Loewy and primary decompositions of D-modules

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Two kinds of decompositions of a $\mathcal{D}$-module are introduced in this article, a Loewy-decomposition and a primary decomposition. Algorithms are described which allow to construct decompositions for finite-dimensional $\mathcal{D}$-modules. Both decompositions rely on the concept of the $\mathcal{D}$-module of relative syzygies introduced in the paper. Furthermore an algorithm is described which tests whether two given $\mathcal{D}$-modules are isomorphic under the relevant type of transformations.

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1. Introduction

Factoring multivariable polynomials over a large class of fields, even in polynomial time, is well-established [6]. On the other hand, much less is known about the differential counterpart of this problem. So far, this question is answered in a satisfactory way only for linear ordinary differential operators (lodo's). Historically, algorithms were first proposed by Beke [1] and Schlesinger [24]. In modern times an algorithm for factoring lodo's was described in [26], and an improved one with a better complexity bound was designed in [7]. A more complete overview of the problem of factorization of lodo's may be found in the book by Singer and van der Put [19]. Unlike the factorization of polynomials, factoring lodo's is not unique. Therefore Loewy [16] introduced the concept of a completely reducible operator which leads to a unique decomposition into largest completely reducible factors.

Factoring lpdo's is understood to a much lesser extent, even the very concept of factoring is not commonly established. It turns out that a more algebraic language is appropriate for dealing with these problems. The objects of interest are ideals or modules over rings of partial differential operators, they are usually called $\mathcal{D}$-modules. To any system of linear partial differential equations there corresponds a unique $\mathcal{D}$-module generated by the lpdo's at the left-hand side of the individual equations. Some background on $\mathcal{D}$-modules may be found, for example, in the book by Sabbah [22] or Coutinho [5]. Algorithmical and complexity problems on $\mathcal{D}$-modules were studied in [7,9,10,17].

If the module generated by the lpdo's under consideration is such that the corresponding system of partial differential equations has a finite-dimensional solution space, Loewy's theory may be generalized in a more or less straightforward way as has been described by Li et al. [15]. The situation is completely different however in the general case. Neither the factorization into irreducible factors is unique, nor the largest completely reducible factors lead to a unique decomposition as the example of Blumberg [2] shows, see Example 3 at the end of Section 3. Partial results have been given in the references of this article. An algorithm for factoring so-called *separable* lpdo's was exhibited in [11]. It was developed further in [33]. A generalized concept of factoring in terms of overideals of the respective $\mathcal{D}$-module was suggested in [12], another approach may be found in [32].

In the present paper we introduce two dual concepts of decompositions of $\mathcal{D}$-modules: Loewy decomposition in Section 4 and primary decomposition in Section 5. They are unique for finite-dimensional $\mathcal{D}$-modules and the algorithms for constructing both are described in Section 7. It is an open question whether there is an algorithm for constructing any of the two decompositions for infinite-dimensional $\mathcal{D}$-modules. The definitions of both decompositions rely on the notion of the *relative syzygies* of $\mathcal{D}$-modules introduced in Section 3 where also some properties of the $\mathcal{D}$-module of relative syzygies are proved. In particular, its invariance is shown and the behavior of its *typical differential dimension*, or in other terms, the leading coefficient of the Hilbert–Kolchin polynomial [13,14] is established.

As in the case of a polynomial ideal one talks about the variety of its solutions, in a similar way one considers the space of solutions of a $\mathcal{D}$-module in Section 2. We prove the invariance of the space and establish the duality between a $\mathcal{D}$-module and the space with respect to isomorphisms under the relevant matrix-type transformations (over universal fields [13]); this resembles the well known duality between the radical ideals and the varieties in algebraic geometry. In Section 8 an algorithm is designed to test the studied isomorphism of $\mathcal{D}$-modules. The algorithms from Sections 7 and 8 rely on the concept of *parametric-algebraic families of $\mathcal{D}$-modules* introduced in Section 6.

An accompanying article in the proceedings [12] deals in more detail with algorithmic questions. Numerous examples and calculations which were based on the software computer algebra package ALLTYPES [27] are given there.

2. Invariance of the space of solutions of a $\mathcal{D}$-module

Let F be a universal differential field [13] with commuting derivatives $d_1, \ldots, d_m$ and $\mathcal{D} = F[d_1, \ldots, d_m]$ be the ring of partial differential operators. Denote by $C \subset F$ its subfield of constants. Introduce differential indeterminates $y_1, \ldots, y_n$ over F. By Θ denote the commutative monoid generated by $d_1, \ldots, d_m$ and by Γ the set of all the derivatives θy_i for $\theta \in \Theta$, $1 \leqslant i \leqslant n$. We fix also an admissible total ordering $\prec$ on the derivatives [14,23]. A background in differential algebra may be found in [3,13,29,30].

Let $I \subset \mathcal{D}^n$ be a left $\mathcal{D}$-module. For vectors $g = (g_1, \ldots, g_n) \in \mathcal{D}^n$, $v = (v_1, \ldots, v_n) \in F^n$ we denote the inner product $gv = (g, v^T) = \sum g_i v_i \in F$. By $V_I = \{v \in F^n\colon\ Iv = 0\} \subset F^n$ we denote the space of solutions of I being a C-vector space. A priori V_I depends on the embedding $I \subset \mathcal{D}^n$. The purpose of this section is to show that actually V_I depends up to an isomorphism just on the factor $\mathcal{D}^n/I$, considered as well up to an isomorphism.

Now let $I_1 \subset \mathcal{D}^{n_1}$, $I_2 \subset \mathcal{D}^{n_2}$. We say that a $n_1 \times n_2$ matrix $A = (a_{ij})$ with $a_{ij} \in \mathcal{D}$ provides a $\mathcal{D}$-homomorphism from $\mathcal{D}^{n_1}/I_1$ to $\mathcal{D}^{n_2}/I_2$ if $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$, i.e. $I_1 A \subset I_2$. Clearly one gets a homomorphism of $\mathcal{D}$-modules.

We call $\mathcal{D}^{n_1}/I_1$ and $\mathcal{D}^{n_2}/I_2$ to be $\mathcal{D}$-isomorphic if in addition there exists a $n_2 \times n_1$ matrix $B = (b_{ij})$ with $b_{ij} \in \mathcal{D}$ such that $(\mathcal{D}^{n_2}/I_2)B \subset D^{n_1}/I_1$ and

$$AB|_{(\mathcal{D}^{n_1}/I_1)} = id, \qquad BA|_{(\mathcal{D}^{n_2}/I_2)} = id. \tag{1}$$

For the spaces of solutions $V_{I_1} \subset F^{n_1}$, $V_{I_2} \subset F^{n_2}$ we say that a matrix A provides a $\mathcal{D}$-*homomorphism* if $A(V_{I_2})^T \subset (V_{I_1})^T$ (more precisely, one should talk about a $\mathcal{D}$-homomorphism of the embeddings $V_{I_1} \subset F^{n_1}$, $V_{I_2} \subset F^{n_2}$). In a similar way, if there exists a $n_2 \times n_1$ matrix B such that $B(V_{I_1})^T \subset (V_{I_2})^T$ and

$$AB|_{V_{I_1}^T} = id, \qquad BA|_{V_{I_2}^T} = id \tag{2}$$

we call V_{I_1}, V_{I_2} to be $\mathcal{D}$-*isomorphic* and denote this by $V_{I_1} \simeq_{\mathcal{D}} V_{I_2}$. The following proposition extends Lemma 2.5 in [28] (established for the ordinary case $m = 1$) to finite-dimensional modules.

