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A 3D DISCRETE DUALITY FINITE VOLUME METHOD FOR NONLINEAR ELLIPTIC EQUATIONS*

YVES COUDIÈRE† AND FLORENCE HUBERT‡

Abstract. Discrete Duality Finite Volume (DDFV) schemes have recently been developed in 2D to approximate on general meshes nonlinear diffusion problems. We propose in this paper a 3D extension of such schemes. The construction of this scheme is investigated. The main properties of the scheme, as well-posedness, error estimates, are also stated.

Key words. Finite-volume methods, Error estimates, Leray-Lions operators.

AMS subject classifications. 35J65, 65N15, 74S10

1. Introduction.

1.1. Nonlinear elliptic equations. In this paper, we are interested in the study of a finite volume approximation of solutions to the nonlinear diffusion problem:

$$-\operatorname{div}(\varphi(z, \nabla u(z))) = f(z), \quad \text{in } \Omega, \quad u = 0, \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^3$. Consider $p \in]1, \infty[$ and $p' = \frac{p}{p-1}$. The flux $\varphi : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ in equation (1.1) is supposed to be a Caratheodory function which is strictly monotonic with respect to $\xi \in \mathbb{R}^3$:

$$(\varphi(z, \xi) - \varphi(z, \eta), \xi - \eta) > 0, \text{ for all } \xi \neq \eta, \text{ for } a.e. \ z \in \Omega\,. \tag{1.2}$$

We also assume that there exist $C_1, C_2 > 0$, $b_1 \in L^1(\Omega)$, $b_2 \in L^{p'}(\Omega)$ such that

$$(\varphi(z, \xi), \xi) \geq C_1|\xi|^p - b_1(z), \text{ for all } \xi \in \mathbb{R}^3, \ a.e.\, z \in \Omega, \tag{1.3}$$

$$|\varphi(z, \xi)| \leq C_2|\xi|^{p-1} + b_2(z), \text{ for all } \xi \in \mathbb{R}^3, \ a.e.\, z \in \Omega. \tag{1.4}$$

These assumptions ensure that $u \mapsto -\operatorname{div}(\varphi(\,\cdot\,, \nabla u))$ is a Leray-Lions operator, and in particular the mapping $G \in (L^p(\Omega))^3 \mapsto \varphi(\cdot, G(\cdot)) \in (L^{p'}(\Omega))^3$ is continuous and Leray Lions [?] proved that

THEOREM 1.1. *Under assumptions* (1.2)*,* (1.3) *and* (1.4)*, for any source term* $f \in W^{-1,p'}(\Omega)$*, the problem* (1.1) *has a unique solution* $u \in W_0^{1,p}(\Omega)$*.*

The homogeneous Dirichlet equation is addressed here for sake of simplicity in the exposition of the scheme. The non homogeneous case can be treated similarly.

1.2. The discrete duality finite volume approaches in 3D. The 2D DDFV method relies on the diamond formula to compute gradients of the unknown u from finite differences in two independent directions, involving four values of u (See [?, ?, ?]). Hence, two finite volumes meshes are needed. They intersect through diamond cells, on which the gradient vectors are computed. Naturally the diamond cells are quadrilateral.

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Three different methods have been proposed in the linear case $\varphi(z,\xi) = G(z)\xi$ ($G(x)$ a symmetric uniformly elliptic matrix). In any case, an additional mesh of dual control volumes is built around the vertexes of the primal mesh and the gradient is piecewise constant on some diamond cells that recover the faces of both the primal and dual control volumes. This approximate gradient yields a natural numerical flux on the faces of the control volumes. The scheme is obtained by integrating eq. (1.1) on both the primal and dual control volumes.

In [?, ?, ?], each diamond cell is composed of a pair of pyramids having as base an interface of the primal mesh and vertexes the two neighboring centers. In [?, ?] a gradient is built from the vertex values of each diamond cell under the condition that the interfaces of the primal mesh are either triangles or quadrangles. This includes locally refined meshes. In [?] the construction is restricted to the case where the primal mesh is a tetraedrization of Ω and verifies an orthogonality constrain. The construction of the dual control volumes is specific to each method. Unlike in the 2D version, the primal and the dual meshes play a different role: in [?] the domain Ω is recovered twice by the dual mesh and in [?] the orthogonality condition means that the dual mesh is the Voronoï mesh associated to the vertexes of the primal mesh.

In [?], diamond cells are constructed in a different way: choosing a point in each face of the mesh, the diamond cell is made of two tetrahedral cells that have a common triangular base with vertexes the endpoints of one edge of the face and the center of the face; and the two neighboring centers as additional vertexes. Two auxiliary unknowns, at the centers of the face and of the edge, are introduced to reconstruct the gradient. With this two additional points, the diamond cell now has 6 vertexes, defining 3 independent directions: between the two new points, between the two neighboring centers and between the to endpoints of the edge. It can be constructed a gradient from the 3 finite differences in these directions. But it remains the auxiliary unknowns to eliminate. F. Hermeline suggests several possibilities to eliminate them. The derived schemes are in general non symmetric. Their convergence seems to be difficult to prove.

Our 3D generalization of the DDFV approach is based on the idea that three finite differences in independent directions are needed to construct a gradient. The diamond mesh constructed in [?] gives naturally these three independent directions. According to our method, the additional unknowns are computed by integrating the equation on a third family of control volumes associated to the new unknowns at the faces and at the edges of the primal mesh. Like in the 2D case, the three meshes play a symmetric role, resulting in a scheme that is quite simple to implement.

Hence, our innovative scheme is based on a three meshes finite volume formulation. The diamond cells have 6 vertexes organized in 3 pairs, defining 3 independent directions in $\mathbb{R}^3$. The approximate gradient is easily obtained by the 3 corresponding finite differences. The scheme is naturally symmetric and easy to implement.

This paper specifies the construction of this 3D DDFV scheme for a nonlinear elliptic equation and states the main properties of this scheme, including some error estimates.

