\tilde{X}, X_n)} \leq (1 + K\epsilon)^n. \quad (14)$$

By Lemma 9, we have $g \in \mathcal{F}_n(C, \epsilon)$. By definition of S_2 , for every $x \in E_1$, the function $y \mapsto g(x, y)$ is even. But as g is globally even we have also $g(-x, y) = g(x, -y) = g(x, y)$. Hence, for any $y \in E_2$, the function $g^y: x \rightarrow g(x, y)$ is even and in $\mathcal{F}_{n-1}(C, \epsilon)$. By the induction assumption

$$\mathbb{E} e^{-(g^y)(\tilde{X})} \mathbb{E} e^{H(g^y)(\tilde{X})} \leq (1 + K\epsilon^3)^{n-1}. \quad (15)$$

The following computation can be found in [11], we recall it for the sake of completeness. We define the operator H_1 by

$$H_1 g(x, y) = H(g^y)(x) = \inf_{u \in E_1} \left(g(x + u, y) + \frac{1}{2}|u|^2 \right).$$

We define a new function ϕ by

$$e^{-\phi(y)} = \mathbb{E} e^{-g(\tilde{X}, y)}.$$

For any $x \in E_1$, the function $y \rightarrow g(x, y)$ is even and belongs to $\mathcal{F}(C, \epsilon)$. By Lemma 5, the same is true for ϕ . Hence we can apply Lemma 3 to get

$$\mathbb{E} e^{-\phi(X_n)} \mathbb{E} e^{H\phi(X_n)} \leq 1 + K\epsilon^3. \quad (16)$$

On the other hand, inequality (15) implies that for any y and u in \mathbb{R}^d we have

$$\mathbb{E} e^{H_1 g(\tilde{X}, y+u) + \frac{1}{2}|u|^2} \leq e^{\phi(y+u) + \frac{1}{2}|u|^2} (1 + K\epsilon^3)^{n-1}. \quad (17)$$

Let $(x, y) \in E_1 \times E_2$. For any $u \in \mathbb{R}^d$ we have $H_1 g(x, y + u) + \frac{1}{2}|u|^2 \geq Hg(x, y)$. Hence, taking the infimum over u in (17), we get

$$\mathbb{E} e^{Hg(\tilde{X}, y)} \leq e^{H\phi(y)} (1 + K\epsilon^3)^{n-1}, \quad (18)$$

which of course implies that

$$\mathbb{E} e^{Hg(\tilde{X}, X_n)} \leq \mathbb{E} e^{H\phi(X_n)} (1 + K\epsilon^3)^{n-1}. \quad (19)$$

Combining (19) with (16) yields (14) and the proof is complete. \square

5 Proof of Theorem 1

Let $n \in \mathbb{N}$, and f be an even function from \mathbb{R}^d to \mathbb{R} belonging to $\mathcal{F}(C, 1)$. Set

$$g: (x_1, \dots, x_n) \in (\mathbb{R}^d)^n \mapsto f\left(\frac{1}{\sqrt{n}}(x_1 + \dots + x_n)\right).$$

It is clear that g is even and belongs to $\mathcal{F}_n(C, \frac{1}{\sqrt{n}})$. Applying Lemma 10, we obtain

$$\mathbb{E} e^{-g(X_1, \dots, X_n)} \mathbb{E} e^{Hg(X_1, \dots, X_n)} \leq (1 + K \frac{1}{n^{3/2}})^n \leq 1 + \frac{K'}{\sqrt{n}}. \quad (20)$$

The convexity of the square of the norm implies that

$$\begin{aligned} Hg(x_1, \dots, x_n) &= \inf_{u_1, \dots, u_n} f\left(\frac{1}{\sqrt{n}}\sum(x_i + u_i)\right) + \frac{1}{2}\sum|u_i|^2 \\ &\geq \inf_{u_1, \dots, u_n} f\left(\frac{1}{\sqrt{n}}\sum x_i + \frac{1}{\sqrt{n}}\sum u_i\right) + \frac{1}{2}\left|\frac{1}{\sqrt{n}}\sum u_i\right|^2, \\ &= Hf\left(\frac{1}{\sqrt{n}}\sum x_i\right). \end{aligned}$$

We combine this inequality with (20), since the random vector $\frac{1}{\sqrt{n}}\sum X_i$ has the same law as X , we obtain

$$\mathbb{E} e^{-f(X)} \mathbb{E} e^{Hf(X)} \leq 1 + \frac{K'}{\sqrt{n}}.$$

Now we let $n \rightarrow \infty$ and we get the result for f . Hence we have the inequality for any even f in $\mathcal{F}(C, 1)$. But this holds for any C , so we can take C as big as we want. In particular, we get the result for any even and \mathcal{C}^2 function with compact support. And a density argument shows that the inequality is valid for any function that is even, continuous and bounded from below.

Remarks

It is also possible to derive the usual property (τ) for the Gaussian measure from the Poincaré inequality. The function f is not assumed to be even anymore, we use the non-symmetric version of Lemma 3 and then perform the tensorisation (just as in Section 4 but without the symmetrisation). Of course, this is a bit more complicated than the usual proof (using the Prékopa-Leindler inequality) but we find it interesting to see that property (τ) has to do with the first eigenvalue of the Laplacian whereas its symmetric version has to do with the second eigenvalue.

Unfortunately we are not able to go farther: what happens if f is orthogonal to constants, linear functionals and polynomials of degree 2? Actually even in the case of a function that is orthogonal to constants and linear functionals our method does not give the result: we had to suppose that our function was even, which is stronger.

Lastly, following Klartag [8], we explain how to extend the symmetric property (τ) to measures that are even and log-concave with respect to the Gaussian. It is well known that property (τ) is stable under 1-Lipschitz image. Clearly, the symmetric property (τ) will have the following stable under pushforward by a 1-Lipschitz odd map. For example, it was proved by Caffarelli (see [5]) that if a probability measure μ is log-concave with respect to the Gaussian measure – meaning that $d\mu = e^{-V} d\gamma$ for some V convex – then the Brenier map transporting the Gaussian measure to μ is 1-Lipschitz. Now suppose additionally that μ is even, then the Brenier

map is automatically odd so μ is the pushforward of the Gaussian measure by a 1-Lipschitz odd map. Therefore, probability measures that are even and log-concave with respect to the Gaussian measure, in particular restrictions of the Gaussian measure to symmetric convex bodies, satisfy the symmetric property (τ).

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