A direct proof of the functional Santalo inequality
Joseph Lehec

To cite this version:

HAL Id: hal-00365764
https://hal.archives-ouvertes.fr/hal-00365764
Submitted on 6 Jan 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
A direct proof of the functional Santaló inequality

Joseph Lehec *

June 2008

Abstract

We give a simple proof of a functional version of the Blaschke-Santaló inequality due to Artstein, Klartag and Milman. The proof is by induction on the dimension and does not use the Blaschke-Santaló inequality.


1 Introduction

For $x, y \in \mathbb{R}^n$, we denote their inner product by $\langle x, y \rangle$ and the Euclidean norm of $x$ by $|x|$. If $A$ is a subset of $\mathbb{R}^n$, we let $A^o = \{ x \in \mathbb{R}^n | \forall y \in A, \langle x, y \rangle \leq 1 \}$ be its polar body. The Blaschke-Santaló inequality states that any convex body $K$ in $\mathbb{R}^n$ with center of mass at 0 satisfies

$$\text{vol}_n(K) \text{vol}_n(K^o) \leq \text{vol}_n(D) \text{vol}_n(D^o) = v_n^2,$$  \hspace{1cm} (1)

where $\text{vol}_n$ stands for the volume, $D$ for the Euclidean ball and $v_n$ for its volume. Let $g$ be a non-negative Borel function on $\mathbb{R}^n$ satisfying $0 < \int g < \infty$ and $\int |x|g(x) \, dx < \infty$, then $\text{bar}(g) = \left( \int g \right)^{-1} \left( \int g(x) \, x \, dx \right)$ denotes its center of mass (or barycenter). The center of mass (or centroid) of a measurable subset of $\mathbb{R}^n$ is by definition the barycenter of its indicator function.

Let us state a functional form of (1) due to Artstein, Klartag and Milman [1]. If $f$ is a non-negative Borel function on $\mathbb{R}^n$, the polar function of $f$ is the log-concave function defined by

$$f^o(x) = \inf_{y \in \mathbb{R}^n} (e^{-\langle x, y \rangle} f(y)^{-1})$$

---

*LAMA (UMR CNRS 8050) Université Paris-Est.
**Theorem 1** (Artstein, Klartag, Milman). If $f$ is a non-negative integrable function on $\mathbb{R}^n$ such that $f^o$ has its barycenter at 0, then

$$\int_{\mathbb{R}^n} f(x) \, dx \int_{\mathbb{R}^n} f^o(y) \, dy \leq \left( \int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2} \, dx \right)^2 = (2\pi)^n.$$

In the special case where the function $f$ is even, this result follows from an earlier inequality of Keith Ball [2]; and in [4], Fradelizi and Meyer prove something more general (see also [5]). In the present note we prove the following:

**Theorem 2.** Let $f$ and $g$ be non-negative Borel functions on $\mathbb{R}^n$ satisfying the duality relation

$$\forall x, y \in \mathbb{R}^n, \quad f(x)g(y) \leq e^{-\langle x, y \rangle}.$$  \hfill (2)

If $f$ (or $g$) has its barycenter at 0 then

$$\int_{\mathbb{R}^n} f(x) \, dx \int_{\mathbb{R}^n} g(y) \, dy \leq (2\pi)^n.$$  \hfill (3)

This is slightly stronger than Theorem 1 in which the function that has its barycenter at 0 should be log-concave. The point of this note is not really this improvement, but rather to present a simple proof of Theorem 1. Theorem 2 yields an improved Blaschke-Santaló inequality, obtained by Lutwak in [6], with a completely different approach.

**Corollary 3.** Let $S$ be a star-shaped (with respect to 0) body in $\mathbb{R}^n$ having its centroid at 0. Then

$$\operatorname{vol}_n(S) \operatorname{vol}_n(S^o) \leq v_n^2.$$  \hfill (4)

**Proof.** Let $N_S(x) = \inf\{r > 0 \mid x \in rS\}$ be the gauge of $S$ and $\phi_S = \exp\left(-\frac{1}{2}N_S^2\right)$. Integrating $\phi_S$ and the indicator function of $S$ on level sets of $N_S$, it is easy to see that $\int_{\mathbb{R}^n} \phi_S = c_n \operatorname{vol}_n(S)$ for some constant $c_n$ depending only on the dimension. Replacing $S$ by the Euclidean ball in this equality yields $c_n = (2\pi)^{n/2}v_n^{-1}$. Therefore it is enough to prove that

$$\int \phi_S \int \phi_{S^o} \leq (2\pi)^n.$$  \hfill (5)

Similarly, it is easy to see that $\operatorname{bar}(\phi_S) = c'_n \operatorname{bar}(S) = 0$. Besides, we have $\langle x, y \rangle \leq N_S(x)N_{S^o}(y) \leq \frac{1}{2}N_S(x)^2 + \frac{1}{2}N_{S^o}(y)^2$, for all $x, y \in \mathbb{R}^n$. Thus $\phi_S$ and $\phi_{S^o}$ satisfy (2), then by Theorem 2 we get (5). \qed


