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A Simple and Efficient Regularization Method for 3D BEM: Application to Frequency-Domain Elastodynamics

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Abstract

An efficient and easy-to-implement method is proposed to regularize integral equations in the 3D Boundary Element Method. The method takes advantage of an assumed three-noded triangle discretization of the boundary surfaces. The method is based on the derivation of analytical expressions of singular integrals. To demonstrate the accuracy of the method, three elastodynamic problems are numerically worked out in frequency domain: cavity under harmonic pressure, diffraction of a plane wave by a spherical cavity, amplification of seismic waves in a semi-spherical alluvial basin (the second one is also investigated in time domain). The numerical results are compared to (semi-) analytical solutions; a close agreement is found for all problems showing the very good accuracy of the proposed method.

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Numerical Modeling in Elastodynamics

To analyze problems in 3D elastodynamics, various numerical methods are available:

- the \textit{finite element method} which is efficient to deal with complex geometries and numerous heterogeneities (Chammas et al., 2003), even for inelastic constitutive models (Bonilla, 2000). It has nevertheless several drawbacks such as numerical dispersion (and damping) (Ihlenburg and Babuška, 1995; Semblat and Brioist, 2000) and (consequently) numerical cost in 3D elastodynamics,

- the \textit{finite difference method} which is very accurate in elastodynamics but is mainly adapted to simple geometries and linear constitutive models (Frankel and Vidale, 1992; Moczo et al., 2002; Virieux, 1986)

- the \textit{boundary element method} which allows a very good description of the radiation conditions but is preferably dedicated to weak heterogeneities and linear constitutive models (Banerjee et al., 1988; Beskos, 1997; Beskos et al., 1986; Bonnet, 1999; Dangla, 1988; Sanchez-Sesma and Luzn, 1995; Yoko, 2003)

- the \textit{spectral element method} which has been increasingly considered to analyse 2D/3D wave propagation in linear media (Faccioli et al., 1996; Komatitsch and Vilotte, 1998)

- the \textit{Aki-Larner method} which takes advantage of the frequency-wavenumber decomposition but is limited to simple geometries (Aki and Larner, 1970; Bouchon et al., 1989)

- \textit{series expansions of wave functions} which give a semi-analytical estimation of the scattered wavefield for simple geometries (Moench-Vaziri and Trifunac, 1985; Sanchez-Sesma, 1983)

Each method has specific advantages and drawbacks. It is consequently often more interesting to combine two methods to take advantage of their peculiarities. One of the most common method in elastodynamics is to couple FEM and BEM allowing an accurate description of the near field (FEM model including complex geometries, heterogeneities and constitutive behaviours) and a reliable estimation of the far-field (BEM model involving radiation conditions).

Integral Equations

This paper is limited to isotropic elastodynamics for time-harmonic problems of circular frequency $\omega$. For
any given body force distribution $F_i(x)$ over $\Omega$, the
governing equations which must be verified by any displace-
ment and stress fields, $u_i(x)$ and $\sigma_{ij}(x)$, take the follow-
ning form:

$$\sigma_{ij} = \lambda u_{ik,k} + \mu (u_{i,j} + u_{j,i})$$

$$\sigma_{ij,j} + \rho \omega^2 u_i + F_i = 0$$

(1) (2)

The fundamental solutions, in time-harmonic elasto-
dynamics, are defined by a force of unit amplitude applied
at a fixed point $y$ and in a fixed coordinate direction $k:$

$$F_i(x) = \delta(x-y)\delta_{ik}.$$  

For infinite body the fundamental
solution, denoted by $u_i(x) = U_k^{\infty}(x,y;\omega),$ is known
as the Helmholtz fundamental solution and is given by
(Eringen and Suhubi, 1975):

$$U_k^\infty(x,y;\omega) = \frac{1}{4\pi \mu} \left[ \frac{1}{k^2_S} \frac{\partial^2}{\partial x_i \partial x_k} \left( \frac{e^{ikkr}}{r} - \frac{e^{ikpr}}{r} \right) + \frac{e^{ikr}}{r^2} \delta_{ik} \right]$$