Proposition 1. (i) *A matrix A provides a $\mathcal{D}$-homomorphism of $\mathcal{D}^{n_1}/I_1$ to $\mathcal{D}^{n_2}/I_2$ if and only if it provides $\mathcal{D}$-homomorphisms of V_{I_2} to V_{I_1}.*

(ii) *$\mathcal{D}^{n_1}/I_1$ and $\mathcal{D}^{n_2}/I_2$ are $\mathcal{D}$-isomorphic if and only if V_{I_1} and V_{I_2} are $\mathcal{D}$-isomorphic.*

Proof. (i) Assume that $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$. We need to verify that $A(V_{I_2})^T \subset (V_{I_1})^T$. The latter is equivalent to the equality $I_1 A(V_{I_2})^T = 0$ which holds because of the inclusion $I_1 A \subset I_2$.

Conversely, assume that $A(V_{I_1})^T \subset (V_{I_1})^T$, then as above $I_1 A(V_{I_2})^T = 0$ which implies $I_1 A \subset I_2$ due to the duality in the differential Zariski topology (see [13, Corollary 1, p. 148], also [29]). Hence $(\mathcal{D}^{n_1}/I_1)A \subset (\mathcal{D}^{n_2}/I_2)$.

(ii) Assume that (1) holds. One has to verify (2), i.e. for any $v \in V_{I_1}$ to show that $ABv^T = v^T$. The latter holds if and only if for any $g \in \mathcal{D}^{n_1}$ the equality $gABv^T = gv^T$ is true. Equation (1) entails that $gABv^T = (g + g_0)v^T = gv^T$ for a certain vector $g_0 \in I_1$.

We mention that $\mathcal{D}$-isomorphism of $\mathcal{D}$-modules implies isomorphism of the spaces of their solutions in a more general setting, see e.g. [18,20] (while the converse essentially uses that we deal with a universal differential field).

Conversely, assume (2) is valid. For any $g \in \mathcal{D}^{n_1}$ (2) implies the equality $(gAB - g)(V_{I_1})^T = 0$, therefore $gAB - g \in I_1$ again due to Corollary 1 in [13, p. 148]. This establishes (1). □

Remark 1. Observe that for any two $\mathcal{D}$-modules $I_1 \subset \mathcal{D}^{n_1}$, $I_2 \subset \mathcal{D}^{n_2}$ such that $\dim_F(\mathcal{D}^{n_1}/I_1) = \dim_F(\mathcal{D}^{n_2}/I_2) < \infty$ we have $\mathcal{D}^{n_1}/I_1 \simeq_{\mathcal{D}} \mathcal{D}^{n_2}/I_2$. On the other hand, in case of infinite-dimensional modules the isomorphism does not always hold, e.g., in case $m = 2$ the modules $\mathcal{D}/(d_1)$ and $\mathcal{D}/(d_2)$ are not $\mathcal{D}$-isomorphic.

3. Relative syzygies of $\mathcal{D}$-modules

In Loewy's original decomposition scheme, the largest completely reducible right factors are removed by exact division. This is a valid procedure because all ideals of ordinary differential operators are principal. In the ring of linear partial differential operators (lpdo's) this is not true any more. In addition to the relations following from the division there are the integrability conditions which guarantee that an ideal or module is generated by a Janet base. The proper generalization of the exact quotient is given by the following

Definition 1. *(Relative syzygies module).* Let $I \subseteq J \subseteq \mathcal{D}^n$ be two $\mathcal{D}$-modules, and let $J = \langle g_1, \ldots, g_t \rangle$. The relative syzygies $\mathcal{D}$-module $Syz(I, J)$ of I and J is $Syz(I, J) = \{(h_1, \ldots, h_t\} \in \mathcal{D}^t \,|\, \sum h_i g_i \in I\}$.

This definition is more general than the definition of the quotient of $\mathcal{D}$-modules in [15] because we do not require $g_1, \ldots, g_t$ to be a Janet basis of J (for a background on Janet basis see e.g. [13,14,23,25]) and in addition it takes into account all relations among $g_1, \ldots, g_t$ which put them in I. We notice that in case when $I = 0$ the module $Syz(0, J)$ coincides with the usual syzygies module $Syz(J)$. Our next goal is to show that Definition 1 does not depend on the choice of generators $g_1, \ldots, g_t$. Another proof may be obtained applying the methods of [20] and [21].

Lemma 1. *Let $I \subseteq I_1 \subseteq J$ be $\mathcal{D}$-modules. Then $Syz(I_1, J)/Syz(I, J) \simeq I_1/I$.*

Proof. First we verify that the mapping $\varphi(h_1, \ldots, h_t) = \sum h_i g_i$ provides a homomorphism $\varphi : Syz(I_1, J)/Syz(I, J) \to I_1/I$ being a monomorphism according to Definition 1. Finally, for any representative $g \in I_1$ of a class $\bar{g} \in I_1/I$ one can write $g = \sum h_i g_i$, then $\varphi(h_1, \ldots, h_t) = g$. $\square$

Corollary 1. (i) $\mathcal{D}^t/Syz(I, J) \simeq J/I$;
(ii) $Syz(I, J)/Syz(J) \simeq I$.

The main goal for introducing the relative syzygies module according to Definition 1 is the following statement proved in [15] in case when $g_1, \ldots, g_t$ being a Janet basis of J, one can find in [21] another proof of it.

Lemma 2. *With the notation above the C-linear spaces $V_{Syz(I,J)}$ and V_I/V_J are isomorphic.*

Proof. The mapping $\psi : v \to (g_1, \ldots, g_t)^T v$ assures the monomorphism $V_I/V_J \hookrightarrow V_{Syz(I,J)} \subset F^t$. To establish that it is an epimorphism, suppose first that $g_1, \ldots, g_t$ constitute a Janet basis of J. Let $y = (y_1, \ldots, y_n)$ be a vector of differential indeterminates. For any vector $(w_1, \ldots, w_t)^T \in V_{Syz(I,J)}$ the system of linear pde's $g_i \bar{y} = w_i$, $1 \leqslant i \leqslant t$ is solvable since $\{g_1 y - w_1, \ldots, g_t y - w_t\}$ is a linear coherent autoreduced set, see [13, p. 136], also [14, Theorem 5.5.6, p. 247] and [15]. Taking any $f \in I$ one can represent $f = \sum h_i g_i$, then

$(h_1,\dots,h_t) \in Syz(I,J)$ and $0=\sum h_i w_i = f\bar{y}$, thus $\bar{y}\in V_I$. This completes the proof that $\psi: V_I/V_J \simeq V_{Syz(I,J)}$ is an isomorphism.