1.3. Outline. The meshes involved in the construction of the scheme are described in section 2. Some discrete divergence $\operatorname{div}^{\mathcal{T}}$ and gradient $\nabla^{\mathcal{T}}$ operators are defined in section 3, that are proved to verify a discrete duality property similar to the Green formula. The approximation scheme for nonlinear elliptic equation (1.1) reads

$$-\operatorname{div}^{\mathcal{T}}(\varphi_{\mathrm{D}}(\nabla^{\mathcal{T}} u^{\mathcal{T}})) = f^{\mathcal{T}}.$$

The main properties of our scheme, well-posedness, *a priori* estimates and some error estimates, are inherited from the discrete duality property and assumptions (1.2), (1.3) and (1.4), as exposed in section 4.

2. Construction of the Meshes. Consider a usual finite volume mesh $\mathcal{M}$ called the primary mesh. We construct two additional finite volumes meshes, with control volumes respectively around the vertexes and the faces and edges of the primary mesh. They are denoted by $\mathcal{N}$ and $\mathcal{FE}$. The diamond cells D are defined in order to contain exactly one interface of each of the three finite volumes meshes, so that three finite differences are available inside D to construct the discrete gradient of u.

The mesh $\mathcal{T}$ is the triple $(\mathcal{M}, \mathcal{N}, \mathcal{FE})$ of meshes on Ω, defined below (see Figures 2.1 and 2.2). We refer as $\mathrm{C} \in \mathcal{T}$ for any of the volumes in $\mathcal{M} \cup \mathcal{N} \cup \mathcal{FE}$.

2.1. The primary mesh. The mesh $\mathcal{M}$ is a set of open disjoint polyhedral control volumes $\mathrm{K} \subset \Omega$ such that $\cup \bar{\mathrm{K}} = \overline{\Omega}$. The interfaces $\bar{\mathrm{K}} \cap \bar{\mathrm{L}}$ of these control volumes[1] are denoted by $\mathrm{F} = \bar{\mathrm{K}} \cap \bar{\mathrm{L}}$ as well as the remaining boundary faces $\mathrm{K} = \partial \mathrm{K} \cap \partial \Omega$. Theses faces are polygons; they are called the *faces of* $\mathcal{M}$. The vertexes of theses faces F are denoted by A and called the *vertexes of* $\mathcal{M}$, while the edges of these faces are called the *edges of* $\mathcal{M}$ and denoted by E.

We associate to each cell K a point $x_{\mathrm{K}} \in \mathrm{K}$, to each face F a point $x_{\mathrm{F}} \in \mathrm{F}$ and finally to each edge E a point $x_{\mathrm{E}} \in \mathrm{E}$. They are for example the isobarycenters of the K, F, E.

For each face $\mathrm{F} \subset \partial\Omega$, we introduce a degenerate boundary control volume K reduced to the face F, with center $x_{\mathrm{K}} = x_{\mathrm{F}}$. The set of boundary control volume is denoted by $\partial\mathcal{M}$.

Definition 2.1. *We defined the relation $\prec$ between respectively vertexes and edges, edges and faces, faces and control volumes as "belongs to the boundary to". In other words*

$$\mathrm{A} \prec \mathrm{E} \prec \mathrm{F} \prec \mathrm{K} \quad \textit{means} \quad \mathrm{A} \subset \partial\mathrm{E}, \quad \mathrm{E} \subset \partial\mathrm{F}, \quad \mathrm{F} \subset \partial\mathrm{K}.$$

This relation is useful to describe for instance the subset of the edges that are connected to a given node, or the subset of the edges that are boundaries of a face, etc.

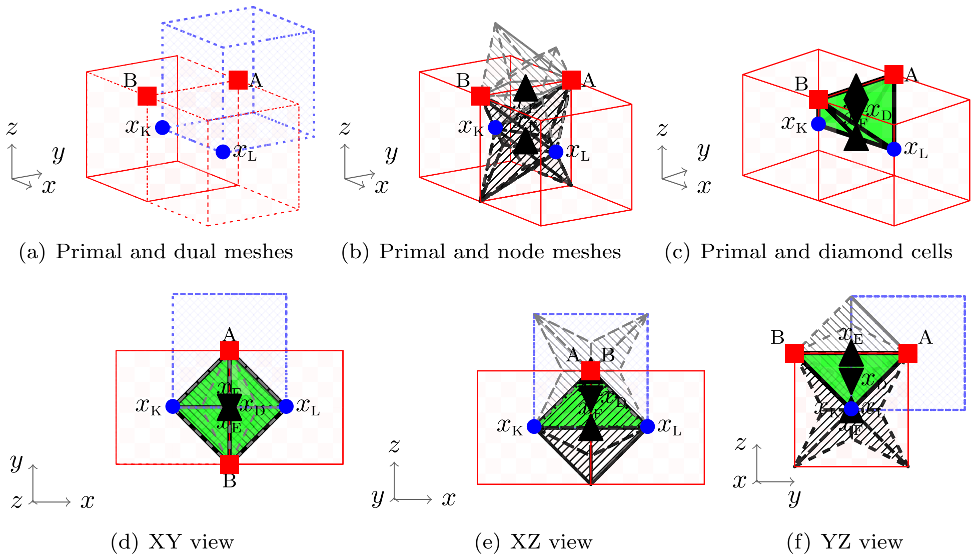
2.2. The node mesh. A control volume denoted by P_{A} is associated to each vertex A of $\mathcal{M}$ located inside the domain Ω. It is uniquely defined by its boundary ∂P_{A} and by $\mathrm{A} \in P_{\mathrm{A}}$. The *node mesh* $\mathcal{N}$ is the set of all these control volumes :

$$\mathcal{N} = \left\{ P_{\mathrm{A}} \text{ such that } \partial P_{\mathrm{A}} = \bigcup_{\mathrm{A}\prec\mathrm{E}\prec\mathrm{F}\prec\mathrm{K}} x_{\mathrm{E}} x_{\mathrm{F}} x_{\mathrm{K}}, \ \mathrm{A} \text{ vertex of } \mathcal{M}, \ \mathrm{A} \in \Omega \right\}.$$

We set $\partial\mathcal{N}$ the set of control volumes around the vertexes $\mathrm{A} \in \partial\Omega$. The specific description of these volumes is not needed here since only homogeneous Dirichlet boundary condition is considered.

