Partitions and functional Santalo inequalities

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Abstract

We give a direct proof of a functional Santaló inequality due to Fradelizi and Meyer. This provides a new proof of the Blaschke-Santaló inequality. The argument combines a logarithmic form of the Prékopa-Leindler inequality and a partition theorem of Yao and Yao.

Introduction

If A is a subset of $\mathbb{R}^n$ we let A° be the polar of A:

$$A^\circ = \{x \in \mathbb{R}^n \,|\, \forall y \in A,\ x \cdot y \leq 1\},$$

where $x \cdot y$ denotes the scalar product of x and y. We denote the Euclidean norm of x by $|x| = \sqrt{x \cdot x}$. Let K be a subset of $\mathbb{R}^n$ with finite measure. The Blaschke-Santaló inequality states that there exists a point z in $\mathbb{R}^n$ such that

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n(K - z)^\circ \leq \mathrm{vol}_n(B_2^n)\,\mathrm{vol}_n(B_2^n)^\circ = v_n^2, \tag{1}$$

where vol_n stands for the Lebesgue measure on $\mathbb{R}^n$, B_2^n for the Euclidean ball and v_n for its volume. It was first proved by Blaschke in dimension 2 and 3 and Santaló [7] extended the result to any dimension. We say that an element z of $\mathbb{R}^n$ satisfying (1) is a *Santaló point* for K.

Throughout the paper a *weight* is an measurable function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any n, the function $x \in \mathbb{R}^n \mapsto \rho(|x|)$ is integrable.

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Definition 1. Let f be a non-negative integrable function on $\mathbb{R}^n$, and ρ be a weight. We say that $c \in \mathbb{R}^n$ is a Santaló point for f with respect to ρ if the following holds: for all non-negative Borel function g on $\mathbb{R}^n$, if

$$\forall x, y \in \mathbb{R}^n, \quad x \cdot y \geq 0 \;\Rightarrow\; f(c+x)g(y) \leq \rho\big(\sqrt{x \cdot y}\big)^2, \tag{2}$$

then

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} g(y)\,dy \leq \Big(\int_{\mathbb{R}^n} \rho(|x|)\,dx\Big)^2. \tag{3}$$

Heuristically, the choice of the weight ρ gives a notion of duality (or polarity) for non-negative functions. Our purpose is give a new proof of the following theorem, due to Fradelizi and Meyer [4].

Theorem 2. *Let f be non-negative and integrable. There exists $c \in \mathbb{R}^n$ such that c is a Santaló point for f with respect to any ρ. Moreover, if f is even then* 0 *is a Santaló point for f with respect to any weight.*

The even case goes back to Keith Ball in [2], this was the first example of a functional version of (1). Later on, Artstein, Klartag and Milman [1] proved that any integrable f admits a Santaló point with respect to the weight $t \mapsto \mathrm{e}^{-t^2/2}$. Moreover in this case the barycenter of f suits (see [5]). Unfortunately this is not true in general; indeed, taking

$$\begin{aligned} f &= \mathbf{1}_{(-2,0)} + 4\mathbf{1}_{(0,1)} \\ g &= \mathbf{1}_{(-0.5,0]} + \tfrac{1}{4}\mathbf{1}_{(0,1)} \\ \rho &= \mathbf{1}_{[0,1]}, \end{aligned}$$

it is easy to check that f has its barycenter at 0, and that $f(s)g(t) \leq \rho\big(\sqrt{st}\big)^2$ as soon as $st \geq 0$. However

$$\int_{\mathbb{R}} f(s)\,ds \int_{\mathbb{R}} g(t)\,dt = \frac{9}{2} > 4 = \Big(\int_{\mathbb{R}} \rho(|r|)\,dr\Big)^2.$$

To prove the existence of a Santaló point, the authors of [4] use a fixed point theorem and the usual Santaló inequality (for convex bodies). Our proof is direct, in the sense that it does not use the Blashke Santaló inequality; it is based on a special form of the Prékopa-Leindler inequality and on a partition theorem due to Yao and Yao [8].
Lastly, the Blaschke-Santaló inequality follows very easily from Theorem 2: we let the reader check that if c is a Santaló point for $\mathbf{1}_K$ with respect to the weight $\mathbf{1}_{[0,1]}$ then c is a Santaló point for K.