To get rid of the supposition that $g_1,\dots,g_t$ constitute a Janet basis take an arbitrary set $g_1^{(1)},\dots,g_{t_1}^{(1)}$ of generators of J and construct the syzygies module $Syz(I,J)^{(1)} \subset \mathcal{D}^{t_1}$; the notation $Syz(I,J)^{(1)}$ is used to distinguish it from the syzygies module $Syz(I,J)$ constructed from a Janet basis $g_1,\dots,g_t$. Take a $t\times t_1$ matrix $W=(w_{i,j})$ over $\mathcal{D}$ such that $(g_1,\dots,g_t)^T = W(g_1^{(1)},\dots,g_{t_1}^{(1)})^T$ and a $t_1\times t$ matrix $W^{(1)}=(w_{i,j}^{(1)})$ such that $(g_1^{(1)},\dots,g_{t_1}^{(1)})^T = W(g_1,\dots,g_t)^T$. We claim that the matrices W, $W^{(1)}$ provide a $\mathcal{D}$-isomorphism from $\mathcal{D}^t/Syz(I,J)$ to $\mathcal{D}^{t_1}/Syz(I,J)^{(1)}$ (see Section 2). Indeed, for the element $(h_1,\dots,h_t)\in\mathcal{D}^t$ we have

$$(h_1,\dots,h_t)WW^{(1)} = \left(\dots, \sum_{1\leqslant i\leqslant t,\ 1\leqslant j\leqslant t_1} h_i w_{i,j} w_{j,k}^{(1)}, \dots\right)$$

where the expression is given for the kth coordinate of the vector. Hence

$$\big((h_1,\dots,h_t)WW^{(1)}, (g_1,\dots,g_t)^T\big) = \big((h_1,\dots,h_t),(g_1,\dots,g_t)^T\big),$$

in particular, $(h_1,\dots,h_t)WW^{(1)} - (h_1,\dots,h_t) \in Syz(I,J)$. The dual calculation for the product $W^{(1)}W$ proves the claim. Proposition 1 entails that $V_{Syz(I,J)} \simeq_{\mathcal{D}} V_{Syz(I,J)^{(1)}}$. Together with the isomorphism ψ this completes the proof. □

Remark 2. As usual, having Janet bases of $I=\langle f_1,\dots,f_s\rangle$ and of $J=\langle g_1,\dots,g_t\rangle$ one can construct a Janet basis of $Syz(I,J)$, e.g. cf. [14, Theorem 5.3.7], also [15]. Briefly to remind, for each f_j there holds $f_j=\sum h_{i,j}g_i$, $1\leqslant j\leqslant s$ for certain $h_{i,j}\in\mathcal{D}$. Furthermore, for each pair (k,j) with $1\leqslant k<j\leqslant t$ we represent the Δ-polynomial of g_k and g_j as $lc(g_j)\theta_1 g_k - lc(g_k)\theta_2 g_j = \sum h_{ijk} g_i$ such that the operators $lc(g_j)\theta_1 g_k$ and $lc(g_k)\theta_2 g_j$ have the same leading terms with the minimal possible leading derivative w.r.t. the applied term ordering $\prec$. Then the basis of $Syz(I,J)$ consists of the vectors $(h_{1,j},\dots,h_{t,j})$, $1\leqslant j\leqslant s$, and of the vectors

$$\big(h_{1jk},\dots,h_{kjk}-lc(g_j)\theta_1,\dots,h_{jjk}-lc(g_k)\theta_2,\dots,h_{tjk}\big),\quad 1\leqslant k<j\leqslant t. \tag{3}$$

In the special case $I=0$, the relative syzygies module $Syz(0,J)$ reduces to the syzygies module of J. Then as in Schreyer's theorem [4, p. 212], one can show that the constructed basis of $Syz(0,J)$ which consists of vectors of the form (3), constitutes in fact, a Janet basis.

We mention also that relying on the algorithm from [8] one can produce a basis of $Syz(I,J)$ starting with arbitrary, not necessarily Janet bases, of I and J, with double-exponential complexity.

Let us denote by H_I the Hilbert–Kolchin polynomial of I w.r.t. the usual filtration by order of derivatives, so $(\mathcal{D}^n)_r=\{f\in\mathcal{D}^n\colon\ ord\,f\leqslant r\}$ (cf. [14, p. 223]). The degree $deg(H_I)$ of H_I is called the *differential type* of [13, p. 130], and [14, p. 229], and its leading coefficient $lc(H_I)$ is called the *typical differential dimension* of I ibid.

The next theorem can be deduced directly from Theorem 5.2.9 in [14], but we give an independent proof following the arguments from [14], cf. also [29, Theorem 4.1].

Theorem 1. *Let again $I\subseteq J\subseteq\mathcal{D}^n$. Then $deg(H_J)\leqslant deg(H_I)$, $deg(H_{Syz(I,J)})\leqslant deg(H_J)$ and $deg(H_{Syz(I,J)})=deg(H_I-H_J)$, $lc(H_{Syz(I,J)})=lc(H_I-H_J)$.*

Proof. We recall that the isomorphism $\varphi : \mathcal{D}^t / Syz(I, J) \hookrightarrow J/I$ in Lemma 1 (putting $I_1 = J$) maps $h_1, \dots, h_t$ to $\sum h_i g_i$. Let $ord(g_i) \leqslant p$, $1 \leqslant i \leqslant t$. Since we have in the filtration $(J/I)_r = J_r/I_r$, $r \geqslant 0$ (cf. [14, Theorem 5.1.8]) we obtain that $\varphi((\mathcal{D}^t/Syz(I, J))_r) \subseteq (J/I)_{r+p}$ and thereby

$$H_{Syz(I,J)}(r) = dim_F\big(\mathcal{D}^t/Syz(I, J)\big)_r \leqslant dim_F(J/I)_{r+p} = H_I(r+p) - H_J(r+p)$$

for sufficiently large r.

Conversely, assuming w.l.o.g. that $g_1, \dots, g_t$ constitute a Janet basis of J we conclude that for any $g \in (J/I)_r$ one can represent $g = \sum h_i g_i$ with $ord(h_i g_i) \leqslant r$, $1 \leqslant i \leqslant t$ and hence

$$H_I(r) - H_J(r) = dim_F V_{(J/I)_r} \leqslant dim_F V_{(\mathcal{D}^t/Syz(I,J))_r} = H_{Syz(I,J)}(r)$$

for sufficiently large r. □

Definition 2 *(Gauge of a Module).* Let I be a $\mathcal{D}$-module. We call the pair $(deg(H_I), lc(H_I))$ the gauge of I. We say that a module I_1 is of lower gauge than another one I_2 if the pair $(deg(H_I), lc(H_{I_1}))$ is less than $(deg(H_{I_2}), lc(H_{I_2}))$ in the lexicographic ordering.

Remark 3. The construction of the relative syzygies allows one to reduce finding a basis of V_I to finding a basis of V_J and joining it with any solution y of the system $g_i y = w_i$, $1 \leqslant i \leqslant t$ (see the proof of Lemma 2) for each element $(w_1, \dots, w_t)$ of a basis of $V_{Syz(I,J)}$. An algorithm for solving the inhomogeneous system $g_i y = w_i$ may be obtained by a proper generalization of Lagrange's variation of constants, see e.g. the textbook [31, pp. 193–195] if the homogeneous system is known to have a finite-dimensional solution space which will always be the case in our applications. Theorem 1 implies that both J and $Syz(I, J)$ have gauges not greater than the gauge of I. Moreover, in the applications in the next section, the gauges of J and $Syz(I, J)$ will be actually lower than the gauge of I. In case of a finite-dimensional ideal I this reduction was exploited in [15].