2.3. The face mesh. A control volume is associated to each center x_{F} of the faces $F \subset \Omega$ of $\mathcal{M}$ and to each center x_{E} of the edges $\mathrm{E} \subset \Omega$ of $\mathcal{M}$, respectively denoted by P_{F} and P_{E} and such that $x_{\mathrm{F}} \in P_{\mathrm{F}}$ and $x_{\mathrm{E}} \in P_{\mathrm{E}}$. The *face mesh* $\mathcal{FE}$ is the

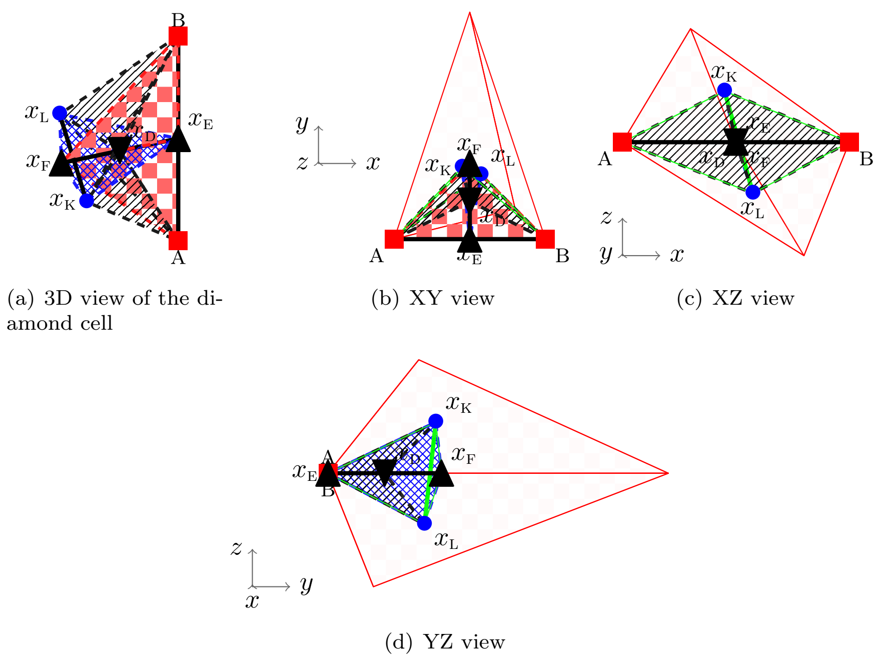
[1] when they have a non zero $d-1$ dimensional measure



(a) Primal and dual meshes (b) Primal and node meshes (c) Primal and diamond cells

(d) XY view (e) XZ view (f) YZ view

FIGURE 2.1. *Example mesh with hexahedrons.*



(a) 3D view of the diamond cell (b) XY view (c) XZ view

(d) YZ view

FIGURE 2.2. *Example mesh with tetrahedrons.*

set of all these control volumes. It is split into the sets $\mathcal{F}$ and $\mathcal{E}$ of the control volumes associated to the faces and edges defined by:

$$\mathcal{F} = \left\{ P_{\text{F}} \text{ such that } \partial P_{\text{F}} = \bigcup_{\text{A}\prec\text{E}\prec\text{F}\prec\text{K}} \text{A}x_{\text{EF}}x_{\text{K}},\ \text{F face of } \mathcal{M},\ \text{F} \subset \Omega \right\},$$

$$\mathcal{E} = \left\{ P_{\text{E}} \text{ such that } \partial P_{\text{E}} = \bigcup_{\text{A}\prec\text{E}\prec\text{F}\prec\text{K}} \text{A}x_{\text{EF}}x_{\text{K}},\ \text{E edge of } \mathcal{M},\ \text{E} \subset \Omega \right\}$$

with

$$x_{\text{EF}} = \theta x_{\text{E}} + (1-\theta)x_{\text{F}} \quad \text{for some fixed } \theta \in]0,1[. \tag{2.1}$$

The set $\partial\mathcal{FE}$ of control volumes around the faces $\text{F} \subset \partial\Omega$ and edges $\text{E} \subset \partial\Omega$ is used to impose the homogeneous Dirichlet boundary condition.

2.4. The diamond cells. Consider an edge E of a face F: $\text{E} \prec \text{F}$. The edge E has two endpoints A, B that are vertexes of $\mathcal{M}$ and is consequently associated to two control volumes in $\mathcal{N}\cup\partial\mathcal{N}$, while the face F is an interface between two control volumes K and L in $\mathcal{M} \cup \partial\mathcal{M}$. They are exactly defined by the relation $\text{A},\text{B} \prec \text{E} \prec \text{F} \prec \text{K},\text{L}$. Finally the control volumes P_{E} and P_{F} in $\mathcal{FE} \cup \partial\mathcal{FE}$ also have an interface for $\text{E} \prec \text{F}$. Consequently, for $\text{E} \prec \text{F}$ the diamond D associated to (E, F) is the polyhedron defined by:

$$\text{D} = \text{D}(\text{E},\text{F}) = \text{hull}(\text{A}, x_{\text{E}}, \text{B}, x_{\text{K}}) \cup \text{hull}(\text{A}, x_{\text{E}}, \text{B}, x_{\text{L}})$$

where hull$(\cdot)$ denotes the convex hull of a set of points. The set of diamond cells, called *the diamond mesh* $\mathcal{D}$, is also defined by

$$\mathcal{D} = \left\{ \text{D such that } \partial\text{D} = \bigcup_{\text{A}\prec\text{E}\prec\text{F}\prec\text{K}} \text{A}x_{\text{F}}x_{\text{K}},\ \text{E} \prec \text{F} \right\}.$$

We associate to each diamond cell a point $x_{\text{D}} \in \text{D}$ called "center".