1 Yao-Yao partitions

In the sequel we consider real affine spaces of finite dimension. If E is such a space we denote by $\vec{E}$ the associated vector space. We say that $\mathcal{P}$ is a partition of E if $\cup\mathcal{P} = E$ and if the interiors of two distinct elements of $\mathcal{P}$ do not intersect. For instance, with this definition, the set $\{(-\infty, a], [a, +\infty)\}$ is a partition of $\mathbb{R}$. We define by induction on the dimension a class of partitions of an n-dimensional affine space.

Definition 3. If $E = \{c\}$ is an affine space of dimension 0, the only possible partition $\mathcal{P} = \{c\}$ is a Yao-Yao partition of E, and its center is defined to be c.
Let E be an affine space of dimension $n \geq 1$. A set $\mathcal{P}$ is said to be a Yao-Yao partition of E if there exists an affine hyperplane F of E, a vector $v \in \vec{E} \backslash \vec{F}$ and two Yao-Yao partitions $\mathcal{P}_+$ and $\mathcal{P}_-$ of F *having the same center* c such that

$$\mathcal{P} = \Big\{A + \mathbb{R}_- v \,|\, A \in \mathcal{P}_-\Big\} \cup \Big\{A + \mathbb{R}_+ v \,|\, A \in \mathcal{P}_+\Big\},$$

The center of $\mathcal{P}$ is then x.

If A is a subset of $\vec{E}$ we denote by $\text{pos}(A)$ the positive hull of A, that is to say the smallest convex cone containing A.
A Yao-Yao partition $\mathcal{P}$ of an n-dimensional space E has 2^n elements and for each A in $\mathcal{P}$ there exists a basis $v_1, \ldots, v_n$ of $\vec{E}$ such that

$$A = c + \text{pos}(v_1, \ldots, v_n), \tag{4}$$

where c is the center of $\mathcal{P}$. Indeed, assume that $\mathcal{P}$ is defined by $F, v, \mathcal{P}_+$ and $\mathcal{P}_-$ (see Definition 3). Let $A \in \mathcal{P}_+$ and assume inductively that there is a basis $v_1, \ldots, v_{n-1}$ of $\vec{F}$ such that $A = c + \text{pos}(v_1, \ldots, v_{n-1})$. Then $A + \mathbb{R}_+ v = c + \text{pos}(v, v_1, \ldots, v_{n-1})$.
A fundamental property of this class of partitions is the following

Proposition 4. *Let $\mathcal{P}$ be a Yao-Yao partition of E and c its center. Let ℓ be an affine form on E such that $\ell(c) = 0$. Then there exists $A \in \mathcal{P}$ such that $\ell(x) \geq 0$ for all $x \in A$. Moreover there is at most one element A of $\mathcal{P}$ such that $\ell(x) > 0$ for all $x \in A \backslash \{c\}$.*

Proof. By induction on the dimension n of E. When $n = 0$ it is obvious, we assume that $n \geq 1$ and that the result holds for all affine spaces of dimension

$n-1$. Let ℓ be an affine form on E such that $\ell(c) = 0$. We introduce $F, v, \mathcal{P}_+, \mathcal{P}_-$ given by Definition 3. By the induction assumption, there exists $A_+ \in \mathcal{P}_+$ and $A_- \in \mathcal{P}_-$ such that

$$\forall y \in A_+ \cup A_- \quad \ell(y) \geq 0.$$

If $\ell(c+v) \geq 0$ then $\ell(x+tv) \geq 0$ for all $x \in A_+$ and $t \in \mathbb{R}_+$, thus $\ell(x) \geq 0$ for all $x \in A_+ + \mathbb{R}_+ v$. If on the contrary $\ell(c+v) \leq 0$ then $\ell(x) \geq 0$ for all $x \in A_- + \mathbb{R}_- v$, which proves the first part of the proposition. The proof of the 'moreover' part is similar. □

The latter proposition yields the following corollary, which deals with dual cones: if C is cone of $\mathbb{R}^n$ the dual cone of C is by definition

$$C^* = \{y \in \mathbb{R}^n \,|\, \forall x \in C,\ x \cdot y \geq 0\}.$$

Corollary 5. *Let $\mathcal{P}$ be a Yao-Yao partition of $\mathbb{R}^n$ centered at 0. Then*

$$\mathcal{P}^* := \{A^* \,|\, A \in \mathcal{P}\}$$

is also a partition of $\mathbb{R}^n$.