4. Loewy decompositions

Let us first study the case of a finite-dimensional module $I \subset \mathcal{D}^n$, i.e. modules of differential type 0. Consider the intersection $R(I) = J^{(0)} = \cap J$ of all maximal modules $J \supseteq I$. Any intersection of maximal modules will be called a *complete intersection*. $R(I)$ plays a role similar to the role of the radical of two-sided ideals in a ring. Note that there exists a finite number of maximal modules $J_1, \dots, J_q$ for which $J_1 \cap \cdots J_q = R(I)$. Indeed, keep taking $J_1, J_2, \dots$ while it is possible to have $dim_C V_{J_1 \cap \cdots \cap J_{i+1}} > dim_C V_{J_1 \cap \cdots \cap J_i}$ for every $i \geqslant 1$. Since $dim_C V_I < \infty$ we arrive finally at $J_1, \dots, J_q$ such that $dim_C V_{J_1 \cap \cdots \cap J_q \cap J} = dim_C V_{J_1 \cap \cdots \cap J_q}$ for any maximal module $J \supseteq I$. Then $J_1 \cap \cdots \cap J_q = R(I)$. Applying this procedure to the relative syzygies module $I^{(1)} = Syz(I, J^{(0)})$, replacing the role of I, which one can compute making use of Remark 2, this yields a complete intersection $J^{(1)}$ such that $J^{(1)} = R(I^{(1)}) \supseteq I^{(1)}$. Continuing this way, one obtains successively the complete intersections $J^{(0)}, J^{(1)}, \dots, J^{(s)}$, and the modules $I^{(1)}, \dots, I^{(s)}$ such that $J^{(l)} = R(I^{(l)})$ and $I^{(l+1)} = Syz(I^{(l)}, J^{(l)})$ for $0 \leqslant l \leqslant s-1$, defining $I^{(0)} = I$. In the last step there holds $J^{(s)} = I^{(s)}$. We have $dim_C V_{I^{(l)}} - dim_C V_{I^{(l+1)}} = dim_C V_{J^{(l)}}$ for $0 \leqslant l \leqslant s$, defining $V_{I^{(s+1)}} = \{0\}$. Thus, $dim_C V_I = \sum_{0 \leqslant l \leqslant s} dim_C V_{J^{(l)}}$, which provides an upper bound $s < dim_C V_I$ on the number of steps of the described procedure. The uniquely defined sequences $J^{(0)}, J^{(1)}, \dots, J^{(s)}$ and $I^{(1)}, \dots, I^{(s)}$ can be viewed as a *Loewy decomposition* of I. To get the spaces of solutions $V_{J^{(l)}}$, $0 \leqslant l \leqslant s$ of the complete intersections $J^{(l)} = \cap_q J_q^{(l)}$ where $J_q^{(l)}$ are maximal modules, we apply Proposition 3.1 [29] (see also the beginning of the proof of Theorem 4.1 in [29, p. 483] and also [3]) which entails that $V_{J^{(l)}} = \sum_q V_{J_q^{(l)}}$. Thus, we have proved the following proposition.

Proposition 2. *Any finite-dimensional module $I \subset \mathcal{D}^n$ has the unique Loewy decomposition.*

Now we proceed to a Loewy decomposition of an infinite-dimensional module $I \subset \mathcal{D}^n$ of differential type $\tau > 0$. To this end, we introduce another concept first.

Definition 3 *(Gauge-equivalence).* We say that two modules $J_1, J_2 \subset \mathcal{D}^n$ are gauge-equivalent if J_1, J_2 and $J_1 \cap J_2$ are of the same gauge.

If J_1 and J_2 are gauge-equivalent, then by Theorem 4.1 in [29] also $J_1 + J_2$ is of the same gauge.

Lemma 3. (i) *Two modules $J_1 \subseteq J_2$ of differential type τ are gauge-equivalent if and only if $deg(H_{J_1} - H_{J_2}) < \tau$;*

(ii) *if each of two modules $J_1, J_2 \subseteq J$ is gauge-equivalent to J then $J_1 \cap J_2$ is also gauge-equivalent to J;*

(iii) *gauge-equivalence is an equivalence relation.*

Proof. (i) follows from Definition 2.

(ii) Since we have in the filtration $(J_1 \cap J_2)_r = (J_1)_r \cap (J_2)_r$ (cf. Section 3) we get for Hilbert–Kolchin polynomials that $H_{J_1 \cap J_2} - H_J \leqslant (H_{J_1} - H_J) + (H_{J_2} - H_J)$ (the inequality for polynomials means the inequality for their values at sufficiently big integer points), which proves (ii).

To prove (iii) assume that each of two modules J_1, J_3 of differential type τ is gauge-equivalent to J_2. Then each of two modules $J_1 \cap J_2$, $J_3 \cap J_2$ is gauge-equivalent to J_2. From (ii) we deduce that $J_1 \cap J_2 \cap J_3$ is gauge-equivalent to J_2. Hence (i) entails that $deg(H_{J_1 \cap J_2 \cap J_3} - H_{J_2}) < \tau$. On the other hand, the assumption and (i) imply that $deg(H_{J_1 \cap J_2} - H_{J_1}) < \tau$, $deg(H_{J_1 \cap J_2} - H_{J_2}) < \tau$, therefore $deg(H_{J_1 \cap J_2 \cap J_3} - H_{J_1}) < \tau$. The latter inequality and the inclusions $J_1 \cap J_2 \cap J_3 \subseteq J_1 \cap J_3 \subseteq J_1$ entail that $deg(H_{J_1 \cap J_3} - H_{J_1}) < \tau$. Together with a similar inequality $deg(H_{J_1 \cap J_3} - H_{J_3}) < \tau$ this completes the proof of (iii). □

The equivalence class of gauge-equivalent modules of a module J is denoted by $[J]$. If the actual value of the differential type of the elements of a class $[J]$ equals to τ, any two members of it are called *τ-equivalent* (below τ is fixed and $|J|$ means a class of τ-equivalence).

Example 1. Let $J_1 = \langle \partial_x \rangle$, $J_2 = \langle \partial_{xx}, \partial_{xy} \rangle$ and $J_3 \langle \partial_y \rangle$. Then $J_1 \cap J_2 = J_2$, $J_1 + J_2 = J_1$ all of which are of gauge $(1, 1)$. Consequently J_1 and J_2 are gauge-equivalent. Notice that although J_3 is also of gauge $(1, 1)$, it is not gauge-equivalent to J_1 because $J_1 \cap J_3 = \langle \partial_{xy} \rangle$ which is of gauge $(1, 2)$.

The generic solution of J_1 is $F(y)$, where F is an "undetermined function," whereas J_2 has generic solution $Cx + F(y)$, C being a generic constant. The generic solution here and below is defined with the help of the defining ideal (see e.g. [13, p. 146] and [14, p. 132]) as follows. For a set of elements of a differential field its defining ideal consists of all lpdo's which annihilate them. A solution of an ideal J is generic if its defining ideal coincides with J. Then above C is a generic constant, i.e. the defining ideal of C coincides with $\langle\partial_x, \partial_y\rangle$, the defining ideal of F is J_1 and the defining ideal of $Cx + F$ coincides with J_2.

We say that $[J_1]$ is *subordinated* to $[J_2]$ if $J_1 \cap J_2$ is τ-equivalent to J_1.

Lemma 4. (i) *If modules J_1, J_1' are τ-equivalent, J_2, J_2' are also τ-equivalent and moreover, $J_1 \cap J_2$ has differential type τ, then $J_1 \cap J_2$, $J_1' \cap J_2'$ are τ-equivalent as well*;

(ii) *under the same assumption $J_1 + J_2$, $J_1' + J_2'$ are τ-equivalent*;

(iii) *the relation of subordination is independent of a choice of representatives J_1, J_2 of the classes of τ-equivalence.*

Proof. (i) From the inclusions $J_1 \cap J_1' \cap J_2 \cap J_2' \subseteq J_1 \cap J_1' \cap J_2 \subset J_1 \cap J_2$ and Lemma 3(i), taking into account that $H_{J_2 \cap J_2' \cap J} - H_{J_2 \cap J} \leqslant H_{J_2 \cap J_2'} - H_{J_2}$ for any module J, we conclude (cf. the proof of Lemma 3(iii)) that $deg(H_{J_1 \cap J_1' \cap J_2 \cap J_2'} - H_{J_1 \cap J_2}) < \tau$. Therefore, $J_1 \cap J_1' \cap J_2 \cap J_2'$ has differential type τ, as well as $J_1' \cap J_2'$. In a similar way one obtains that $deg(H_{J_1 \cap J_1' \cap J_2 \cap J_2'} - H_{J_1' \cap J_2'}) < \tau$. Then (i) follows from Lemma 3(i), (iii).