2 Main results

**Theorem 4.** Let \( f \) be a non-negative Borel function on \( \mathbb{R}^n \) having a barycenter. Let \( H \) be an affine hyperplane splitting \( \mathbb{R}^n \) into two half-spaces \( H_+ \) and \( H_- \). Define \( \lambda \in [0, 1] \) by \( \lambda \int_{\mathbb{R}^n} f = \int_{H_+} f \). Then there exists \( z \in \mathbb{R}^n \) such that for every non-negative Borel function \( g \)

\[
(\forall x, y \in \mathbb{R}^n, \ f(z+x)g(y) \leq e^{-\langle x, y \rangle}) \Rightarrow \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq \frac{1}{4\lambda(1-\lambda)} (2\pi)^n.
\]

(6)

In particular, in every median \( H \) (\( \lambda = \frac{1}{2} \)) there is a point \( z \) such that for all \( g \)

\[
(\forall x, y \in \mathbb{R}^n, \ f(z+x)g(y) \leq e^{-\langle x, y \rangle}) \Rightarrow \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g \leq (2\pi)^n.
\]

(7)

A similar result concerning convex bodies (instead of functions) was obtained by Meyer and Pajor in [7].

Let us derive Theorem 2 from the latter. Let \( f, g \) satisfy (2). Assume for example that \( \text{bar}(g) = 0 \), then 0 cannot be separated from the support of \( g \) by a hyperplane, so there exists \( x_1, \ldots, x_{n+1} \in \mathbb{R}^n \) such that 0 belongs to the interior of \( \text{conv}\{x_1, \ldots, x_{n+1}\} \) and \( g(x_i) > 0 \) for \( i = 1, \ldots, n+1 \). Then (2) implies that \( f(x) \leq Ce^{-\|x\|} \), for some \( C > 0 \), where \( \|x\| = \max\{\langle x, x_i \rangle | i \leq n+1\} \). Assume also that \( \int f > 0 \), then \( f \) has a barycenter. Apply the “\( \lambda = 1/2 \)” part of Theorem 4 to \( f \). There exists \( z \in \mathbb{R}^n \) such that (7) holds. On the other hand, by (2)

\[
f(z+x)g(y)e^{\langle y, z \rangle} \leq e^{-\langle z+x, y \rangle}e^{\langle y, z \rangle} = e^{-\langle x, y \rangle}
\]

for all \( x, y \in \mathbb{R}^n \). Therefore

\[
\int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} g(y)e^{\langle y, z \rangle} dy \leq (2\pi)^n.
\]

(8)

Integrating with respect to \( g(y)dy \) the inequality \( 1 \leq e^{\langle y, z \rangle} - \langle y, z \rangle \) we get

\[
\int_{\mathbb{R}^n} g(y) dy \leq \int_{\mathbb{R}^n} g(y)e^{\langle y, z \rangle} dy - \int_{\mathbb{R}^n} \langle y, z \rangle g(y) dy.
\]

Since \( \text{bar}(g) = 0 \), the latter integral is 0 and together with (8) we obtain (3). Observe also that this proof shows that Theorem 4 in dimension \( n \) implies Theorem 2 in dimension \( n \).

In order to prove Theorem 4, we need the following logarithmic form of the Prékopa-Leindler inequality. For details on Prékopa-Leindler, we refer to [3].
Lemma 5. Let \( \phi_1, \phi_2 \) be non-negative Borel functions on \( \mathbb{R}_+ \). If \( \phi_1(s)\phi_2(t) \leq e^{-st} \) for every \( s, t \) in \( \mathbb{R}_+ \), then

\[
\int_{\mathbb{R}_+} \phi_1(s) \, ds \int_{\mathbb{R}_+} \phi_2(t) \, dt \leq \frac{\pi}{2}.
\]  

(9)

Proof. Let \( f(s) = \phi_1(e^s)e^s \), \( g(t) = \phi_2(e^t)e^t \) and \( h(r) = \exp(-e^{2r}/2)e^r \). For all \( s, t \in \mathbb{R} \) we have \( \sqrt{f(s)g(t)} \leq h(\frac{r}{2}) \), hence by Prékopa-Leindler

\[
\int_{\mathbb{R}_+} f \int_{\mathbb{R}_+} g \leq \left( \int_{\mathbb{R}_+} h \right)^2.
\]

By change of variable, this is the same as \( \int_{\mathbb{R}_+} \phi_1 \int_{\mathbb{R}_+} \phi_2 \leq (\int_{\mathbb{R}_+} e^{-a^2/2} \, du)^2 \) which is the result. \( \Box \)

3 Proof of Theorem 4

Clearly we can assume that \( \int f = 1 \). Let \( \mu \) be the measure with density \( f \).