(3)

where $r^2 = (x - y)^2$ and where $k_p = \omega \sqrt{\rho/(\lambda + 2\mu)}$
and $k_S = \omega \sqrt{\rho/\mu}$ are the longitudinal and transvers-
al wave numbers respectively. The stress tensor associated
with $U_k^\infty(x,y;\omega),$ defined by (3), is denoted by

$$\Sigma^\infty_k(x,y;\omega)$$

while the stress vector applied to the
surface boundary of $\Omega$ is $T_k^\infty(x,y;\omega)$ $\Sigma^\infty_k(x,y;\omega) u_i.$

For sake of simplicity let us assume no body force
from now on. Application of the Maxwell-Betti recipro-
ality theorem leads to the following displacement
integral representation at point $y \in \mathcal{R}^3$ (Bonnet, 1999):

$$\kappa(y) u_k(y) = \int_{\partial \Omega} \left[ t_i(x) U_k^\infty(x,y;\omega) - u_i(x) T_k^\infty(x,y;\omega) \right] dS_x$$

(4)

where $\kappa = 1$ ($y \in \Omega$) or $\kappa = 0$ ($y \notin \Omega$).

Let $y$ denote a fixed point on the boundary surface
$\partial \Omega$. For a given small $\varepsilon > 0$, introduce a spherical
shaped neighbourhood $v_\varepsilon(y)$ of $y$, called an exclusion
neighbourhood (Guiggiani et al., 1992). The domain
$\Omega_\varepsilon(y) = \Omega - v_\varepsilon(y)$ obtained by removing $v_\varepsilon(y)$ from $\Omega$ is
such that the point $y$ is exterior to $\Omega_\varepsilon(y)$. Its boundary
is $\partial \Omega_\varepsilon = (\partial \Omega - v_\varepsilon) + s_\varepsilon$, where $s_\varepsilon$ $= \partial \Omega \cap v_\varepsilon$, $s_\varepsilon = \Omega \cap \partial v_\varepsilon$.
The classical form of the integral equation consists in
taking the limit $\varepsilon \rightarrow 0$ in the representation formula (4)
taken for the domain $\Omega_\varepsilon$. The limiting expression thus
obtained is known as the Somigliana identity:

$$C_k^\infty(y) u_i(y) = \int_{\partial \Omega} \left[ t_i(x) U_k^\infty(x,y;\omega) - u_i(x) T_k^\infty(x,y;\omega) \right] dS_x$$

(5)

The notation $\int$ stands for the Cauchy principal value
of a singular integral, i.e. the limit:

$$\int_{\partial \Omega} (\cdot) = \lim_{\varepsilon \rightarrow 0} \int_{(\partial \Omega - \varepsilon \varepsilon)} (\cdot)$$

(6)

The free term $C_k^\infty(y)$ appearing in (5), is defined by:

$$C_k^\infty(y) = \lim_{{\varepsilon \rightarrow 0}} \int_{s_\varepsilon} T_k^\infty(x,y;\omega) dS_x$$

(7)

It is found to be equal to $1/2\delta_{ik}$ when $\Omega$ is smooth at $y$.

Discretization and Regularization Principle

The boundary surface $\partial \Omega$ and the boundary variables $(u_i,t_i)$ are discretized by using three-noded trian-
gular elements. A finite set of equations is generated
by enforcing equation (5) at the nodes of the surface
mesh (collocation method). Thus the boundary surface
consists of the set of $N$ boundary surface elements $E_n:$

$$\partial \Omega = \{ E_n, e = 1...N \}.$$  

The integral appearing in (5) then assumes the form of a sum of $N$ element integrals:

$$\int_{\partial \Omega} (\cdot) = \sum_{e=1}^{N} \int_{E_n} (\cdot)$$

(8)

The numerical evaluation of non singular element inte-
grals that appear in (5) is usually based, like in finite element
methods, on Gaussian quadrature formulas. The approximate value of an element integral can be given
formally by:

$$\int_{E_n} f(x) dS_x \approx \frac{n}{N} \sum_{i=1}^{N} w_i f(x_i)$$

(9)

where $x_i$ and $w_i$ are the coordinates and weights of the
Gauss points. The notation $\int$ stands for the numerical
approximation of integrals. This special notation has been adopted to emphasize that in case of singular
integral $\int \neq \int.$