A diamond cell $\text{D} = \text{D}(\text{E},\text{F})$ is uniquely defined by the data of (E, F) such that $\text{E} \prec \text{F}$. Unless specified explicitly, the diamond cell associated to $\text{E} \prec \text{F}$ is simply denoted by D, and its vertexes by $x_{\text{K}}, x_{\text{L}}$, A, B and $x_{\text{E}}, x_{\text{F}}$, supposed to be order in such a way that

$$\Delta_{\text{EF}} := \det(\text{B}-\text{A}, x_{\text{F}} - x_{\text{E}}, x_{\text{L}} - x_{\text{K}}) > 0.$$

With this orientation the measure of D is $|\text{D}| = \frac{1}{6}\Delta_{\text{EF}}$ and the subsets of the interfaces between pairs control volumes of the three meshes $\mathcal{M}, \mathcal{N}, \mathcal{FE}$ included in the diamond cell D are as follows:

- The intersection $\bar{\text{K}} \cap \bar{\text{L}} \cap \text{D}$ is composed of the two triangles $(\text{A}, x_{\text{E}}, x_{\text{F}})$ and $(\text{B}, x_{\text{E}}, x_{\text{F}})$ and the vector $N_{\text{KL}} = \frac{1}{2}(\text{B}-\text{A}) \times (x_{\text{F}} - x_{\text{E}}) = \int_{\bar{\text{K}}\cap\bar{\text{L}}\cap\text{D}} n_{\text{K,L}}\, ds$ where $n_{\text{K,L}}$ stands for the unit normal to $\bar{\text{K}} \cap \bar{\text{L}} \cap \text{D}$ oriented from K to L;
- The intersection $\bar{P}_{\text{A}} \cap \bar{P}_{\text{B}} \cap \text{D}$ is composed of the two triangles $(x_{\text{F}}, x_{\text{E}}, x_{\text{K}})$ and $(x_{\text{F}}, x_{\text{E}}, x_{\text{L}})$ and the vector $N_{\text{AB}} = \frac{1}{2}(x_{\text{F}} - x_{\text{E}}) \times (x_{\text{L}} - x_{\text{K}}) = \int_{\bar{P}_{\text{A}}\cap\bar{P}_{\text{B}}\cap\text{D}} n_{\text{A,B}}\, ds$ where $n_{\text{A,B}}$ stands for the unit normal to $\bar{P}_{\text{A}} \cap \bar{P}_{\text{B}} \cap \text{D}$ oriented from A to B;
- The intersection $\bar{P}_{\text{E}} \cap \bar{P}_{\text{F}} \cap \text{D}$ is composed of the four triangles $(x_{\text{K}}, \text{A}, x_{\text{EF}})$, $(x_{\text{K}}, \text{B}, x_{\text{EF}})$, $(x_{\text{L}}, \text{B}, x_{\text{EF}})$ and $(x_{\text{L}}, \text{A}, x_{\text{EF}})$ and the vector $N_{\text{EF}} = \frac{1}{2}(x_{\text{L}} - x_{\text{K}}) \times (\text{B}-\text{A}) = \int_{\bar{P}_{\text{E}}\cap\bar{P}_{\text{F}}\cap\text{D}} n_{\text{E,F}}\, ds$ where $n_{\text{E,F}}$ stands for the unit normal to $\bar{P}_{\text{E}} \cap \bar{P}_{\text{F}} \cap \text{D}$ oriented from E to F.

Note that

$$N_{\text{KL}} \cdot (x_{\text{L}} - x_{\text{K}}) = N_{\text{AB}} \cdot (\text{B} - \text{A}) = N_{\text{EF}} \cdot (x_{\text{F}} - x_{\text{E}}) = \frac{1}{2}\Delta_{\text{EF}} = 3|\text{D}|. \tag{2.2}$$

Remark that x_{EF} defined in eq. (2.1) is a natural choice for the center x_{D} of the diamond cell D.

3. The discrete spaces and operators.

3.1. The discrete spaces. Consider the data $(u^{\mathcal{M}}, u^{\mathcal{N}}, u^{\mathcal{FE}})$ of three functions piecewise constant respectively on the $\text{K} \in \mathcal{M}$, $P_{\text{A}} \in \mathcal{N}$ and $P_{\text{E/F}} \in \mathcal{FE}$:

$$u^{\mathcal{M}} = \sum_{\text{K}\in\mathcal{M}} u_{\text{K}}\chi_{\text{K}}, \quad u^{\mathcal{N}} = \sum_{A\in\mathcal{N}} u_{\text{A}}\chi_{A}, \quad u^{\mathcal{FE}} = \sum_{\text{F}\in\mathcal{F}} u_{\text{F}}\chi_{P_{\text{F}}} + \sum_{\text{E}\in\mathcal{E}} u\chi_{P_{\text{E}}}.$$

The sets of functions piecewise constant on the $K \in \mathcal{M}$, $P_{\text{A}} \in \mathcal{N}$ and $P_{\text{E/F}} \in \mathcal{FE}$ are respectively denoted by $X^{\mathcal{M}}$, $X^{\mathcal{N}}$ and $X^{\mathcal{FE}}$. The finite volume unknown is generally a element of the space $X = X^{\mathcal{M}} \times X^{\mathcal{N}} \times X^{\mathcal{FE}}$.

The finite volume unknown is supplemented with boundary values

$$\delta u^{\mathcal{T}} = ((u_{\text{K}})_{\text{K}\in\partial\mathcal{M}},\ (u_{\text{A}})_{\text{A}\in\partial\Omega},\ (u_{\text{F}})_{\text{F}\subset\partial\Omega},\ (u_{\text{E}})_{\text{E}\subset\partial\Omega})$$

that define a linear space ∂X. For homogeneous Dirichlet boundary conditions, it is set $\delta u^{\mathcal{T}} = 0$ and the construction of the scheme and proofs are carried out simply in X.

REMARK 1. *For non homogeneous Dirichlet conditions, $\delta u^{\mathcal{T}} \neq 0$, the unknown belongs to an affine subspace of $X \times \partial X$, the inner product and subsequent properties are handled in $X \times \partial X$, while the discrete problem is posed in X. The proofs and scheme becomes more technical while the difficulties are the same as the one encountered for homogeneous conditions.*

For Neumann boundary conditions, it must be defined the control volumes associated to boundary nodes, edges and faces, which is again an additional technicality.

