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One-dimensional Anderson localization in certain correlated random potentials

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We study Anderson localization of ultracold atoms in weak, one-dimensional speckle potentials, using perturbation theory beyond Born approximation. We show the existence of a series of sharp crossovers (effective mobility edges) between energy regions where localization lengths differ by orders of magnitude. We also point out that the correction to the Born term explicitly depends on the sign of the potential. Our results are in agreement with numerical calculations in a regime relevant for experiments. Finally, we analyze our findings in the light of a diagrammatic approach.

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I. INTRODUCTION

Anderson localization (AL) of single electron wave functions, first proposed to understand certain metal-insulator transitions, is now considered an ubiquitous phenomenon, which can happen for any kind of waves propagating in a medium with random impurities. It can be understood as a coherent interference effect of waves multiply scattered from random defects, yielding localized waves with exponential profile, and resulting in complete suppression of the usual diffusive transport associated with incoherent wave scattering. So far, AL has been reported for light waves in diffusive media and photonic crystals, sound waves or microwaves. Ultracold atoms have allowed studies of AL in momentum space and recently direct observation of localized atomic matter waves.

In one-dimensional (1D) systems, all states are localized, and the localization length is simply proportional to the transport mean-free path. However, this strong property should not hide that long-range correlations can induce subtle effects in 1D models of disorder, in particular those whose power spectrum has a finite support. Examples are random potentials resulting from laser speckle and used in experiments with ultracold atoms or microwaves. Indeed, by construction, speckles have no Fourier component beyond a certain value $2k_c$, and the Born approximation predicts no back-scattering and no localization for atoms with momentum $hk > bk_c$. This defines an effective mobility edge at $k = k_c$, clear evidence of which has been reported.

Beyond this analysis –relevant for systems of moderate size – study of AL in correlated potentials beyond the effective mobility edge requires more elaborated approaches. In Ref. 21, disorder with symmetric probability distribution was studied, and examples were exhibited, for which exponential localization occurs even for $k > k_c$ although with a much longer localization length than for $k < k_c$. It was also concluded that for Gaussian disorder, there is a second effective mobility edge at $2k_c$, while for non-Gaussian disorder, it is generally not so. These results do not apply to speckle potentials whose probability distribution is asymmetric. Moreover, although speckle potentials are not Gaussian, they derive from the squared modulus of a Gaussian field, and, as we will show, the conclusions of Ref. 21 must be re-examined. Hence, considering speckle potentials presents a twofold interest. First, they form an original class of non-Gaussian disorder which can inherit properties of an underlying Gaussian process. Second, they are easily implemented in experiments with ultracold atoms where the localization length can be directly measured.

Figure 1: (Color online) Lyapunov exponent $\gamma$ calculated two orders beyond the Born approximation for particles in 1D speckle potentials created with a square diffusive plate, versus the particle momentum $hk$ and the strength of disorder $\epsilon_R = 2m\sigma_R V_R/h^2$ ($V_R$ and $\sigma_R$ are the amplitude and correlation length of the disorder). The solid blue lines correspond to $\epsilon_R = 0.1$ and $\epsilon_R = 0.02$. 
In this work, we study AL in speckle potentials beyond the Born approximation, using perturbation theory, numerical calculations, and diagrammatic methods. We find that there exist several effective mobility edges at \( k_{c}^{(p)} = p\kappa \), with integer \( p \), such that AL in the successive intervals \( k_{c}^{(p-1)} < k < k_{c}^{(p)} \) results from scattering processes of increasing order. Effective mobility edges are thus characterized by sharp crossovers in the \( k \) dependence of the Lyapunov exponent (see Fig. 1). We prove this for the first two effective mobility edges by calculating the three lowest-order terms, and give general arguments for any \( p \). In addition, we discuss the effect of odd terms that appear in the Born series due to the asymmetric probability distribution of speckle potentials.

II. SPECKLE POTENTIALS

Let us first recall the main properties of speckle potentials. Optical speckle is obtained by transmission of a laser beam through a medium with a random phase profile, such as a ground glass plate [21]. The resulting complex electric field \( \mathcal{E} \) is a sum of independent random variables and forms a Gaussian process. In such a light field, atoms experience a random potential proportional to the intensity \( |\mathcal{E}|^2 \). Defining the zero of energies so that \( \langle V \rangle = 0 \), the random potential is thus

\[
V(z) = V_n \times (a(z/\sigma_n))^2 - \langle a(z/\sigma_n)^2 \rangle
\]

