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Abstract

We give a simple proof of a functional version of the Blaschke-Santaló inequality due to Artstein, Klartag and Milman. The proof is by induction on the dimension and does not use the Blaschke-Santaló inequality.


1 Introduction

For $x, y \in \mathbb{R}^n$, we denote their inner product by $\langle x, y \rangle$ and the Euclidean norm of $x$ by $|x|$. If $A$ is a subset of $\mathbb{R}^n$, we let $A^o = \{x \in \mathbb{R}^n | \forall y \in A, \langle x, y \rangle \leq 1 \}$ be its polar body. The Blaschke-Santaló inequality states that any convex body $K$ in $\mathbb{R}^n$ with center of mass at 0 satisfies

$$\text{vol}_n(K) \text{vol}_n(K^o) \leq \text{vol}_n(D) \text{vol}_n(D^o) = v_n^2, \tag{1}$$

where $\text{vol}_n$ stands for the volume, $D$ for the Euclidean ball and $v_n$ for its volume. Let $g$ be a non-negative Borel function on $\mathbb{R}^n$ satisfying $0 < \int g < \infty$ and $\int |x|g(x) \, dx < \infty$, then $\text{bar}(g) = \left(\int g \right)^{-1} \left(\int g \cdot x \, dx \right)$ denotes its center of mass (or barycenter). The center of mass (or centroid) of a measurable subset of $\mathbb{R}^n$ is by definition the barycenter of its indicator function.

Let us state a functional form of (1) due to Artstein, Klartag and Milman [1]. If $f$ is a non-negative Borel function on $\mathbb{R}^n$, the polar function of $f$ is the log-concave function defined by

$$f^o(x) = \inf_{y \in \mathbb{R}^n} \left(e^{-\langle x, y \rangle} f(y)^{-1} \right)$$

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Theorem 1 (Artstein, Klartag, Milman). If $f$ is a non-negative integrable function on $\mathbb{R}^n$ such that $f^o$ has its barycenter at 0, then
\[
\int_{\mathbb{R}^n} f(x) \, dx \int_{\mathbb{R}^n} f^o(y) \, dy \leq \left( \int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2} \, dx \right)^2 = (2\pi)^n.
\]

In the special case where the function $f$ is even, this result follows from an earlier inequality of Keith Ball [2]; and in [4], Fradelizi and Meyer prove something more general (see also [5]). In the present note we prove the following:

Theorem 2. Let $f$ and $g$ be non-negative Borel functions on $\mathbb{R}^n$ satisfying the duality relation
\[
\forall x, y \in \mathbb{R}^n, \quad f(x)g(y) \leq e^{-\langle x, y \rangle}.
\]
If $f$ (or $g$) has its barycenter at 0 then
\[
\int_{\mathbb{R}^n} f(x) \, dx \int_{\mathbb{R}^n} g(y) \, dy \leq (2\pi)^n.
\]

This is slightly stronger than Theorem 1 in which the function that has its barycenter at 0 should be log-concave. The point of this note is not really this improvement, but rather to present a simple proof of Theorem 1. Theorem 2 yields an improved Blaschke-Santaló inequality, obtained by Lutwak in [6], with a completely different approach.

Corollary 3. Let $S$ be a star-shaped (with respect to 0) body in $\mathbb{R}^n$ having its centroid at 0. Then
\[
\text{vol}_n(S) \text{vol}_n(S^o) \leq c_n^2.
\]

Proof. Let $N_S(x) = \inf\{r > 0 \mid x \in rS\}$ be the gauge of $S$ and $\phi_S = \exp\left(-\frac{1}{2}N_S^2\right)$. Integrating $\phi_S$ and the indicator function of $S$ on level sets of $N_S$, it is easy to see that $\int_{\mathbb{R}^n} \phi_S = c_n \text{vol}_n(S)$ for some constant $c_n$ depending only on the dimension. Replacing $S$ by the Euclidean ball in this equality yields $c_n = (2\pi)^{n/2}v_n^{-1}$. Therefore it is enough to prove that
\[
\int \phi_S \int \phi_{S^o} \leq (2\pi)^n.
\]

Similarly, it is easy to see that $\text{bar}(\phi_S) = c'_n \text{bar}(S) = 0$. Besides, we have $\langle x, y \rangle \leq N_S(x)N_{S^o}(y) \leq \frac{1}{2}N_S(x)^2 + \frac{1}{2}N_{S^o}(y)^2$, for all $x, y \in \mathbb{R}^n$. Thus $\phi_S$ and $\phi_{S^o}$ satisfy (2), then by Theorem 2 we get (5). \qed
2 Main results