Actually the dual partition is also a Yao-Yao partition centered at 0 but we will not use this fact.

Proof. Let $x \in \mathbb{R}^n$ and $\ell : y \in \mathbb{R}^n \mapsto x \cdot y$. By the previous proposition there exists $A \in \mathcal{P}$ such that $\ell(y) \geq 0$ for all $y \in A$. Then $x \in A^*$. Thus $\cup \mathcal{P}^* = \mathbb{R}^n$. Moreover if x belongs to the interior of A^*, then for all $y \in A \backslash \{0\}$ we have $\ell(y) > 0$. Again by the proposition above there is at most one such A. Thus the interiors of two distinct elements of $\mathcal{P}^*$ do not intersect. □

We now let $\mathcal{M}(E)$ be the set of Borel measure μ on E which are finite and which satisfy $\mu(F) = 0$ for any affine hyperplane F.

Definition 6. Let $\mu \in \mathcal{M}(E)$, a Yao-Yao equipartition $\mathcal{P}$ for μ is a Yao-Yao partition of E satisfying

$$\forall A \in \mathcal{P}, \quad \mu(A) = 2^{-n}\mu(E). \tag{5}$$

We say that $c \in E$ is a Yao-Yao center of μ if c is the center of a Yao-Yao equipartition for μ.

Here is the main result concerning those partitions.

Theorem 7. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$, there exists a Yao-Yao equipartition for μ. Moreover, if μ is even then 0 is a Yao-Yao center for μ.*

It is due to Yao and Yao [8]. They have some extra hypothesis on the measure and their paper is very sketchy, so we refer to [6] for a proof of this very statement.

2 Proof of the Fradelizi-Meyer inequality

In this section, all integrals are taken with respect to the Lebesgue measure. Let us recall the Prékopa-Leindler inequality, which is a functional form of the famous Brunn-Minkowski inequality, see for instance [3] for a proof and selected applications. If $\varphi_1, \varphi_2, \varphi_3$ are non-negative and integrable functions on $\mathbb{R}^n$ satisfying $\varphi_1(x)^\lambda \varphi_2(y)^{1-\lambda} \leq \varphi_3(\lambda x + (1-\lambda)y)$ for all x, y in $\mathbb{R}^n$ and for some fixed $\lambda \in (0,1)$, then

$$\Big(\int_{\mathbb{R}^n} \varphi_1\Big)^\lambda \Big(\int_{\mathbb{R}^n} \varphi_2\Big)^{1-\lambda} \leq \int_{\mathbb{R}^n} \varphi_3.$$

The following lemma is a useful (see [4, 2]) logarithmic version of Prékopa-Leindler. We recall the proof for completeness.

Lemma 8. *Let f_1, f_2, f_3 be non-negative Borel functions on $\mathbb{R}^n_+$ satisfying*

$$f_1(x) f_2(y) \leq \Big(f(\sqrt{x_1 y_1}, \ldots, \sqrt{x_n y_n}) \Big)^2.$$

for all x, y in $\mathbb{R}^n_+$. Then

$$\int_{\mathbb{R}^n_+} f_1 \int_{\mathbb{R}^n_+} f_2 \leq \Big(\int_{\mathbb{R}^n_+} f_3 \Big)^2. \tag{6}$$

Proof. For $i = 1, 2, 3$ we let

$$g_i(x) = f_i(\mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}) \mathrm{e}^{x_1 + \cdots + x_n}.$$

Then by change of variable we have

$$\int_{\mathbb{R}^n} g_i = \int_{\mathbb{R}^n_+} f_i.$$

On the other hand the hypothesis on f_1, f_2, f_3 yields

$$g_1(x)g_2(y) \le g_3(\tfrac{x+y}{2}),$$

for all x, y in $\mathbb{R}^n$. Then by Prékopa-Leindler

$$\int_{\mathbb{R}^n} g_1 \int_{\mathbb{R}^n} g_2 \le \Big(\int_{\mathbb{R}^n} g_3\Big)^2. \qquad \square$$

Theorem 9. *Let f be a non-negative Borel integrable function on $\mathbb{R}^n$, and let c be a Yao-Yao center for the measure with density f. Then c is a Santaló point for f with respect to any weight.*

Combining this result with Theorem 7 we obtain a complete proof of the Fradelizi-Meyer inequality.