The space X is supplied with the natural inner product

$$\begin{aligned}(u^{\mathcal{T}}, v^{\mathcal{T}})_X &= \frac{1}{3}\left(\int_\Omega u^{\mathcal{M}}v^{\mathcal{M}} + \int_\Omega u^{\mathcal{N}}v^{\mathcal{N}} + \int_\Omega u^{\mathcal{FE}}v^{\mathcal{FE}}\right) \\ &= \frac{1}{3}\left(\sum_{\text{K}\in\mathcal{M}} u_{\text{K}}v_{\text{K}}|K| + \sum_{P_{\text{A}}\in\mathcal{N}} u_{\text{A}}v_{\text{A}}|P_{\text{A}}| + \sum_{P_{\text{E}}\in\mathcal{E}} u_{\text{E}}v_{\text{E}}|P_{\text{E}}| + \sum_{P_{\text{F}}\in\mathcal{F}} u_{\text{F}}v_{\text{F}}|P_{\text{F}}|\right).\end{aligned} \tag{3.1}$$

A linear space of vector fields will be associated to the finite volumes gradient. It is the space $\mathbf{Q}$ of functions piecewise constant on the $\text{D} \in \mathcal{D}$ with value in $\mathbb{R}^3$:

$$\xi^{\mathcal{D}} \in \mathbf{Q} \Leftrightarrow \xi^{\mathcal{D}} = \sum_{\text{D}\in\mathcal{D}} \xi_{\text{D}}\chi_{\text{D}}, \quad \forall \text{D} \in \mathcal{D},\ \xi_{\text{D}} \in \mathbb{R}^3.$$

Like for elements $u^{\mathcal{T}} \in X$, an element $\xi^{\mathcal{D}} \in \mathbf{Q}$ may also be denoted by the sequence of its degrees of freedom: $\xi^{\mathcal{D}} = (\xi_{\text{D}})_{\text{D}\in\mathcal{D}}$. It is endowed with the natural inner product

$$(\xi^{\mathcal{D}}, \eta^{\mathcal{D}})_{\mathbf{Q}} = \int_\Omega \xi^{\mathcal{D}} \cdot \eta^{\mathcal{D}} = \sum_{\text{D}\in\mathcal{D}} \xi_{\text{D}} \cdot \eta_{\text{D}}|\text{D}|. \tag{3.2}$$

3.2. The discrete gradient. Given $u^{\mathcal{T}} = (u^{\mathcal{M}}, u^{\mathcal{N}}, u^{\mathcal{FE}}) \in X$, its gradient is the element $\nabla^{\mathcal{T}} u^{\mathcal{T}} = (\nabla_{\mathrm{D}} u^{\mathcal{T}})_{\mathrm{D}\in\mathcal{D}}$ in $\mathbf{Q}$ defined by

$$\forall \mathrm{D} \in \mathcal{D}, \quad \nabla_{\mathrm{D}} u^{\mathcal{T}} = \frac{1}{3|\mathrm{D}|}\left((u_{\mathrm{L}} - u_{\mathrm{K}})N_{\mathrm{KL}} + (u_{\mathrm{B}} - u_{\mathrm{A}})N_{\mathrm{AB}} + (u_{\mathrm{F}} - u_{\mathrm{E}})N_{\mathrm{EF}}\right). \tag{3.3}$$

Unlike the discrete unknown, the discrete gradient is defined up to the boundary with this relation, assuming that the Dirichlet condition is imposed on the boundary: $u_{\mathrm{F}} = u_{\mathrm{L}} = 0$ if $\mathrm{F} \subset \partial\Omega$, $u_{\mathrm{A}} = 0$ (resp. $u_{\mathrm{B}} = 0$) if $\mathrm{A} \in \partial\Omega$ (resp. $\mathrm{B} \in \partial\Omega$) and $u_{\mathrm{E}} = u_{\mathrm{A}} = u_{\mathrm{B}} = 0$ if $\mathrm{E} \subset \partial\Omega$.

By construction, for each $\mathrm{D} \in \mathcal{D}$, the vector $\nabla_{\mathrm{D}} u^{\mathcal{T}}$ is the unique vector of $\mathbb{R}^3$ such that

$$\nabla_{\mathrm{D}} u^{\mathcal{T}} \cdot (x_{\mathrm{L}} - x_{\mathrm{K}}) = u_{\mathrm{L}} - u_{\mathrm{K}}, \quad \nabla_{\mathrm{D}} u^{\mathcal{T}} \cdot (\mathrm{B} - \mathrm{A}) = u_{\mathrm{B}} - u_{\mathrm{A}}, \quad \nabla_{\mathrm{D}} u^{\mathcal{T}} \cdot (x_{\mathrm{F}} - x_{\mathrm{E}}) = u_{\mathrm{F}} - u_{\mathrm{E}},$$

because of the relation (2.2).

3.3. The discrete divergence. Given $\xi^{\mathcal{D}} \in X^{\mathcal{D}}$, its discrete divergence is the element $\mathrm{div}^{\mathcal{T}}\, \xi^{\mathcal{D}} = (\mathrm{div}^{\mathcal{M}}\, \xi^{\mathcal{D}}, \mathrm{div}^{\mathcal{N}}\, \xi^{\mathcal{D}}, \mathrm{div}^{\mathcal{FE}}\, \xi^{\mathcal{D}})$ in X defined by

$$\mathrm{div}^{\mathcal{M}}\, \xi^{\mathcal{D}} = (\mathrm{div}_{\mathrm{K}}\, \xi^{\mathcal{D}})_{\mathrm{K}\in\mathcal{M}}, \quad \mathrm{div}^{\mathcal{N}}\, \xi^{\mathcal{D}} = (\mathrm{div}_{\mathrm{A}}\, \xi^{\mathcal{D}})_{\mathrm{A}\in\mathcal{N}},$$
$$\mathrm{div}^{\mathcal{FE}}\, \xi^{\mathcal{D}} = \left\{(\mathrm{div}_{\mathrm{E}}\, \xi^{\mathcal{D}})_{\mathrm{E}\in\mathcal{E}}, (\mathrm{div}_{\mathrm{F}}\, \xi^{\mathcal{D}})_{\mathrm{F}\in\mathcal{F}}\right\}$$