(ii) From the inclusions $J_1 + J_2 \subseteq J_1 + J_1' + J_2 \subseteq J_1 + J_1' + J_2 + J_2'$ and Lemma 3(i), taking into account the inequality $H_{J_1+J} - H_{J_1+J_1'+J} \leqslant H_{J_1} - H_{J_1+J_1'}$ for any module J, we conclude (as in the proof of (i)) that $deg(H_{J_1+J_2} - H_{J_1+J_1'+J_2+J_2'}) < \tau$. Since $J_1 + J_2$ has differential type τ due to Theorem 4.1 in [29], $J_1 + J_1' + J_2 + J_2'$ has also differential type τ, as well as $J_1' + J_2'$. In a similar way one obtains that $deg(H_{J_1'+J_2'} - H_{J_1+J_1'+J_2+J_2'}) < \tau$. Then (ii) follows from Lemma 3(i), (iii).

(iii) Under the assumption of (i) and making use of that J_1 is τ-equivalent to $J_1 \cap J_2$ (thereby, the assumption that $J_1 \cap J_2$ has differential type τ, is fulfilled automatically), we obtain (iii) due to Lemma 3(iii). □

Remark 4. The proof of (i) shows that $J_1 \cap J_2$, $J_1' \cap J_2'$ are gauge-equivalent without the assumption that $J_1 \cap J_2$ has differential type τ because the differential type of $J_1 \cap J_2$ is greater or equal to τ.

We denote the relation of subordination by $[J_1] \trianglelefteq [J_2]$. Then $lc(H_{J_1}) \geqslant lc(H_{J_2})$. If in addition $[J_1] \neq [J_2]$ (we denote this by $[J_1] \triangleleft [J_2]$) then $lc(H_{J_1}) > lc(H_{J_2})$. Hence any increasing chain of τ-equivalence classes stops and one can consider maximal τ-equivalence classes.

For any τ-equivalence classes $[J_1]$, $[J_2]$ satisfying $[J] \trianglelefteq [J_1]$, $[J] \trianglelefteq [J_2]$ one can uniquely define the class $[J_1 \cap J_2]$ such that $[J] \trianglelefteq [J_1 \cap J_2]$. One can verify that $deg(H_{J_1 \cap J_2}) = \tau$ and the class $[J_1 \cap J_2]$ does not depend on the representatives J_1, J_2.

Example 2. Let $J = \langle\partial_{xyy}\rangle$ with gauge $(1, 3)$, $J_1 = \langle\partial_x\rangle$ and $J_2 = \langle\partial_y\rangle$, both with gauge $(1, 1)$. Because $J \cap J_1 = J \cap J_2 = J$ there holds $[J] \trianglelefteq [J_1]$ and $[J] \trianglelefteq [J_2]$. Furthermore $J_1 \cap J_2 = \langle\partial_{xy}\rangle \equiv J_3$ with gauge $(1, 2)$ and $[J] \trianglelefteq [J_3]$. Because $lc(H_J) = 3$, $lc(H_{J_3}) = 2$ and $lc(H_{J_1}) = lc(H_{J_2}) = 1$, both $[J_1]$ and $[J_2]$ are maximal.

Now take all τ-maximal classes $[J]$ such that $[I] \trianglelefteq [J]$. Since $J + I$ is τ-equivalent to J (again due to Theorem 4.1 in [29]) we can assume without loss of generality that the representatives are chosen in such a way that $I \subseteq J$. We choose consecutively such classes $[J_1], [J_2], \ldots, [J_p]$ while it is possible to have

$$[J_1] \rhd [J_1 \cap J_2] \rhd \cdots \rhd \ J^{(0)} = J_1 \cap J_2 \cap \cdots \cap J_p \ .$$

Clearly, $p \leqslant lc(H_I)$. Then for any maximal class $[J]$ for which $[I] \unlhd [J]$, we obtain $[J^{(0)}] \unlhd [J]$. Hence for any finite family $[J_1'], \ldots, [J_q']$ of τ-maximal classes for which $[I] \unlhd [J_l']$, $1 \leqslant l \leqslant q$, we conclude that $[J^{(0)}] \unlhd [J_1' \cap \cdots \cap J_q']$. Therefore, the class $[J^{(0)}]$ is defined uniquely and in addition $I \subseteq J^{(0)}$ holds. We say that $J^{(0)} = J_1 \cap J_2 \cap \cdots \cap J_p$ is *completely τ-reducible*.

We define a Loewy decomposition of I by induction on the gauge of I. As a base of induction when the τ-class $[I]$ is maximal then I provides a Loewy decomposition of itself. When $[I]$ is not maximal one can further apply the described inductive definition of a Loewy decomposition (thereby, replacing the role of I) to the relative syzygies module $I^{(1)} = Syz(I, J^{(0)})$ (see Section 3) taking into account that either $deg(H_{I^{(1)}}) < \tau$ or $deg(H_{I^{(1)}}) = \tau$, and in the latter case $lc(H_{I^{(1)}}) = lc(H_I) - lc(H_{J^{(0)}}) < lc(H_J)$ due to Theorem 1; in other words, $I^{(1)}$ is of a lower gauge than I. In case when $deg(H_{I^{(1)}}) < \tau$ we have $[I] = [J^{(0)}]$ again due to Theorem 1 and $[I]$ being completely τ-reducible.

Continuing this way we arrive at a sequence of modules $J^{(0)}, J^{(1)}, \ldots, J^{(q)}$ with non-decreasing differential types such that each module $J^{(l)}$, $0 \leqslant l \leqslant q$ is completely $deg(H_{J^{(l)}})$-reducible. We notice that this sequence is not necessarily unique unlike the Loewy decomposition of a finite-dimensional module. The obtained sequence could be called a *generalized Loewy decomposition* of I. At present we don't possess an algorithm to construct it in general. The following example due to Blumberg [2] shows the non-uniqueness of the decomposition described above.

Example 3. In his dissertation Blumberg [2] gave the following example of a non-unique factorization of a third-order partial differential operator.

$$\begin{aligned} &\partial_{xxx} + x\partial_{xxy} + 2\partial_{xx} + 2(x+1)\partial_{xy} + \partial_x + (x+2)\partial_y \\ = \ & \begin{array}{l} (\partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y)(\partial_x + 1) \\ (\partial_{xx} + 2\partial_x + 1)(\partial_x + x\partial_y). \end{array} \end{aligned}$$

The second-order factor in the second line at the right hand side may be written as $\partial_{xx} + 2\partial_x + 1 = lclm(\partial_x + 1, \partial_x + 1 - 1/x)$, i.e. it is completely reducible, the remaining operators are absolutely irreducible. Consequently in the notation introduced above there holds

$$\begin{aligned} J_1^{(0)} &= \langle \partial_x + 1 \rangle, & J_1^{(1)} &= \langle \partial_{xx} + x\partial_{xy} + \partial_x + (x+2)\partial_y \rangle, \\ J_2^{(0)} &= \langle \partial_x + x\partial_y \rangle, & J_2^{(1)} &= \langle \partial_{xx} + 2\partial_x + 1 \rangle. \end{aligned}$$

The result may also be obtained by applying the algorithm from Section 5 of [12].