Since some of element integrals are singular, a straight-
forward evaluation of (5) based on Gaussian quadrature
formulas will inevitably lead to some significant error.
To correct this error, a new term $R_n^k(y)$ must be intro-
duced in the numerical evaluation of (5):

$$C_k^\infty(y) u_i(y) = R_k^\infty(y) + \int_{\partial \Omega} \left[ t_i(x) U_k^\infty(x,y;\omega) - u_i(x) T_k^\infty(x,y;\omega) \right] dS_x$$

(10)

The regularization method proposed in this paper
consists in deriving analytically the correction term by
taking advantage of the simple shape of the triangle
elements. To do so, let us introduce the Kelvin’s funda-
mental solution:

$$U_k^\infty(x,y) = \frac{1}{16\pi \mu (1 - \nu)} \left[ \delta_{ij} + \nu r_i r_j \right][3 - 4\nu] \delta_{ik}$$

(11)

and note $\Sigma_{ij}^\infty(x,y)$ the stress tensor associated with solution $U_k^\infty(x,y)$. It is noticed that the Helmholtz and
Kelvin solutions have identical singularities:

\[
(x \to y) \quad \begin{cases} 
U^k_i(x, y; \omega) - U^k_i(x, y) = O(1) \\
(\forall \omega > 0) \quad \Sigma^k_{ij}(x, y; \omega) - \Sigma^k_{ij}(x, y) = O(1)
\end{cases}
\] (12)

Thanks to this property the correction term \( R^k(y) \) only needs to involve the Kelvin fundamental solutions. For a given point \( y \in \partial \Omega \), introduce the index subset \( I(y) = \{e \in [1, N], y \in E_e\} \) such that the integral over \( E_e \) is singular for \( e \in I(y) \) and non singular for \( e \notin I(y) \). Introduce \( \partial \Omega_y = \{E_e, e \in I(y)\} \) the set of the neighborhood elements of \( y \). Thus the correction term can be formulated in the following form:

\[
R^k(y) = \left( \int_{\partial \Omega_y} U^k_i(x, y) dS_x - \sum_{e \in I(y)} U^k_i(x, y) dS_x \right) t_i(y)
\] - \left( \int_{\partial \Omega_y} T^k_i(x, y) dS_x - \sum_{e \in I(y)} T^k_i(x, y) dS_x \right) u_i(y)
\] (13)

It can be noticed that formulation (13) is independent of the interpolation order since \( t_i(y) \) and \( u_i(y) \) only need to be evaluated at point \( y \). Taking advantage of the simple shape of the three-noded triangle elements, we can derive analytical expressions of the singular integrals appearing in (13).

In particular, it can be shown that they are the sum of elementary contributions involving elements of \( I(y) \):

\[
I^k_i(y) = \int_{\partial \Omega_y} T^k_i(x, y) dS_x = \sum_{e \in I(y)} I^k_i(y; E_e)
\] (14)

\[
J^k_i(y) = \int_{\partial \Omega_y} U^k_i(x, y) dS_x = \sum_{e \in I(y)} J^k_i(y; E_e)
\] (15)

The analytical derivations of \( I^k_i(y; E_e) \) and \( J^k_i(y; E_e) \) are proposed in the appendix (equations (30) to (35)).

In a similar manner, the free term involves the Kelvin fundamental solutions and can be assumed in the form of a sum of free term elements involving elements of \( I(y) \):

\[
C^k_i(y) = \lim_{\varepsilon \to 0} \int_{s_{\varepsilon}} T^k_i(x, y) dS_x = \sum_{e \in I(y)} C^k_i(y; E_e)
\] (16)

where the exact derivation of \( C^k_i(y; E_e) \) is given in the appendix (equations (22) to (23)).

Since the method of derivation of the correction term \( R^k(x, y) \) is now established, the formulation (13) can be considered as the regularized form of the initial integral equation (3).