with, for any $\mathrm{K}, \mathrm{A}, \mathrm{E}, \mathrm{F}$

$$|K|\, \mathrm{div}_{\mathrm{K}}\, \xi^{\mathcal{D}} = \sum_{\mathrm{D}\in\mathrm{D}_{\mathrm{K}}} \xi_{\mathrm{D}} N_{\mathrm{KL}}, \quad |P_{\mathrm{A}}|\, \mathrm{div}_{\mathrm{A}}\, \xi^{\mathcal{D}} = \sum_{\mathrm{D}\in\mathrm{D}_{\mathrm{A}}} \xi_{\mathrm{D}} N_{\mathrm{AB}}, \tag{3.4}$$

$$|P_{\mathrm{E}}|\, \mathrm{div}_{\mathrm{E}}\, \xi^{\mathcal{D}} = \sum_{\mathrm{D}\in\mathrm{D}_{\mathrm{E}}} \xi_{\mathrm{D}} N_{\mathrm{EF}}, \quad |P_{\mathrm{F}}|\, \mathrm{div}_{\mathrm{F}}\, \xi^{\mathcal{D}} = \sum_{\mathrm{D}\in\mathrm{D}_{\mathrm{F}}} \xi_{\mathrm{D}}\left(-N_{\mathrm{EF}}\right), \tag{3.5}$$

where the subsets of $\mathcal{D}$ are defined by $\mathrm{D}_{\mathrm{K}} = \{\mathrm{D}(\mathrm{E},\mathrm{F}) : \mathrm{E} \prec \mathrm{F} \prec \mathrm{K}\}$, $\mathrm{D}_{\mathrm{A}} = \{\mathrm{D}(\mathrm{E},\mathrm{F}) : \mathrm{A} \prec \mathrm{E} \prec \mathrm{F}\}$, $\mathrm{D}_{\mathrm{E}} = \{\mathrm{D}(\mathrm{E},\mathrm{F}) : \mathrm{E} \prec \mathrm{F}\}$ and $\mathrm{D}_{\mathrm{F}} = \{\mathrm{D}(\mathrm{E},\mathrm{F}) : \mathrm{E} \prec \mathrm{F}\}$. Remark that we have for all $\mathrm{C} \in \mathcal{T}$

$$|\mathrm{C}|\, \mathrm{div}_{\mathrm{C}}\, \xi^{\mathcal{D}} = \int_{\partial \mathrm{C}} \xi^{\mathcal{D}}(x) n_{\mathrm{C}}(x) d\sigma(x), \tag{3.6}$$

where n_{C} is the unit normal to $\partial\mathrm{C}$ outward of C.

4. The Discrete Duality Formula and other Basic Properties.

4.1. The Discrete Duality Relationship. We first state a discrete version of the Green formula:

$$\langle \mathrm{div}\, q, u\rangle_{L^{p'},L^p} + \langle q, \nabla u\rangle_{(L^{p'})^3,(L^p)^3} = 0, \, \forall u \in W_0^{1,p}(\Omega), \, \forall q \in \left(W^{1,p'}(\Omega)\right)^3.$$

Theorem 4.1 (Discrete duality). *For any $u^{\mathcal{T}} \in X$ and $\xi^{\mathcal{D}} \in \mathbf{Q}$, the gradient $\nabla^{\mathcal{T}} u^{\mathcal{T}}$ and divergence $\mathrm{div}^{\mathcal{T}}\, \xi^{\mathcal{D}}$ verify the discrete duality relation*

$$\left(\mathrm{div}^{\mathcal{T}}\, \xi^{\mathcal{D}}, u^{\mathcal{T}}\right)_X + \left(\xi^{\mathcal{D}}, \nabla^{\mathcal{T}} u^{\mathcal{T}}\right)_{\mathbf{Q}} = 0.$$

Proof. From the definitions (3.4) and (3.5) of the divergence and (3.1) of the inner product in X, one have

$$\begin{aligned}\left(\operatorname{div}^{\mathcal{T}} \xi^{\mathcal{D}}, u^{\mathcal{T}}\right)_X &= \frac{1}{3}\Bigg(\sum_{\mathrm{K}\in\mathcal{M}}\sum_{\mathrm{D}\in\mathrm{D}_\mathrm{K}} \xi_\mathrm{D}\cdot N_\mathrm{KL} u_\mathrm{K} + \sum_{\mathrm{A}\in\mathcal{N}}\sum_{\mathrm{D}\in\mathrm{D}_\mathrm{A}} \xi_\mathrm{D}\cdot N_\mathrm{AB} u_\mathrm{A} \\ &\qquad + \sum_{\mathrm{F}\in\mathcal{F}}\sum_{\mathrm{D}\in\mathrm{D}_\mathrm{F}} \xi_\mathrm{D}\cdot(-N_\mathrm{EF}) u_\mathrm{F} + \sum_{\mathrm{E}\in\mathcal{E}}\sum_{\mathrm{D}\in\mathrm{D}_\mathrm{E}} \xi_\mathrm{D}\cdot N_\mathrm{EF} u_\mathrm{E}\Bigg) \\ &= -\frac{1}{3}\sum_{\mathrm{D}\in\mathcal{D}} \xi_\mathrm{D}\cdot\left(N_\mathrm{KL}(u_\mathrm{L}-u_\mathrm{K}) + N_\mathrm{AB}(u_\mathrm{B}-u_\mathrm{A}) + N_\mathrm{EF}(u_\mathrm{F}-u_\mathrm{E})\right) \\ &= -\sum_{\mathrm{D}\in\mathcal{D}} |\mathrm{D}|\xi_\mathrm{D}\cdot\nabla_\mathrm{D} u^{\mathcal{T}} = -\left(\xi^{\mathcal{D}}, \nabla^{\mathcal{T}} u^{\mathcal{T}}\right)_{\mathbf{Q}},\end{aligned}$$

using also eq. (3.3) and (3.2) for the discrete gradient and the inner product in $\mathbf{Q}$, and with the homogeneous Dirichlet condition. ∎

4.2. The Inequality of Poincaré. The discrete space X is analog to $W_0^{1,p}(\Omega)$ and then it is expected that $\nabla^{\mathcal{T}} u^{\mathcal{T}} = 0 \Rightarrow u^{\mathcal{T}} = 0$ and that a discrete inequality of Poincaré holds.