where the quantities \( a(u) \) are complex Gaussian variables proportional to the electric field \( \mathcal{E} \), and \( \sigma_n \) and \( V_n \) feature characteristic length and strength scales of the random potential (The precise definition of \( V_n \) and \( \sigma_n \) may depend on the model of disorder; see below). In contrast, \( V(z) \) is not a Gaussian variable and its probability distribution is a decaying exponential, i.e. asymmetric. The sign of \( V_n \) is thus relevant and can be either positive or negative for "blue"- and "red"-detuned laser light respectively. However, the random potential \( V(z) \) inherits properties of the underlying Gaussian field \( a(u) \). For instance, all potential correlators \( c_n \) are completely determined by the field correlator \( a(u) \) via

\[
(a_1 a_2 \cdots a_p) = \sum_{\Pi} (a_{1\Pi(1)} a_{2\Pi(2)} \cdots a_{p\Pi(p)}),
\]

where \( a_{p'} = a(z_{p'}/\sigma_n) \) and \( \Pi \) describes the \( p! \) permutations of \( \{1, \ldots, p\} \). Hence, \( c_2(u) = \langle a(u)^2 \rangle \) and defining \( a(u) \) so that \( \langle a(u)^2 \rangle = 1 \), we have \( \sqrt{\langle V(z)^2 \rangle} = \langle V_n \rangle \). Also, since speckle results from interference between light waves of wavelength \( \lambda_L \) coming from a finite-size aperture of angular width \( 2\alpha \), the Fourier transform of the field correlator has no component beyond \( k_c = 2\pi \sin \alpha/\lambda_L \), and \( c_n \) has always a finite support:

\[
\hat{c}_n(q) = 0 \quad \text{for } |q| > k_c, \sigma_n \equiv 1.
\]

As a consequence, the Fourier transform of the potential correlator also has a finite support: \( \hat{c}_2(q) = 0 \) for \( |q| > 2 \).

III. PHASE FORMALISM

Consider now a particle of energy \( E \) in a 1D random potential \( V(z) \) with zero statistical average \( \langle V(z) \rangle \) need not be a speckle potential here]. The particle wave function \( \phi \) can be written in phase-amplitude representation

\[
\phi(z) = r(z) \sin \left[ \theta(z) \right]; \quad \partial_z \phi = k r(z) \cos \left[ \theta(z) \right],
\]

which proves convenient to capture the asymptotic decay of the wave function (here \( k = \sqrt{2mE/\hbar^2} \) is the particle wave vector in the absence of disorder). It is easily checked that the Schrödinger equation is then equivalent to the coupled equations

\[
\partial_z \theta(z) = k \left[ 1 - (E/V(z))/E \right] \sin^2 \left[ \theta(z) \right]
\]

Since Eq. (3) is a closed equation for the phase \( \theta \), it is straightforward to develop the perturbative series of \( \theta \) in increasing powers of \( V \). Reintroducing the solutions at different orders into Eq. (3) yields the corresponding series for the amplitude \( r(z) \) and the Lyapunov exponent:

\[
\gamma(k) = \lim_{|z| \to \infty} \frac{\ln |r(z)|}{|z|} = \sum_{n \geq 2} \gamma(n)(k).
\]

The \( n \)-th-order term \( \gamma(n) \) is thus expressed as a function of the \( n \)-point correlator \( C_n(z_1, \ldots, z_{n-1}) = \langle V(0)V(z_1) \cdots V(z_{n-1}) \rangle \) of the random potential, which we write \( C_n(z_1, \ldots, z_{n-1}) = V_n^n c_n \left( z_1/\sigma_n, \ldots, z_{n-1}/\sigma_n \right) \). Up order \( n = 4 \), we find

\[
\gamma(n) = \frac{\epsilon_n}{k \sigma_n} \int f_n(k \sigma_n)
\]

where \( \epsilon_n = 2m \sigma_n^2 V_n/\hbar^2 \) and

\[
f_2(k) = -\int_{-\infty}^{\infty} du \ c_2(u) \cos(2\kappa u)
\]

\[
f_3(k) = -\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} \ dv \ c_3(u, v) \sin(2\kappa v)
\]

\[
f_4(k) = \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} w \ c_4(u, v, w)
\]