**Numerical Implementation**

Both boundary and unknowns are discretized using three-noded flat triangles and interpolation techniques initially developed for the Finite Element Method. The discretization of the geometry and the unknowns is thus written, respectively, as follows (Bonnet, 1999):

\[
x(\xi) = \sum_{k=1}^{3} N_k(\xi)x^k \quad a(x) = \sum_{k=1}^{3} N_k(\xi)a^k
\] (17)

with \( x^k \): the node coordinates, \( N_k \): the linear interpolation functions and \( a^k \): the nodal values of the displacement or traction unknowns.

Thus, the set of scalar equations resulting from the discretization of equations (10), enforced at the nodes of the mesh, has the following matrix structure:

\[
[A]{u} + [B]{t} = 0
\] (18)

where \([A]\) and \([B]\) are fully populated non symmetric matrices. \( \{u\}, \{t\} \) are the “vectors” containing, respectively, the nodal values of \( u_i(y) \) and \( t_i(y) \). The incorporation of the boundary conditions consists in substituting the prescribed nodal values of \( \{u_i(t_i)\} \) into \( \{u\}, \{t\} \) in Eq. (18). The columns of this matrix equation are reordered as so to have a matrix equation of the form:

\[
[K]{v} = \{f\}
\] (19)

where the vector \( \{v\} \) consists of the unknown components of \( \{u\}, \{t\} \). The matrix \([K]\) contains the columns of \([A], [B]\) associated with those unknown components while the right-hand side \( \{f\} \) results from the multiplication of the known components of \( \{u\}, \{t\} \) by the corresponding columns of the matrices \([A], [B]\). As shown in the following for unbounded media, the right hand side \( \{f\} \) can also involve a contribution due to an incident wavefield. The method has been implemented into the computer code CESAR-LCPC (Humbert et al., 2005) of the Laboratoire Central des Ponts et Chausses (French Public Works Research Laboratory, Paris, France).

**Validation in Frequency-Domain Elastodynamics**

**Example 1** Spherical cavity under harmonic internal pressure.

**Description of the problem and analytical solution.**

The first example (figure 1) concerns a spherical cavity of radius \( R \) in a full elastic isotropic space undergoing an internal harmonic pressure. The cavity mesh includes 320 triangular boundary elements (that is 162 nodes) and a special generation process is considered.
to have a regular triangular mesh of the sphere starting from an icosahedron (Edouard et al., 1996) (also see next sections). Using the regularization method proposed herein, we have computed the displacement field around the cavity at various (normalized) frequencies.

The validation of the numerical results is made considering the analytical solution in terms of radial displacement $u(r, \omega)$ given by Eringen and Suhubi (1975) as follows:

$$u(r, \omega) = -\frac{P(\omega)R^4 (ik_p - 1/r) \exp (ik_p (r - R))}{4\nu (1 - ik_p R - k_S^2 R^2 / 4)}$$

where $k_p$ and $k_S$ are the longitudinal and transverse wavenumbers.

This equation can be rewritten using normalized distance $\chi = r/R$, normalized frequency $\eta = k_p R / \pi$ (that is $\eta = 2R / \lambda_p$, $\lambda_p$ being the longitudinal wavelength) and considering $v(\chi, \eta_p) = \mu u(r, \omega) P(\omega) R$. It leads to:

$$v(\chi, \eta_p) = -\left( \frac{i\pi \eta_p - 1}{1 - i\pi \eta_p - \pi^2 \zeta^2 \eta_p^2 / 4} \right) \chi^2$$

with $\zeta = k_p / k_S = \sqrt{(1 - 2\nu)/(2 - 2\nu)} = \sqrt{3}$ (that is $\nu = 0.25$).

Figure 2: Normalized radial displacement $v(\chi, \eta_p)$ (real part) vs normalized distance $\chi$: comparison between numerical and analytical results for normalized frequencies $\eta_p = 0.01, 0.50, 1.00$ and $2.00$.