5. Primary decompositions

At first let $I \subset \mathcal{D}^n$ be a finite-dimensional module. Denote by $J^{(0)} = N(I) = \bigcap_{J \supset I} J$ the intersection of all modules J properly containing I (we mention that $N(I)$ plays a role similar to the role of the nil-radical of two-sided ideals in a ring). We call I *primary* if $N(I) \neq I$. In the latter case $N(I)$ is the minimal module which properly contains I.

Lemma 5. *The relative syzygies module $Syz(I, N(I))$ is a maximal module in $\mathcal{D}^t$ (see Definition 1).*

Proof. Assume the contrary. Due to Corollary 1 there is an isomorphism $\phi : \mathcal{D}^t / Syz(I, J^{(0)}) \to J^{(0)}/I$. Due to the assumption there exists a $\mathcal{D}$-module M such that

$$Syz\big(I, J^{(0)}\big) \subset M \subset \mathcal{D}^t$$

and both quotient modules $M/Syz(I, J^{(0)})$ and $\mathcal{D}^t/M$ are non-zero. Then the quotient modules are the same in the sequence

$$\{0\} \subset \phi\big(M/Syz\big(I, J^{(0)}\big)\big) \subset J^{(0)}/I.$$

Considering the epimorphism $\psi : J^{(0)} \to J^{(0)}/I$, we get one more sequence

$$I \subset \psi^{-1}\big(\phi\big(M/Syz\big(I, J^{(0)}\big)\big)\big) \subset J^{(0)}$$

with the same quotient modules, but this contradicts to the choice of $J^{(0)} = N(I)$, which proves the lemma. □

Lemma 6. *Any finite-dimensional module I is an intersection of a finite number of primary modules.*

Proof. Proof goes by induction on $dim_C(I)$. The base of induction for a maximal module is obvious because it is primary. For the inductive step in case when I is not primary one can represent it as a finite intersection $I = J_1 \cap \cdots \cap J_q$ with $J_i \supset I$ (cf. Section 4). Then lemma follows from the inductive hypothesis applied to J_i. □

Therefore, by recursion on $dim_C(I)$ one can define a *primary decomposition* of I. If I is not primary then one takes $I = J_1 \cap \cdots \cap J_q$ from Lemma 6 and the primary decomposition of I is obtained as the collection of primary decompositions of $J_1, \ldots, J_q$ by the recursive hypothesis. For a primary module I its primary decomposition consists of a pair of the relative syzygies module $Syz(I, N(I))$ (being a maximal module) and a primary decomposition of $N(I)$ by the recursive hypothesis. One can make a primary decomposition unique taking in the intersection $I = J_1 \cap \cdots \cap J_q$ all the primary modules $J_i \supset I$ being minimal with respect to the inclusion. Although, it is unclear whether one could take $I = J_{i_1} \cap \cdots \cap J_{i_{q'}}$ uniquely for a proper subfamily $1 \leqslant i_1 < \cdots < i_{q'} \leqslant q$. Thus, we have proved the following proposition.

Proposition 3. *Any finite-dimensional module $I \subset \mathcal{D}^n$ has a unique primary decomposition.*

One can view as an advantage of a primary decomposition versus the Loewy decomposition from Section 4 that the expensive operation of taking the relative syzygies module leads to a maximal module, and so taking relative syzygies modules do not iterate each other. On the other hand, the primary decomposition still allows one to find the space V_I by combining the already cited result from [3,29], and also obtaining V_I from V_J and the space V_I/V_J (see Remark 3).

Now let $I \subset \mathcal{D}^n$ be a $\mathcal{D}$-module of differential type τ. We follow the notations from Section 4. We choose consecutively classes $[J_1], [J_2], \ldots, [J_p]$ such that $[J_i] \rhd [I]$ for each i (again one can assume that $I \subseteq J_i$) while it is possible such that

$$[J_1] \rhd [J_1 \cap J_2] \rhd \cdots \rhd \ J^{(0)} = J_1 \cap J_2 \cap \cdots \cap J_p \ .$$

Then for any class $[J]$ for which $[I] \lhd [J]$ we have $[J^{(0)}] \unlhd [J]$. We denote $[J^{(0)}] = N_\tau([I])$. If $N_\tau([I]) \rhd [I]$ we call $[I]$ τ*-primary*.

We define a *primary decomposition* of I by induction on the gauge of I. For the base of induction when $[I]$ is a maximal τ-equivalence class then I constitutes its own primary decomposition. For the inductive step a primary decomposition of I consists of the ones of the modules $J_1, \ldots, J_p$ and in addition of the relative syzygies module $Syz(I, J^{(0)})$ which has a gauge less than the gauge of I (due to Theorem 1, cf. also Section 4). We observe that when the differential type $dim(Syz(I, J^{(0)})) = \tau$ then $Syz(I, J^{(0)})$ is τ-maximal and provides its own primary decomposition (one can prove this similar to Lemma 5). Else $dim(Syz(I, J^{(0)})) < \tau$ and one deals further in the induction with modules of differential types less than τ.

As a result we arrive at a set of modules $\{J\}$ such that each $[J]$ is a $dim(J)$-maximal class, which one can view as a *primary decomposition* of I. It would be interesting to design an algorithm which constructs a primary decomposition.

6. Parametric-algebraic families of $\mathcal{D}$-modules

For the rest of the paper, dealing with the design of algorithms, we assume that the coefficients of the input operators belong to the differential field $F_0 = \overline{\mathbf{Q}}(X_1, \ldots, X_m)$ (cf. Remark 1) with derivatives $d_k = \partial/\partial X_k$, $1 \leqslant k \leqslant m$ and $\mathcal{D}_0 = F_0[d_1, \ldots, d_m]$, $\mathcal{D} = F[d_1, \ldots, d_m]$ where F is a universal extension of F_0.

In the sequel we suppose that all the considered algebraic (affine) varieties $W \subset \overline{\mathbf{Q}}^N$ are given in an efficient way, say as in [6]. Namely, $W = \cup W_j$ where W_j are irreducible over $\mathbf{Q}$ components of W, and the algorithms from [6] represent each W_j (of dimension s) in two following ways.

First, we represent W_j by means of a *generic point*, i.e. an isomorphism

$$\mathbf{Q}(t_1, \ldots, t_s)[\alpha] \simeq \mathbf{Q}(W_j) \tag{4}$$

where $\mathbf{Q}(W_j)$ is the field of rational functions on W_j. The elements $t_1, \ldots, t_s \subset \{Z_1, \ldots, Z_N\}$ constitute a basis of transcendency of $\mathbf{Q}(W_j)$ over $\mathbf{Q}$ which can be taken among the coordinates $Z_1, \ldots, Z_N$ of the affine space $\overline{\mathbf{Q}}^N$. The element $\alpha = \sum_{1 \leqslant l \leqslant N} \alpha_l Z_l$ for suitable integers α_l is algebraic over the field $\mathbf{Q}(t_1, \ldots, t_s)$ with a minimal polynomial $\phi \in \mathbf{Q}(t_1, \ldots, t_s)[Z]$. The algorithms from [6] yield the ingredients of (4) explicitly, in other words, $t_1, \ldots, t_s; \alpha_1, \ldots, \alpha_N; \phi$ and the rational expressions of Z_l via $t_1, \ldots, t_s, \alpha$, i.e. the rational functions of the form $g_l(t_1, \ldots, t_s, Z)/g(t_1, \ldots, t_s)$ where the polynomials $g(t_1, \ldots, t_s), g_l(t_1, \ldots, t_s, Z) \in \mathbf{Q}[t_1, \ldots, t_s, Z]$ being such that the equality $Z_l = g_l(t_1, \ldots, t_s, Z)/g(t_1, \ldots, t_s)$ holds everywhere on W_j.