In the sequel we let \( f_z(x) = f(z + x) \) for all \( x, z \).

We prove the theorem by induction on the dimension. Let \( f \) be a non-negative Borel function on the line, let \( r \in \mathbb{R} \) and \( \lambda = \mu([r, \infty)) \in [0, 1] \). Let \( g \) satisfy \( f(r + s)g(t) \leq e^{-st} \), for all \( s, t \). Apply Lemma 5 twice: first to \( \phi_1(s) = f(r + s) \) and \( \phi_2(t) = g(t) \) then to \( \phi_1(s) = f(r - s) \) and \( \phi_2(t) = g(-t) \). Then

\[
\int_{\mathbb{R}_+} f \int_{\mathbb{R}_+} g \leq \frac{\pi}{2}
\]

and

\[
\int_{\mathbb{R}_-} g \leq \frac{\pi}{2}.
\]

Therefore \( \int_{\mathbb{R}_+} g \leq \frac{\pi}{2\lambda} \) and \( \int_{\mathbb{R}_-} g \leq \frac{\pi}{2(1-\lambda)} \), which yields the result in dimension 1.

Assume the theorem to be true in dimension \( n - 1 \). Let \( H \) be an affine hyperplane splitting \( \mathbb{R}^n \) into two half-spaces \( H_+ \) and \( H_- \) and let \( \lambda = \mu(H_+) \).

Provided that \( \lambda \neq 0, 1 \) we can define \( b_+ \) and \( b_- \) to be the barycenters of \( \mu|_{H_+} \) and \( \mu|_{H_-} \), respectively. Since \( \mu(H) = 0 \), the point \( b_+ \) belongs to the interior of \( H_+ \), and similarly for \( b_- \). Hence the line passing through \( b_+ \) and \( b_- \) intersects \( H \) at one point, which we call \( z \). Let us prove that \( z \) satisfies (6), for all \( g \). Clearly, replacing \( f \) by \( f_z \) and \( H \) by \( H - z \), we can assume that \( z = 0 \). Let \( g \) satisfy

\[
\forall x, y \in \mathbb{R}^n, \quad f(x)g(y) \leq e^{-(x,y)}.
\]  

(10)

Let \( e_1, \ldots, e_n \) be an orthonormal basis of \( \mathbb{R}^n \) such that \( H = e_n^\perp \) and \( \langle b_+, e_n \rangle > 0 \). Let \( v = b_+/\langle b_+, e_n \rangle \) and \( A \) be the linear operator on \( \mathbb{R}^n \) that maps \( e_n \) to \( v \) and \( e_i \) to itself for \( i = 1 \ldots n - 1 \) and let \( B = (A^{-1})^t \). Define

\[
F_+: y \in H \mapsto \int_{\mathbb{R}_+} f(y + sv) \, ds \quad \text{and} \quad G_+: y' \in H \mapsto \int_{\mathbb{R}_+} g(By' + te_n) \, dt.
\]
By Fubini, and since $A$ has determinant 1, $\int_H F_+ = \int_{H_+} f \circ A = \mu(H_+) = \lambda$. Also, letting $P$ be the projection with range $H$ and kernel $\mathbb{R}v$, we have

$$\text{bar}(F_+) = \frac{1}{\lambda} \int_{H_+} P(Ax)f(Ax) \, dx = \frac{1}{\lambda} P\left( \int_{H_+} xf(x) \, dx \right) = P(b_+),$$

and this is 0 by definition of $P$. Since $\langle Ax, Bx' \rangle = \langle x, x' \rangle$ for all $x, x' \in \mathbb{R}^n$, we have $\langle y + sv, By' + te_n \rangle = \langle y, y' \rangle + st$ for all $s, t \in \mathbb{R}$ and $y, y' \in H$. So (10) implies

$$f(y + sv)g(By' + te_n) \leq e^{-st - \langle y, y' \rangle}.$$

Applying Lemma 5 to $\phi_1(s) = f(y + sv)$ and $\phi_2(t) = g(By' + te_n)$ we get $F_+(y)G_+(y') \leq \frac{\pi}{2} e^{-\langle y, y' \rangle}$ for every $y, y' \in H$. Recall that $\text{bar}(F_+) = 0$, then by the induction assumption (which implies Theorem 2 in dimension $n - 1$)

$$\int_H F_+ \int_H G_+ \leq \frac{\pi}{2} (2\pi)^{n-1}. \quad (11)$$

hence $\int_{H_+} g(Bx) \, dx \leq \frac{1}{\lambda} (2\pi)^n$. In the same way $\int_{H_-} g(Bx) \, dx \leq \frac{1}{\lambda(1 - \lambda)} (2\pi)^n$, adding these two inequalities, we obtain

$$\int_{\mathbb{R}^n} g(Bx) \, dx \leq \frac{1}{4\lambda(1 - \lambda)} (2\pi)^n$$

which is the result since $B$ has determinant 1.

References