Comparisons between numerical and analytical results. In figure 2, the real part of the normalized radial displacement $v(\chi, \eta_p)$ defined by equation (21) is displayed vs normalized distance $\chi$ for both analytical and numerical solutions at normalized frequencies $\eta_p = 0.01, 0.50, 1.00$ and $2.00$. For the nearly static case ($\eta_p = 0.01$) as well as the fully dynamic cases, the agreement between the numerical results and the analytical ones is very good at all normalized distances. From this first simple example, the reliability and accuracy of the proposed method then appear very good.

Efficiency of the regularization method. We will then investigate the efficiency of the regularization method itself by evaluating the correction term for the same mechanical problem (figure 3). A non regularized solution is computed by dropping the correction term in (21). In figure 3, this non regularized solution is compared with both the regularized one and the analytical solution at normalized frequencies $\eta_p = 0.50$ (left) and $2.00$ (right). These comparisons show that the numerical results without the analytical correction are far from both analytical and corrected numerical solutions. The efficiency of the regularization method then appears very good since the direct computation of the singular integrals leads to very bad results.

Example 2: Diffraction of a plane wave by a spherical cavity.

Description of the problem and analytical solution. The second example deals with the diffraction of a plane P-wave ($u_{inc} = U_0 \exp [i(k_p x - \omega t)]$ with $U_0 = 1$), propagating along $x$ axis, by a spherical cavity. The numerical results are firstly computed in frequency domain and compared with analytical results. They are afterwards converted into time domain to characterize the scattered wavefield. As shown in figure 3, we have computed the wave field around the cavity for various directions. The boundary element mesh of the cavity (2562 nodes) is generated the same way as in the previous case (Edouard et al., 1996). This mesh has been refined since the wave field has much stronger variations compared to the previous example. In figure 4, the radial displacement $u(r, \theta, \phi)$ is displayed vs distance $\chi = r / R$ for both analytical and numerical solutions at two different normalized frequencies $\eta_p = 2R / \lambda_p$. Different azimuths are also considered. The analytical solution in terms of radial displacement $u_\alpha$ is given by Pao and Mow (1973) as well as Eringen and Suhubi (1975).

Comparisons between numerical and analytical results in frequency domain. The results are computed for various azimuths ($\theta_i = (i - 1) \times 45^\circ$, $1 \leq i \leq 5$) and figure 4 displays the real part of the radial displacement vs normalized distance $\chi = r / R$ ($1 \leq \chi \leq 2$). There are two mistakes in the original book of Eringen and Suhubi (1975) which have to be corrected as follows. The original expression of $T_{11}^{(3)}$ in (Eringen and Suhubi, 1975) (page 914, Eq. (9.12.11)) is:

$$T_{11}^{(3)}(ar) = (a^2 - n - 0.5 \beta^2 r^2) b_n^{(1)}(ar) + 2ar b_n^{(1)}(ar)$$

and should be replaced by the following expression:

$$T_{11}^{(3)}(ar) = (a^2 - n - 0.5 \beta^2 r^2) b_n^{(1)}(ar) + 2ar b_n^{(1)}(ar)$$

The original expression of $IC_n$ in (Eringen and Suhubi, 1975) (page 914, Eq. (9.12.13)) is:

$$IC_n = \left( 1 - \Delta_n \right) \alpha \omega^{2n+1} \left[ T_1^{(1)}(aa) \bar{T}_1^{(3)}(aa) - T_1^{(1)}(aa) \bar{T}_1^{(3)}(aa) \right]$$

and should be replaced by:

$$IC_n = \left( 1 - \Delta_n \right) \alpha \omega^{2n+1} \left[ T_1^{(1)}(aa) \bar{T}_1^{(3)}(aa) - T_1^{(1)}(aa) \bar{T}_1^{(3)}(aa) \right]$$

where $a$ is the cavity radius, denoted $R$ in this paper.
is lower because it does not coincide with the direction of propagation. Whereas for the scattered wavefield, radial directions correspond to the direction of propagation and the time domain numerical results show a large amplitude for both $X$ and $Y$ components. For the $Y$ component of the scattered wavefield, both $P$ and $S$-wave components can be identified in figure 5. The velocity values estimated from the numerical results are found very close to theoretical ones. The velocity discrepancy between the downstream $S$ component of the scattered wavefield and the transmitted $P$-wave is only due to the change of the apparent velocity of the latter which is azimuth dependent.