Second, the algorithms from [6] yield polynomials $h_1, \dots, h_M \in \mathbf{Q}[Z_1, \dots, Z_N]$ such that W_j coincides with the variety of all the points from $\overline{\mathbf{Q}}^N$ which satisfy the system of equations $h_1 = \cdots = h_M = 0$.

The algorithms from [6] allow one to produce the union, intersection, complement of varieties, to get the dimension of W_j, to project a variety (in other words, to eliminate quantifiers), to find all the points of W_j in case when it is finite (i.e. zero-dimensional) or to yield as many points as one wishes in case when W_j is infinite (positive-dimensional). Moreover, one extends these algorithms from varieties to constructive sets, i.e. the unions of the sets of the form $W' \setminus W''$ where W', W'' are varieties (in other terms, constructive sets constitute the boolean algebra generated by all the varieties).

Definition 4. *(Parametric-algebraic $\mathcal{D}$-modules)* We say that a family of $\mathcal{D}$-modules $\mathcal{J} = \{J\} \subset \mathcal{D}^n$ is parametric-algebraic if there is a constructive set $V = \cup V_j \subset \overline{\mathbf{Q}}^N$ for an appropriate N such that $\mathcal{J} = \cup \mathcal{J}_j$ and for any fixed j the following holds. A Janet basis of any $J \in \mathcal{J}_j$ has fixed leading derivatives $lder(J) = lder_j$ and the parametric derivatives $pder(J) = pder_j$, see [15]. Moreover, any element of the Janet basis of J has the form

$$\gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma \qquad (5)$$

where $\gamma_0 \in lder_j$ and $A_\gamma \in \mathbf{Q}(Z_1, \dots, Z_N)(X_1, \dots, X_m)$.

When $(Z_1, \dots, Z_N)$ ranges over the constructive set V_j, the set of linear differential operators of the form (5) for all $\gamma_0 \in lder_j$ ranges over the Janet basis for all modules J from $\mathcal{J}_j$. Thus, we have a bijective correspondence between the points of V_j and the modules (or rather their Janet basis) from $\mathcal{J}_j$.

We rephrase in our terms the following proposition which was actually proved in [15].

Proposition 4. [15] *One can design an algorithm which for any finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}^n$ finds a parametric-algebraic family of all factors of I, i.e. the modules $J \subset \mathcal{D}^n$ such that $I \subset J$.*

Lemma 7. *One can design an algorithm which for a pair of parametric-algebraic families $\mathcal{I}$, $\mathcal{J}$ of $\mathcal{D}$-modules yields the parametric-algebraic family of all the pairs (I, J) where $I \in \mathcal{I}$, $J \in \mathcal{J}$ such that $I \subseteq J$.*

Proof. Let

$$\left\{ \gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma \right\}_{\gamma_0 \in lder_j}$$

be a Janet basis of $\mathcal{J}_j$ and

$$\left\{ \lambda_0 + \sum_{\lambda \in pder_s} B_\lambda(Z_1, \dots, Z_N)\lambda \right\}_{\lambda_0 \in lder_s}$$

be a Janet basis of $\mathcal{I}_s$. Then the condition that $I \subseteq J$ for $I \in \mathcal{I}_s$, $J \in \mathcal{J}_j$ can be expressed as the existence for each $\lambda_0 \in lder_s$ of operators of the form $\sum_\theta C_{\theta,\gamma_0,\lambda_0}\theta \in \mathcal{D}$ where $\theta \prec \theta_0$ and $\lambda_0 = \theta_0 y_i$ for a certain $1 \leqslant i \leqslant n$ such that

$$\lambda_0 + \sum_{\lambda \in pder_s} B_\lambda(Z_1, \dots, Z_N)\lambda = \sum_{\gamma_0 \in lder_j} \Big(\sum_\theta C_{\theta,\gamma_0,\lambda_0}\theta\Big)\Big(\gamma_0 + \sum_{\gamma \in pder_j} A_\gamma(Z_1, \dots, Z_N)\gamma\Big) \tag{6}$$

where the external summation in the right-hand side ranges over the elements of the Janet basis of $\mathcal{J}_j$.

Clearly, one can rewrite (6) as a system of linear (algebraic) equations, in the unknowns $C_{\theta,\gamma_0,\lambda_0}$, the entries of which are the rational functions from $\mathbf{Q}(X_1, \dots, X_m)(Z_1, \dots, Z_N)$. One can find the constructive set $U = U_{j,s} \subset \overline{\mathbf{Q}}^N$ such that just for $(Z_1, \dots, Z_N) \in U$ this linear system is solvable. Combining this for all pairs l, s completes the proof of the lemma. □

Corollary 2. *For a finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}^n$ one can find a parametric-algebraic family $\mathcal{I}_{\max}$ of all maximal $\mathcal{D}$-modules J which contain I.*

Proof. Among the family of all the factors J of I produced in Proposition 4 one can relying on Lemma 7 distinguish all J_0 such that if $J_0 \subseteq J$ then $J_0 = J$ holds. □

7. Constructing Loewy and primary decompositions

Now we are able for a finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}_0^n$ to construct its Loewy (see Section 4) and primary decompositions (see Section 5). First, in order to obtain Loewy decomposition we apply Corollary 2. After that the purpose is to find the intersection $R(I)$ of all maximal modules from $\mathcal{I}_{\max}$. To this end we conduct the (internal) recursion on $dim(R(I))$. Assume that a current (complete) intersection J_0 of several maximal modules from $\mathcal{I}_{\max}$ is already constructed. Applying Lemma 7 we test whether there exists a maximal module $J \in \mathcal{I}_{\max}$ which does not contain J_0. Then we replace J_0 by the (complete) intersection $J \cap J_0$ and continue the (internal) recursion. Finally, we arrive at $R(I)$ and thereupon (by the external recursion) proceed to the relative syzygies module $Syz(I, R(I))$ (see Section 3), provided that the latter is not zero, else halt.

In order to construct a primary decomposition of I we use Proposition 4 and Lemma 7 in a similar way and (by the internal recursion) compute the intersection $N(I)$ of all modules strictly containing I in the form $N(I) = \cap J$ where the latter intersection is finite. Thereupon we proceed (by the external recursion) to primary decompositions of all non-maximal J from this intersection joined by the relative syzygies module $Syz(I, N(I))$ (provided that the latter does not vanish). If all J are maximal then halt.