Theorem 4. Let \( f \) be a non-negative Borel function on \( \mathbb{R}^n \) having a barycenter. Let \( H \) be an affine hyperplane splitting \( \mathbb{R}^n \) into two half-spaces \( H_+ \) and \( H_- \). Define \( \lambda \in [0, 1] \) by \( \lambda \int_{\mathbb{R}^n} f = \int_{H_+} f \). Then there exists \( z \in \mathbb{R}^n \) such that for every non-negative Borel function \( g \)

\[
(\forall x, y \in \mathbb{R}^n, \; f(z + x)g(y) \leq e^{-\langle x, y \rangle}) \; \Rightarrow \; \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq \frac{1}{4\lambda(1-\lambda)}(2\pi)^n.
\]

In particular, in every median \( H (\lambda = \frac{1}{2}) \) there is a point \( z \) such that for all \( g \)

\[
(\forall x, y \in \mathbb{R}^n, \; f(z + x)g(y) \leq e^{-\langle x, y \rangle}) \; \Rightarrow \; \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq (2\pi)^n.
\]

A similar result concerning convex bodies (instead of functions) was obtained by Meyer and Pajor in [7].

Let us derive Theorem 2 from the latter. Let \( f, g \) satisfy (2). Assume for example that \( \text{bar}(g) = 0 \), then 0 cannot be separated from the support of \( g \) by a hyperplane, so there exists \( x_1, \ldots, x_{n+1} \in \mathbb{R}^n \) such that 0 belongs to the interior of \( \text{conv}\{x_1, \ldots, x_{n+1}\} \) and \( g(x_i) > 0 \) for \( i = 1, \ldots, n+1 \). Then (2) implies that \( f(x) \leq Ce^{-\|x\|} \), for some \( C > 0 \), where \( \|x\| = \max\{\langle x, x_i \rangle \mid i \leq n+1\} \). Assume also that \( \int f > 0 \), then \( f \) has a barycenter. Apply the “\( \lambda = 1/2 \)” part of Theorem 4 to \( f \). There exists \( z \in \mathbb{R}^n \) such that (7) holds. On the other hand, by (2)

\[
f(z + x)g(y)e^{\langle y, z \rangle} \leq e^{-\langle z + x, y \rangle}e^{\langle y, z \rangle} = e^{-\langle x, y \rangle}
\]

for all \( x, y \in \mathbb{R}^n \). Therefore

\[
\int_{\mathbb{R}^n} f(x) \, dx \int_{\mathbb{R}^n} g(y)e^{\langle y, z \rangle} \, dy \leq (2\pi)^n.
\]

Integrating with respect to \( g(y)dy \) the inequality \( 1 \leq e^{\langle y, z \rangle} - \langle y, z \rangle \) we get

\[
\int_{\mathbb{R}^n} g(y) \, dy \leq \int_{\mathbb{R}^n} g(y)e^{\langle y, z \rangle} \, dy - \int_{\mathbb{R}^n} \langle y, z \rangle g(y) \, dy.
\]

Since \( \text{bar}(g) = 0 \), the latter integral is 0 and together with (8) we obtain (3). Observe also that this proof shows that Theorem 4 in dimension \( n \) implies Theorem 2 in dimension \( n \).

In order to prove Theorem 4, we need the following logarithmic form of the Prékopa-Leindler inequality. For details on Prékopa-Leindler, we refer to [3].
Lemma 5. Let $\phi_1, \phi_2$ be non-negative Borel functions on $\mathbb{R}_+$. If $\phi_1(s)\phi_2(t) \leq e^{-st}$ for every $s, t$ in $\mathbb{R}_+$, then

$$\int_{\mathbb{R}_+} \phi_1(s) \, ds \int_{\mathbb{R}_+} \phi_2(t) \, dt \leq \frac{\pi}{2}.$$  

Proof. Let $f(s) = \phi_1(e^s) e^s$, $g(t) = \phi_2(e^t) e^t$ and $h(r) = \exp(-e^{2r}/2) e^r$. For all $s, t \in \mathbb{R}$ we have $\sqrt{f(s)g(t)} \leq h(\frac{s+t}{2})$, hence by Prékopa-Leindler

$$\int_{\mathbb{R}} f(s) g(t) \leq (\int_{\mathbb{R}} h)^2.$$  

By change of variable, this is the same as $\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \phi_1(t) \phi_2(s) \leq (\int_{\mathbb{R}^+} e^{-u^2/2} \, du)^2$ which is the result. 

3 Proof of Theorem 4

Clearly we can assume that $\int f = 1$. Let $\mu$ be the measure with density $f$.