- $\theta = 90^\circ$ (bottom): for this azimuth, the apparent velocity of the incident $P$-wave is zero because it is perpendicular to the direction of propagation. The $X$ and $Y$ components of the displacement are displayed on one side of the cavity only since they are symmetrical on the opposite side. The $X$ component clearly shows the $S$-wave part of the scattered wavefield. The estimation of its velocity is as good as in previous cases. For this azimuth, the $Y$ component shows that the interaction between the plane wave and the cavity is particularly complex since we have a grazing incidence on the cavity wall.

Figure 4: Diffraction of a plane wave by a spherical cavity: comparison with analytical results for various azimuthes at normalized frequencies $\eta_P=1.00$ and $\eta_P=2.00$.

Figure 5: Diffraction of a plane $P$-wave by a spherical cavity: numerical results in time domain (Ricker signal) for various azimuthes at normalized frequency $\eta_P=0.50$.

Example 3: Amplification of a plane seismic wave by a semi-spherical alluvial basin.

Description of the problem and reference solution. The third example investigates the amplification of a plane seismic wave in an alluvial basin. In seismology and earthquake engineering, this phenomenon is known as "site effects" and generally leads to a strong amplification of the seismic motion in soft alluvial deposits (Bard and Bouchon, 1985; Bielak et al., 1999; Chavez-Garcia et al., 2000; Moeen-Vaziri and Trifunac, 1985; Semblat et al., 2000, 2003a, 2005). The example considered herein corresponds to a semi spherical alluvial basin (that is a soft elastic inclusion) in an elastic half space. Numerous papers have investigated the 3D wave diffraction by a semi spherical canyon (Lee, 1978; Liao et al., 2004; Yokoi, 2003) or 3D seismic wave amplification by surface heterogeneities (Dravinski, 2003; Komatitsch and Vilote, 1998; Moczo et al., 2002; Sanchez-Sesma, 1983; Sanchez-Sesma and Luzn, 1995).

Several results have been published for the case of a semi spherical alluvial basin (Dravinski, 2003; Lee, 2004).
1984; Sánchez-Sesma, 1983). The 3D BEM model considered herein for purpose of validation is depicted in figure 6. The mesh includes the semi-spherical basin of radius R (same type of triangular meshing as in the previous section (Edouard et al., 1996)) and part of the free-surface (for r ≤ 5R). The contribution of the free surface r ≥ 5R in the BIE is neglected. Therefore, the BIE are enforced at the nodes of the mesh except those located at its boundary. The model is excited by a vertical plane P-wave. For the comparison, we will consider the results of Sánchez-Sesma (1983) derived thanks to a series expansion method. We will then investigate the amplification of the motion at the surface of the alluvial basin (i.e. soft inclusion).

For the semi-spherical basin and the half-space, the mechanical parameters are chosen identical to Sánchez-Sesma’s values as follows:

- shear moduli: \( \mu_R/\mu_E = 0.3 \)
- mass densities: \( \rho_R/\rho_E = 0.6 \)
- Poisson’s ratios: \( \nu_R = 0.30 \) and \( \nu_E = 0.25 \)

where subscript \( R \) refers to the alluvial basin and subscript \( E \) to the half-space.

Similarly to the previous section, we consider for the computations the same normalized frequency as Sánchez-Sesma corresponding to the diameter-to-wavelength ratio \( \eta_P = 2R/\Lambda_P \) where \( \Lambda_P \) is the P wavelength in the alluvial basin.