THEOREM 4.2 (Inequality of Poincaré). *There exists a constant $C = C(Reg^{\mathcal{T}})$ depending only on $Reg^{\mathcal{T}}$ such that*

$$\forall u^{\mathcal{T}} \in X, \quad \|u^{\mathcal{M}}\|_{L^p} + \|u^{\mathcal{N}}\|_{L^p} + \|u^{\mathcal{FE}}\|_{L^p} \le C\|\nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^p}.$$

Proof. The proof relies on the remark that for any diamond $\mathrm{D} \in \mathcal{D}$,

$$\begin{aligned}\left|\nabla_\mathrm{D} u^{\mathcal{T}}\right|^2 |\mathrm{D}| &= \frac{1}{9|\mathrm{D}|}\delta u_\mathrm{D} G_\mathrm{D} \delta u_\mathrm{D}{}^T \\ &\ge \lambda_m |\mathrm{D}| \left((u_\mathrm{L}-u_\mathrm{K})^2\left|\frac{N_\mathrm{KL}}{3|\mathrm{D}|}\right|^2 + (u_\mathrm{A}-u_\mathrm{B})^2\left|\frac{N_\mathrm{AB}}{3|\mathrm{D}|}\right|^2 + (u_\mathrm{E}-u_\mathrm{F})^2\left|\frac{N_\mathrm{EF}}{3|\mathrm{D}|}\right|^2\right)\end{aligned}$$

where $\delta u_\mathrm{D} = ((u_\mathrm{L}-u_\mathrm{K})N_\mathrm{KL}, (u_\mathrm{B}-u_\mathrm{A})N_\mathrm{AB}, (u_\mathrm{F}-u_\mathrm{E})N_\mathrm{EF})$ and $\lambda_m > 0$ is the smallest eigenvalue of the Gram matrix

$$G_\mathrm{D} = \left(\frac{N_\mathrm{KL}}{|N_\mathrm{KL}|}, \frac{N_\mathrm{AB}}{|N_\mathrm{AB}|}, \frac{N_\mathrm{EF}}{|N_\mathrm{EF}|}\right)\left(\frac{N_\mathrm{KL}}{|N_\mathrm{KL}|}, \frac{N_\mathrm{AB}}{|N_\mathrm{AB}|}, \frac{N_\mathrm{EF}}{|N_\mathrm{EF}|}\right)^T.$$

This eigenvalue is naturally uniformly bounded below under the geometrical assumption the diamond cells are non-degenerate.The inequality of Poincaré is derived from the 3 usual discrete inequalities of Poincaré in the spaces $X^{\mathcal{M}}$, $X^{\mathcal{N}}$ and $X^{\mathcal{FE}}$ [?, ?]. ∎

5. The Finite Volume Scheme.

5.1. Formulation of the scheme. The *Discrete Duality Finite Volume* scheme is obtained by integrating equation (1.1) on all the control volumes of the three meshes, $\mathrm{K} \in \mathcal{M}$, $P_\mathrm{A} \in \mathcal{N}$, $P_\mathrm{E} \in \mathcal{E}$ and $P_\mathrm{F} \in \mathcal{F}$ [?, ?, ?]. The exact solution u verifies for all $\mathrm{C} \in \mathcal{T}$:

$$-\int_{\partial\mathrm{C}} \varphi(s, \nabla u(s))\cdot n_\mathrm{C} ds = \int_\mathrm{C} f(x)dx. \tag{5.1}$$

For any $\text{D} \in \mathcal{D}$, consider the spatial approximation $\varphi_{\text{D}} : \mathbb{R}^3 \to \mathbb{R}^3$ of the flux φ defined by

$$\forall \text{D} \in \mathcal{D},\ \forall \xi \in \mathbb{R}^3, \quad \varphi_{\text{D}}(\xi) = \frac{1}{|\text{D}|} \int_{\text{D}} \varphi(z, \xi)\, dz. \tag{5.2}$$

The flux $\varphi(\cdot, \nabla u(\cdot))$ is approximated by the function $\varphi_{\mathcal{T}}(\nabla^{\mathcal{T}} u^{\mathcal{T}}) = (\varphi_{\text{D}}(\nabla_{\text{D}} u^{\mathcal{T}}))_{\text{D}\in\mathcal{D}}$ in $\mathbf{Q}$ where $\nabla^{\mathcal{T}} u^{\mathcal{T}}$ has been defined in section 3.2. With this approximation of the flux, and using eq. (5.1), the DDFV scheme reads

$$-\operatorname{div}^{\mathcal{T}}\left(\varphi_{\mathcal{T}}(\nabla^{\mathcal{T}} u^{\mathcal{T}})\right) = \pi^{\mathcal{T}} f \tag{5.3}$$

where the discrete divergence $\operatorname{div}^{\mathcal{T}}$ is defined in section 3.3 and the projection $\pi^{\mathcal{T}} f = \{(f_{\text{K}})_{\text{K}\in\mathcal{M}}, (f_{\text{A}})_{\text{A}\in\mathcal{M}}, (f_{\text{E}}, f_{\text{F}})_{\text{E}\in\mathcal{E},\text{F}\in\mathcal{F}}\} \in X$ is defined by

$$\forall \text{K} \in \mathcal{M}, \quad f_{\text{K}} = \frac{1}{|\text{K}|}\int_{\text{K}} f(x)dx, \quad \forall \text{A} \in \mathcal{N}, \quad f_{\text{A}} = \frac{1}{|P_{\text{A}}|}\int_{P_{\text{A}}} f(x)dx,$$
$$\forall \text{E} \in \mathcal{E}, \quad f_{\text{E}} = \frac{1}{|P_{\text{E}}|}\int_{P_{\text{E}}} f(x)dx, \quad \forall \text{F} \in \mathcal{F}, \quad f_{\text{F}} = \frac{1}{|P_{\text{F}}|}\int_{P_{\text{F}}} f(x)dx.$$