In the sequel we let $f_z(x) = f(x + z)$ for all $x, z$.

We prove the theorem by induction on the dimension. Let $f$ be a non-negative Borel function on the line, let $r \in \mathbb{R}$ and $\lambda = \mu([r, \infty)) \in [0, 1]$. Let $g$ satisfy $f(r + s) g(t) \leq e^{-st}$, for all $s, t$. Apply Lemma 5 twice: first to $\phi_1(s) = f(r + s)$ and $\phi_2(t) = g(t)$ then to $\phi_1(s) = f(r - s)$ and $\phi_2(t) = g(-t)$. Then

$$\int_{\mathbb{R}^+} f_r \int_{\mathbb{R}^+} g \leq \frac{\pi}{2} \quad \text{and} \quad \int_{\mathbb{R}^+} f_r \int_{\mathbb{R}^-} g \leq \frac{\pi}{2}.$$  

Therefore $\int_{\mathbb{R}^+} g \leq \frac{\pi}{2\lambda}$ and $\int_{\mathbb{R}^-} g \leq \frac{\pi}{2(1-\lambda)}$, which yields the result in dimension 1.

Assume the theorem to be true in dimension $n - 1$. Let $H$ be an affine hyperplane splitting $\mathbb{R}^n$ into two half-spaces $H_+$ and $H_-$ and let $\lambda = \mu(H_+)$. Provided that $\lambda \neq 0, 1$ we can define $b_+$ and $b_-$ to be the barycenters of $\mu|_{H_+}$ and $\mu|_{H_-}$, respectively. Since $\mu(H) = 0$, the point $b_+ \in H_+$ and similarly for $b_-$. Hence the line passing through $b_+$ and $b_-$ intersects $H$ at one point, which we call $z$. Let us prove that $z$ satisfies (6), for all $g$. Clearly, replacing $f$ by $f_z$ and $H$ by $H - z$, we can assume that $z = 0$. Let $g$ satisfy

$$\forall x, y \in \mathbb{R}^n, \quad f(x)g(y) \leq e^{-\langle x, y \rangle}.$$  

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $H = e_n^\perp$ and $\langle b_+, e_n \rangle > 0$. Let $v = b_+/(b_+, e_n)$ and $A$ be the linear operator on $\mathbb{R}^n$ that maps $e_n$ to $v$ and $e_i$ to itself for $i = 1 \ldots n - 1$ and let $B = (A^{-1})^t$. Define

$$F_+: y \in H \mapsto \int_{\mathbb{R}^+} f(y + sv) \, ds \quad \text{and} \quad G_+: y' \in H \mapsto \int_{\mathbb{R}^+} g(By' + te_n) \, dt.$$  

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By Fubini, and since $A$ has determinant 1, $\int_H F_+ = \int_{H_+} f \circ A = \mu(H_+) = \lambda$.

Also, letting $P$ be the projection with range $H$ and kernel $Rv$, we have

$$\text{bar}(F_+) = \frac{1}{\lambda} \int_{H_+} P(Ax)f(Ax)\,dx = \frac{1}{\lambda} P\left(\int_{H_+} xf(x)\,dx\right) = P(b_+),$$

and this is 0 by definition of $P$. Since $\langle Ax, Bx' \rangle = \langle x, x' \rangle$ for all $x, x' \in \mathbb{R}^n$, we have $\langle y + sv, By' + te_n \rangle = \langle y, y' \rangle + st$ for all $s, t \in \mathbb{R}$ and $y, y' \in H$. So (10) implies

$$f(y + sv)g(By' + te_n) \leq e^{-st - \langle y, y' \rangle}.$$

Applying Lemma 5 to $\phi_1(s) = f(y + sv)$ and $\phi_2(t) = g(By' + te_n)$ we get $F_+(y)G_+(y') \leq \frac{\pi}{2} e^{-\langle y, y' \rangle}$ for every $y, y' \in H$. Recall that $\text{bar}(F_+) = 0$, then by the induction assumption (which implies Theorem 2 in dimension $n - 1$)

$$\int_H F_+ \int_H G_+ \leq \frac{\pi}{2} (2\pi)^{n-1}.$$  \hfill (11)

hence $\int_{H_+} g(Bx)\,dx \leq \frac{1}{4\lambda}(2\pi)^n$. In the same way $\int_{H_-} g(Bx)\,dx \leq \frac{1}{4(1-\lambda)}(2\pi)^n$, adding these two inequalities, we obtain

$$\int_{\mathbb{R}^n} g(Bx)\,dx \leq \frac{1}{4\lambda(1-\lambda)}(2\pi)^n$$

which is the result since $B$ has determinant 1.

References


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