Comparison between numerical and reference results. In figure 6, the amplification of the seismic motion is computed at the free-surface (vertical displacement) and displayed vs normalized distance (0 ≤ \( \chi \) ≤ 3). It is compared with Sánchez-Sesma’s results (1983) at normalized frequency \( \eta_P = 0.50 \). The amplification of the vertical motion at the center of the semi-spherical basin is very well estimated by our numerical approach: 2.81 for our numerical approach (i.e. 5.63 in amplitude) to be compared to 2 for the half-space) and 2.82 for Sánchez-Sesma’s results (i.e. 5.64 in amplitude). The computed displacement/distance curve from our numerical approach is very close to Sánchez-Sesma’s semi-analytical results (figure 6). This amplification value is larger than for the constant depth layer case (1D) since, for the semi-spherical basin, focusing effects are very strong (Sánchez-Sesma, 1983; Semblat et al., 2000, 2005).

It should be noticed that the normalized frequency \( \eta_P = 0.50 \) corresponds to the fundamental frequency of the 1D case (the wavelength being \( \Lambda_P = 4R \) with \( R \) the depth of the basin). Nevertheless, at this frequency, the variation of the amplification factor vs frequency is strong: for the 3D semi-spherical basin, this frequency is rather far from the maximum amplification peak. If we compute the amplification factor at the centre of the semi-spherical basin for various frequencies, the largest site effects are found at normalized frequency \( \eta_P = 0.57 \). At this frequency and for the mechanical properties chosen herein, the corresponding amplification factor is about 4.76 (i.e. 9.52 in amplitude), that is 70% larger than for \( \eta_P = 0.50 \). For sake of comparisons, around normalized frequency \( \eta_P = 0.57 \) the amplitude variation with frequency is smaller (resp. basin properties).

Figure 7: Computed vertical motion showing the amplification at the basin surface and comparison with Sánchez-Sesma’s result for normalized frequency \( \eta_P = 0.50 \).

Conclusion

In this paper, a simple and efficient method to regularize singular integrals in 3D boundary integral equations has been presented. The regularization method is based on the derivation of analytical terms which correct the error due to the straightforward estimation of singular integrals through classical Gaussian quadrature formulas. The analytical derivation of the correction term has assumed a three-noded triangle discretization of the boundary surfaces. However, the method described in the appendix can be easily generalized to any flat element such as quadrangle. This method has been implemented in a BEM code and applied to 3D frequency domain elastodynamics.

Some comparisons have been made with (semi-)analytical results for simple problems:

- cavity under harmonic pressure: the agreement between our numerical results and the analytical solution is very good for various frequencies even with a small number of nodes/elements. The efficiency of the regularization method proposed in this paper is also discussed for this example.
- diffraction of a plane-wave by a spherical cavity: the agreement between our numerical results and the analytical solution is very good for various azimuths and frequencies. In time domain, the numerical results are also found to be satisfactory.
- wave amplification in a semispherical alluvial basin (soft inclusion): the comparison of our numerical results with Sánchez-Sesma’s semi-analytical results (Sánchez-Sesma, 1983) is also satisfactory. Further comparisons are planned with other current
numerical approaches and more complex geometries.

Considering these good results, future work will then concern more realistic cases in the field of seismology. For sake of numerical efficiency, the regularization method could also be implemented in a Symmetric Galerkin boundary element formulation (Bonnet et al., 1998) or in the framework of a Fast Multipole Method (Greenberg et al., 1998). Our main goal is to have a detailed description of the 3D geological structure of a given area to perform reliable computations of seismic wave propagation and amplification (Bard and Bouchon, 1985; Bouchon et al., 1989; Chvez-Garca et al., 2000; Frankel and Vidale, 1992; Moczo et al., 2002; Semblat et al., 2000, 2003a,b, 2005).

Appendix

Calculation of $C_i^k(y; E_c)$

Let us calculate the free term $C_i^k(y)$ defined by:

$$C_i^k(y) = \lim_{\varepsilon \to 0} \int_{s_{x}} T_{ik}^k(x, y) \, dS_x$$

In Eq. (22), $s_x$ is a spherical surface of radius $\varepsilon$. Let $x$ be a point on $s_x$. The unit outward normal to $s_x$ is given by $n = (y - x)/\varepsilon$. Thus the stress vector of the Kelvin fundamental solution applied to $s_x$ has the form:

$$T_i^k(x, y) = \frac{1}{8\pi(1 - \nu) \varepsilon^2} ((1 - 2\nu) \delta_{ik} + 3n_in_k)$$