Thus, we have shown the following

Corollary 3. *For a finite-dimensional $\mathcal{D}$-module $I \subset \mathcal{D}_0^n$ one can construct its Loewy and primary decompositions.*

8. Testing isomorphism of finite-dimensional $\mathcal{D}$-modules

We follow the notations of Section 2. We assume that the field of constants $C \subset F$ coincides with $\overline{\mathbf{Q}}$ and the modules $I_1 \subset \mathcal{D}^{n_1}$, $I_2 \subset \mathcal{D}^{n_2}$ are defined over the field $F_0 = \overline{\mathbf{Q}}(X_1, \dots, X_m)$. We design an algorithm to test whether $\mathcal{D}^{n_1}/I_1 \simeq_{\mathcal{D}_0} \mathcal{D}^{n_2}/I_2$. W.l.o.g. one can suppose that $dim(\mathcal{D}^{n_1}/I_1) = dim(\mathcal{D}^{n_2}/I_2) = l$. Then $dim_C V_{I_1} = dim_C V_{I_2} = l$ [13, p. 151]. Let $I_1 = \langle g_1, \dots, g_q \rangle$, $I_2 = \langle f_1, \dots, f_p \rangle$ be Janet bases of I_1 and I_2 respectively. The condition that a matrix $A = (a_{i,j})$ with $a_{i,j} \in \mathcal{D}$ provides a $\mathcal{D}$-homomorphism of $\mathcal{D}^{n_1}/I_1$ and $\mathcal{D}^{n_2}/I_2$ can be expressed as a system

$$\sum_{1 \leqslant i \leqslant n_1} g_{s,i} a_{i,j} = \sum_{1 \leqslant t \leqslant p} h_{s,t} f_{t,j}, \quad 1 \leqslant s \leqslant q,\ 1 \leqslant j \leqslant n_2 \tag{7}$$

of lpde's with unknowns $a_{i,j}, h_{s,t} \in \mathcal{D}$. Since $a_{i,j}$ are taken modulo f_j, $1 \leqslant j \leqslant p$ one can assume $a_{i,j}$ to be reduced modulo f_j, $1 \leqslant j \leqslant p$. Let $\mathrm{ord}(g_s), \mathrm{ord}(f_j) \leqslant r$, $1 \leqslant s \leqslant q$, $1 \leqslant j \leqslant p$, then $\mathrm{ord}(a_{i,j}) \leqslant r$, $\mathrm{ord}(g_{s,i} a_{i,j}) \leqslant 2r$ and therefore $\mathrm{ord}(h_{s,t} f_{t,j}) \leqslant 2r$ as well because $f_1, \dots, f_p$ is a Janet basis. Thus writing $a_{i,j} = \sum_K a_{i,j,K} d^K$, $h_{s,t} = \sum_K h_{s,t,K} d^K$ with the weights of multiindices $|K| \leqslant 2r$, one can treat (7) as a system of lpde's in the indeterminates $a_{i,j,K}$ and $h_{s,t,K}$.

By virtue of Proposition 1 the matrix A provides a C-linear transformation of l-dimensional C-vector spaces $(V_{I_2})^T$, $(V_{I_1})^T$. If A provides a zero transformation then $\mathcal{D}^{n_1} A \subset I_2$ due to the duality in the Zariski topology (cf. the proof of Proposition 1). Hence the $a_{i,j}$-components of all solutions of (7) constitute a C-linear subspace of $l \times l$ matrices representing C-linear transformations between $(V_{I_2})^T$ and $(V_{I_1})^T$. In other words, the space of all $\mathcal{D}$-isomorphisms $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{n_1}/I_1, \mathcal{D}^{n_2}/I_2)$ can be viewed as a C-linear subspace of $l \times l$ matrices over C (this generalizes the considerations of the ordinary case $m = 1$, see [19, pp. 42–44]).

In terms close to Definition 4 $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{n_1}/I_1, \mathcal{D}^{n_2}/I_2)$ can be represented as *parametric-linear* family $A(Z) = (a_{i,j}(Z))$ where the parameters $Z = (\{Z_u\}_{1 \leqslant u \leqslant N})$ range over the space C^N, and $a_{i,j}(Z)$ depend on Z linearly.

The algorithm finds this parametric-linear family $A(Z)$ by producing a Janet basis of system (7). We have already established that $A(Z)$ lies in a finite-dimensional C-vector space of dimension at most $N \leqslant l^2$, therefore one obtains from the Janet basis an ideal of $A(Z)$ and thereupon making use of [15, p. 448], finds a basis of all rational solutions $A(Z)$ over the field F_0. Slightly changing the notation, we keep the notation $A(Z)$ for the parametric-linear family of all elements from $\mathrm{Hom}_{\mathcal{D}_0}(\mathcal{D}_0^{n_1}/I_1, \mathcal{D}_0^{n_2}/I_2)$ with rational coefficients, in other words, $\mathcal{D}_0$-homomorphisms.

In a similar way the algorithm yields a parametric-linear family $B(Z')$ of all the elements from $\mathrm{Hom}_{\mathcal{D}_0}(\mathcal{D}_0^{n_2}/I_2, \mathcal{D}_0^{n_1}/I_1)$ with rational coefficients. Then $\mathcal{D}_0^{n_1}/I_1$ and $\mathcal{D}_0^{n_2}/I_2$ are $\mathcal{D}_0$-isomorphic if and only if there exist elements of the form $A = A(Z)$, $B = B(Z')$ such that $AB|_{\mathcal{D}_0^{n_2}/I_2} = \mathrm{id}$, $BA|_{\mathcal{D}_0^{n_1}/I_1} = \mathrm{id}$ that can be rewritten as a system

$$BAe_i - e_i = \sum_{1 \leqslant t \leqslant q} h_t g_t, \quad 1 \leqslant i \leqslant n_1, \qquad ABe'_j - e'_j = \sum_{1 \leqslant s \leqslant p} h'_s f_s, \quad 1 \leqslant j \leqslant n_2 \tag{8}$$

with unknowns h_t, h'_s, where $e_1, \dots, e_{n_1}$ (respectively $e'_1, \dots, e'_{n_2}$) form a basis of the free module $\mathcal{D}_0^{n_1}$ (respectively $\mathcal{D}_0^{n_2}$).

We have already seen that ord(A), ord$(B) \leqslant r$, hence ord$(h_t g_t)$, ord$(h'_s f_s) \leqslant 2r$, taking into account that $g_1, \ldots, g_q$ and $f_1, \ldots, f_p$ constitute Janet bases. Denote $h_t = \sum h_{t,K} d^K$, $h'_s = \sum h'_{s,K} d^K$ where $|K| \leqslant 2r$. Thus one can treat (8) as a parametric linear *algebraic* system in the indeterminates $h_{t,K}$, $h'_{s,K}$ with parameters Z, Z'. One can solve such a parametric system using an algorithm described e.g. in [7]. The algorithm outputs the constructive set of all parameters Z, Z' for which system (8) is solvable, i.e. which provide an isomorphism $A(Z)$, $B(Z')$, in particular this constructive set is not empty if an only if $\mathcal{D}_0^{n_1}/I_1$ and $\mathcal{D}_0^{n_2}/I_2$ are $\mathcal{D}_0$-isomorphic. We summarize the results of the present section in the following theorem.

Theorem 2. *There is an algorithm which finds for any pair of finite-dimensional $\mathcal{D}_0$-modules all $\mathcal{D}_0$-homomorphisms* (*respectively isomorphisms*) *of $\mathcal{D}_0^{n_1}/I_1$ and $\mathcal{D}_0^{n_2}/I_2$ as a parametric-linear* (*respectively parametric-algebraic*) *family. By the same token the algorithm can yield the* (*algebraic*) *groups of all $\mathcal{D}_0$-automorphisms of the $\mathcal{D}_0$-module $\mathcal{D}_0^{n_1}/I_1$.*

It would be interesting to design an algorithm to test $\mathcal{D}_0$-isomorphism or even $\mathcal{D}$-isomorphism of infinite-dimensional $\mathcal{D}$-modules.

9. Conclusion

The results presented in this article allow decomposing modules of partial differential operators into components of lower gauge. If such a decomposition is found, it may be applied to determine the general solution of the corresponding pde, or at least some parts of it. It is highly desirable to develop a similar scheme for larger classes of modules of partial differential operators. Such a scheme, combined with the algorithms described in [11], could very well establish a new subarea in the realm of linear partial differential equations when closed form solutions are searched for.

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