Proof. It is enough to prove that if 0 is a Yao-Yao center for f then 0 is a Santaló point. Indeed, if c is a center for f then 0 is a center for

$$f_c : x \mapsto f(c+x).$$

And if 0 is a Santaló point for f_c then c is a Santaló point for f.
Let $\mathcal{P}$ be a Yao-Yao equipartition for f with center 0. Let g and ρ be such that (2) holds (with $c = 0$). Let $A \in \mathcal{P}$, by (4), there exists an operator T on $\mathbb{R}^n$ with determinant 1 such that $A = T\big(\mathbb{R}^n_+\big)$. Let $S = (T^{-1})^*$, then $S\big(\mathbb{R}^n_+\big) = A^*$. Let $f_1 = f \circ T$, $f_2 = g \circ S$ and $f_3(x) = \rho(|x|)$. Since for all x, y we have $T(x) \cdot S(y) = x \cdot y$, we get from (2)

$$f_1(x)f_2(y) \le \rho(\sqrt{x \cdot y})^2 = f_3(\sqrt{x_1y_1}, \dots, \sqrt{x_ny_n})^2,$$

for all x, y in $\mathbb{R}^n_+$. Applying the previous lemma we get (6). By change of variable it yields

$$\int_A f \int_{A^*} g \le \Big(\int_{\mathbb{R}^n_+} \rho(|x|)\, dx\Big)^2.$$

Therefore

$$\sum_{A \in \mathcal{P}} \int_A f \int_{A*} g \le 2^n \Big(\int_{\mathbb{R}^n_+} \rho(|x|)\, dx\Big)^2. \tag{7}$$

Since $\mathcal{P}$ is an equipartition for f we have for all $A \in \mathcal{P}$

$$\int_A f = 2^{-n} \int_{\mathbb{R}^n} f.$$

By Corollary 5, the set $\{A^*, A \in \mathcal{P}\}$ is a partition of $\mathbb{R}^n$, thus

$$\sum_{A \in \mathcal{P}} \int_{A^*} g = \int_{\mathbb{R}^n} g.$$

Inequality (7) becomes

$$\int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq 4^n \Big(\int_{\mathbb{R}^n_+} \rho(|x|)\, dx \Big)^2,$$

and of course the latter is equal to $\big(\int_{\mathbb{R}^n} \rho(|x|)\, dx \big)^2$. □

References

[1] S. Artstein, B. Klartag, and V. Milman. The Santaló point of a function, and a functional form of Santaló inequality. *Mathematika* 51 (2005), 33–48.

[2] K. Ball. Isometric problems in ℓ_p and sections of convex sets. PhD dissertation, University of Cambridge, 1986.

[3] K. Ball. An elementary introduction to modern convex geometry. In *Flavors of geometry*, edited by S. Levy, Cambridge University Press, 1997.

[4] M. Fradelizi and M. Meyer. Some functional forms of Blaschke-Santaló inequality. *Math. Z.*, 256 (2007), no. 2, 379–395.

[5] J. Lehec. A simple proof of the functional Santaló inequality. Preprint (2007).

[6] J. Lehec. 0n the Yao-Yao partition theorem. Preprint (2008).

[7] L.A. Santaló. Un invariante afin para los cuerpos convexos del espacio de n dimensiones. *Portugaliae Math.* 8 (1949), 155–161.

[8] A.C. Yao and F.F. Yao, A general approach to d-dimensional geometric queries, in *Proceedings of the seventeenth annual ACM symposium on Theory of computing*, ACM Press, 1985, pp.163–168.