Note that the compact form (4) is valid provided that oscillating terms, which may appear from terms in \( c_4 \) that can be factorized as \( c_2 \) correlators, are appropriately regularized at infinity. Note also that in Eq. (4), the coefficients \( \epsilon_n/k \sigma_n \) diverge for \( k \to 0 \), while the exact \( \gamma(k) \) remains finite for any \( \epsilon_n \). This signals a well-known breakdown of the perturbative approach. Conversely, the perturbative expansion is valid when \( \gamma(k) \ll 1 \) for \( k \to 0 \), i.e. when the localization length exceeds the particle wavelength, a physically satisfactory criterion.
Second, we find a very narrow logarithmic divergence, singularity of the perturbative approach (note that it is a discontinuity of the derivative of $f_3$ at $\kappa = 1$, which signals a singularity of the perturbative approach (note that it does not appear in Fig. 1 due to finite resolution of the plot). Finally, the value $\kappa = 2$ corresponds to the boundary of the support of $f_4$, showing explicitly the existence of a second effective mobility edge at $k = 2\sigma_n^{-1}$. Hence, while $\gamma(k)\sigma_n \sim (\epsilon_n/k\sigma_n)^4$ for $k \lesssim 2\sigma_n^{-1}$, we have $\gamma(k)\sigma_n = O(\epsilon_n/k\sigma_n)^6$ for $k \gtrsim 2\sigma_n^{-1}$, since $f_4(\kappa)$ as well as $f_3(\kappa)$ vanish for $\kappa \gtrsim 2$.

B. Numerics

In order to test the validity of the perturbative approach for experimentally relevant parameters, we have performed numerical calculations using a transfer matrix approach. The results are plotted in Fig. 3 for $\epsilon_n = 0.02$ corresponds to $V_n/h = 2\pi \times 16\text{Hz}$ in Fig. 3 of Ref. [13] and $\epsilon_n = 0.1$ to $V_n/h = 2\pi \times 80\text{Hz}$ in Fig. 3 and to Fig. 4 of Ref. [14]. For $\epsilon_n = 0.02$, the agreement between analytical and numerical results is excellent. The effective mobility edge at $k = \sigma_n^{-1}$ is very clear: we find a sharp step for $\gamma(k)$ of about 2 orders of magnitude. For $\epsilon_n = 0.1$, we find the same trend but with a smoother and smaller step (about one order of magnitude). In this case, although the Born term for $k \lesssim \sigma_n^{-1}$ and the fourth-order term for $k \gtrsim \sigma_n^{-1}$ provide reasonable estimates (within a factor of 2), higher-order terms—which may depend on the sign of $V_n$—contribute significantly.

The contribution of the odd terms can be extracted by taking $\gamma_+ - \gamma_-$, where $\gamma_\pm$ are the Lyapunov exponents obtained for positive and negative disorder amplitude of same modulus $|V_n|$, respectively. As shown in the inset of Fig. 3, the odd terms range from 30% to 70% of the Born term for $0.6 \lesssim k\sigma_n \lesssim 0.9$ and $\epsilon_n = 0.1$, and are of the order of $\gamma_\pm^3$ in weak disorder and away from the divergence at $k = \sigma_n^{-1}$. This shows that the first correction $\gamma_\pm^3$ to the Born term can be relevant in experiments.

For completeness, we have calculated the $f_n(\kappa)$ as the
coefficients of fits in powers of $\epsilon_n/k\sigma_n$ using series of calculations of $\gamma(k)$ at fixed $k$ and various $\epsilon_n$. As shown in Fig. 4 the agreement with the analytic formulas is excellent. In particular, the numerics reproduce the predicted kink at $\kappa = 1/2$. The logarithmic singularity around $\kappa = 1$ being very narrow, we did not attempt to study it.

V. DIAGRAMMATICAL ANALYSIS

Let us finally complete our analysis using diagrammatic methods, which allow us to exhibit momentum exchange in scattering processes as compact graphics, and thus to identify effective mobility edges in a quite general way. In 1D, the localization length can be calculated thus to identify effective mobility edges in a quite general way . In particular, the numerics reproduce the predicted kink at $\kappa = 1/2$. The logarithmic singularity around $\kappa = 1$ being very narrow, we did not attempt to study it.

Many diagrams contribute to order $\epsilon_n^3$. First there are the usual backscattering contributions with pure intensity correlations [Fig. 4(a)-(c)]. Both Figs. 4(a) and 4(c) have a single vertical intensity correlation and vanish for $k > \sigma_n^{-1}$. In contrast, the crossed diagram [4(b)] has two vertical intensity correlation lines and can thus accommodate momenta up to $k = 2\sigma_n^{-1}$. Performing the integration, we find that this diagram reproduces those contributions to $f_4(k)$ for $\kappa \in [1, 2]$. Carrying out the three-loop integration, we recover exactly the non-factorizable contributions to $f_4(k)$ for $\kappa \in [1, 2]$.