5.2. A word on Practical Implementation. Note that the implementation of such a scheme does not require the construction of the node mesh $\mathcal{N}$ and the face mesh $\mathcal{F}$. If the primal mesh is given with the format

$$\text{A} \prec \text{E} \prec \text{F} \prec \text{K}$$

that is a control volume is defined by its faces, a face by its edges and an edge by its vertexes, it is easy to construct a diamond cell structure that contains the reference to its vertexes $\text{A}, \text{B}, x_{\text{K}}, x_{\text{L}}, x_{\text{E}}, x_{\text{F}}$, the values $N_{\text{KL}}, N_{\text{AB}}, N_{\text{EF}}$ and the measures of the 8 tetrahedral cells that compose the diamond cell: $(x_{\text{EF}}, x_{\text{K}}, \text{A}, x_{\text{F}})$,$(x_{\text{EF}}, x_{\text{K}}, \text{A}, x_{\text{E}})$, ... The system involved in the resolution of the scheme, can be easily implemented by going through this diamond cell structure.

6. Convergence and error estimates. In this section, the nonlinear system of equations (5.3) is proved to be well-posed; uniform *a priori* estimates are found on its solutions; and finally error estimates are given.

THEOREM 6.1 (*A priori* estimate and existence of a solution to (5.3)).

Assume that the flux φ satisfies assumptions (1.2), (1.3) and (1.4). For any $f \in L^{p'}(\Omega)$ and any mesh $\mathcal{T}$ on Ω, the finite volume scheme (5.3) admits a unique solution $u^{\mathcal{T}} \in X$ and there exists a uniform constant $C > 0$ depending only on C_1 and $Reg^{\mathcal{T}}$, such that

$$\left(\int_{\Omega} |\nabla^{\mathcal{T}} u^{\mathcal{T}}|^p\right)^{\frac{1}{p}} \le C\left(\|f\|_{L^{p'}}^{\frac{1}{p-1}} + \|b_1\|_{L^1}^{\frac{1}{p}}\right). \tag{6.1}$$

Proof. The *a priori* estimate is a consequence of theorems 4.1 and 4.2 and of assumption 1.3. A solution of the discrete problem is found as a Brouwer fixed point of the mapping $u^{\mathcal{T}} \mapsto -\operatorname{div}^{\mathcal{T}}(\varphi^{\mathcal{T}}(u^{\mathcal{T}})) - \pi^{\mathcal{T}} f$. Uniqueness is recovered from thm 4.1 and assumption (1.2). □

REMARK 2. *In the case where the flux φ derives from a convex potential Φ:*

$$\varphi(z, \xi) = \nabla_{\xi}\Phi(z, \xi)\ \forall \xi \in \mathbb{R}^2, \text{ for a.e. } z \in \Omega, \text{ and } \Phi(z, 0) = 0 \text{ for a.e. } z \in \Omega \tag{6.2}$$

the solution $u^{\mathcal{T}}$ of the scheme (5.3) is also the unique minimizer of the discrete energy $J^{\mathcal{T}}$ associated to the scheme by $J^{\mathcal{T}}(u^{\mathcal{T}}) = \int_\Omega \Phi(z, \nabla^{\mathcal{T}} u^{\mathcal{T}}) - \int_\Omega u^{\mathcal{T}} \pi^{\mathcal{T}} f$.

The scheme is well-posed, and the discrete solution might be proved to converge under assumptions (1.2), (1.3) and (1.4) only. Anyway, in order to compute some error estimates, in the case $\mathbf{p \geq 2}$, the following additional assumptions are needed: there exists constants $C_3, C_4, C_5 > 0$, $b_4 \in L^{\frac{p}{p-2}}(\Omega)$ and a function $b_5 \in L^{p'}(\Omega)$ such that for all $(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3$ and almost every $z \in \Omega$,

$$(\varphi(z,\xi) - \varphi(z,\eta), \xi - \eta) \geq C_3 |\xi - \eta|^p, \tag{6.3}$$

$$|\varphi(z,\xi) - \varphi(z,\eta)| \leq C_4 \left(b_4(z) + |\xi|^{p-2} + |\eta|^{p-2}\right) |\xi - \eta|, \tag{6.4}$$

and for all $x \in \mathbb{R}^3$ and almost every $z \in \Omega$,

$$\left|\frac{\partial \varphi}{\partial z}(z,\xi)\right| \leq C_5 \left(b_5(z) + |\xi|^{p-1}\right). \tag{6.5}$$

Our main result is the following.

THEOREM 6.2. *Assume that the flux φ satisfies assumptions (1.3), (1.4), (6.3), (6.4) and (6.5). For $p \geq 2$ consider $f \in L^{p'}(\Omega)$ and assume that the solution u to (1.1) belongs to $W^{2,q}(\Omega) \cap W_0^{1,p}(\Omega)$, with $q = p$ for $p > 2$ and $q > 2$ for $p = 2$.*

For any mesh $\mathcal{T}$ on Ω there exists a constant $C > 0$ depending on the norm $\|u\|_{W^{2,p}}$, the regularity parameter $Reg^{\mathcal{T}}$, the data f, $(b_i)_{1 \leq i \leq 6}$ and $(C_i)_{1 \leq i \leq 5}$, such that

$$\|u - u^{\mathcal{M}}\|_{L^p} + \|u - u^{\mathcal{N}}\|_{L^p} + \|u - u^{\mathcal{FE}}\|_{L^p} + \|\nabla u - \nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^p} \leq C h^{\frac{1}{p-1}}, \tag{6.6}$$

where $u^{\mathcal{T}} = (u^{\mathcal{M}}, u^{\mathcal{N}}, u^{\mathcal{FE}}) \in X$ is the solution to eq. (5.3).

Proof. The proof is long and technical. It involves the consistency of the discrete gradient, the computation of error estimates between the discrete and exact flux inside the diamond cells, and all the assumptions on the flux function φ. ∎