VI. CONCLUSION

We have developed perturbative and diagrammatic approaches beyond the Born approximation, suitable to study 1D AL in correlated disorder with possibly asymmetric probability distribution. In speckles, the $k$ dependence of the Lyapunov exponent exhibits sharp crossovers (effective mobility edges) separating regions where AL is due to scattering processes of increasing order. We have shown it explicitly for $k = \sigma_n^{-1}$ and $k = 2\sigma_n^{-1}$, and we infer that there is a series of effective mobility edges at $k = p\sigma_n^{-1}$ with integer $p$ since, generally, diagrams with $2p$ field correlations or $p$ intensity correlations can contribute up to $k = p\sigma_n^{-1}$. This is because, although speckles are not Gaussian, they derive from a Gaussian field. Finally, exact numerics support our analysis for experimentally relevant parameters, and indicate the necessity to use higher-order terms in the Born series, even for $k < \sigma_n^{-1}$. Hence, important features that we have pointed out, such as odd terms in the Born series for $k < \sigma_n^{-1}$ and exponential localization for $k > \sigma_n^{-1}$, should be observable experimentally.

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APPENDIX

Here, we give the explicit formula of the function $f_4(k)$ for a speckle potential created by a square diffusive plate, such that the fourth-order term in the Born expansion of the Lyapunov exponent $\gamma$ reads $\gamma^{(4)} = \sigma_n^{-1} \left( \frac{\epsilon_n}{k\sigma_n} \right)^4 f_4(k\sigma_n)$. The function $f_4(k)$ is the sum of

Figure 4: Relevant fourth-order backscattering contributions. Contrary to the case of uncorrelated potentials [24, 25], the sum of diagrams (a)-(c) does not give zero for speckle potentials; only diagrams (b) and (d) contribute for $k\sigma_n \in [1, 2]$.
three terms with different supports,

\[ f_4(\kappa) = f_4^{[0,1/2]}(\kappa) + f_4^{[0,1]}(\kappa) + f_4^{[1,2]}(\kappa), \]

where \( f_4^{[\alpha,\beta]}(\kappa) \) lives on the interval \( \kappa \in [\alpha, \beta] \), and

\[
\begin{align*}
\frac{f_4^{[0,1/2]}(\kappa)}{\kappa^2} & = -\frac{\pi^3}{16} (1 - 2\kappa) \\
\frac{f_4^{[0,1]}(\kappa)}{\kappa^2} & = \frac{\pi}{64} \left\{ 4 - 6\kappa - \frac{10\pi^2}{3} (1 - 2\kappa) - (4 - 2\kappa) \ln(\kappa) - \left( \frac{5}{\kappa} - 3\pi \right) \ln(1 - \kappa) + \left( \frac{1}{\kappa} + \kappa \right) \ln(1 + \kappa) \right. \\
& \quad - (4 - 8\kappa) \ln^2(\kappa) + 22(1 - \kappa) \ln^2(1 - \kappa) + (18 + 14\kappa) \ln^2(1 + \kappa) \\
& \quad - 16(1 - \kappa) \ln(1 - \kappa) \ln(\kappa) - 4(1 - \kappa) \ln(1 - \kappa) \ln(1 + \kappa) - 32(1 + \kappa) \ln(\kappa) \ln(1 + \kappa) \\
& \quad - 24(1 + \kappa) \text{Li}_2(\kappa) + 32(1 + \kappa) \text{Li}_2 \left( \frac{\kappa}{1 + \kappa} \right) - 8\kappa \text{Li}_2 \left( \frac{2\kappa}{1 + \kappa} \right) - 8(1 - 2\kappa) \text{Li}_2 \left( 2 - \frac{1}{\kappa} \right) \\
\left. + 64 \right\} \\
\frac{f_4^{[1,2]}(\kappa)}{\kappa^2} & = \frac{\pi}{32} \left\{ -2 + \left( 1 + \frac{\pi^2}{3} \right) \kappa + 4\kappa \text{Li}_2(1 - \kappa) - \left( \frac{2}{\kappa} - 2 + \kappa \right) \ln(\kappa - 1) - 2(\kappa - 1) \ln^2(\kappa - 1) + 4\kappa \ln(\kappa - 1) \ln(\kappa) \right\}
\end{align*}
\]

where \( \text{Li}_2(z) = \int_0^z \frac{dt}{t} \ln(1 - t) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \) is the dilogarithm function.

[24] Our results can be extended to any 1D random potential that fulfill conditions (1)-(3) with similar conclusions.