Substituting this expression for $T_i^k$ in (22) yields:

$$C_i^k(y) = \frac{1}{8\pi(1 - \nu)} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{s_x} \sum_{n} n_in_k \, dS_x$$

where $|s_x|$ is the surface area of $s_x$. Here $|s_x| = \varepsilon^2 \psi$, where $\psi$ is the solid angle. A small amount of calculations allows to derive the following expression:

$$\int_{s_x} n_in_k \, dS_x = \frac{|s_x|}{3} \delta_{ik} - \frac{2}{3} \varepsilon^2 \sum_{e \in I(y)} \sin \left( \frac{\theta_e}{2} \right) b_{ik}^e n_k^e$$

where $\theta_e$ is the angle formed by the edges of $E_c$ at $y$, $b_{ik}^e$ is the unit vector of the bisecting line and $n_k^e$ is the unit outward normal to $E_c$. The symmetry with respect to subscripts $i$ and $k$ in Eq. (24) shows the following identity:

$$\sum_{e \in I(y)} \sin \left( \frac{\theta_e}{2} \right) b_{ik}^e = \sum_{e \in I(y)} \sin \left( \frac{\theta_e}{2} \right) b_{ik}^e$$

Combining (24) and (25) yields:

$$C_i^k(y) = \frac{\psi}{4\pi} \delta_{ik} - \frac{1}{8\pi(1 - \nu)} \sum_{e \in I(y)} \sin \left( \frac{\theta_e}{2} \right) (b_{ik}^e n_k^e + b_{ik}^e n_i^e)$$

Finally the solid angle is assumed to be the sum of element solid angles:

$$\psi = \sum_{e \in I(y)} \psi_e$$

Pratically, as shown in figure 5, $\psi_e$ can be defined by the solid angle of the trihedron of apex $y$ formed by the two edges of element $E_c$ and the semi-axis in the direction of $-n(y)$, where $n(y)$ is an arbitrary outward unit vector at $y$. In this case $\psi_e = (\varphi_1^e + \varphi_2^e + \varphi_3^e - \pi)$ where the $\varphi_i^e$ are the three angles formed by the plane of the trihedron (figure 5). The calculation of the solid angles $\psi_e$ relies
on the knowledge of an outward unit vector at each node of the mesh. Practically for each node of coordinate \( y \), \( \mathbf{n}(y) \) can be calculated as the mean of unit normals to each element of \( I(y) \). It should be noticed that the averaging of the normal is only conventional. It results from an arbitrary choice in order to perform the calculation of solid angles \( \psi_n \). The accuracy of the method does not depend on this averaging procedure since the value of the solid angle \( \psi \) (equation (28)) is eventually recovered whatever the choice of \( \mathbf{n}(y) \). Therefore, the free term \( C^k(y) \) is really the sum of free term elements \( C^k_i(y; E_e) \) of the form:

\[
C^k_i(y; E_e) = \frac{\psi^e_n}{4\pi} \delta_{ik} - \frac{1}{8\pi(1-\nu)} \sin \left( \frac{\theta^e}{2} \right) (b_k^i n_k^e + b_k^e n_k^i)
\]

(29)

Figure 8: Conical surface with apex at \( y \)

Calculation of \( I^k_i(y; E_e) \)

The integral \( I^k_i(y) \) can be written in the form:

\[
I^k_i(y) = \lim_{\varepsilon \to 0} \sum_{e \in I(y)} \int_{(E_e - e^\varepsilon)} T^k_i(x, y) dS_x
\]

(30)

where \( e^\varepsilon_e = E_e \cap v_\varepsilon \) (with of course \( \sum_{e \in I(y)} e^\varepsilon_e = e_e \)). Given \( \varepsilon > 0 \), let us calculate the element integral appearing in (30). Let \( x \) be a current point on \( E_e \) and note \( n^e \) the unit outward normal to \( E_e \). The stress vector of the Kelvin fundamental solution applied to \( E_e \) is given by:

\[
T^k_i(x, y) = \frac{1 - 2\nu}{8\pi(1-\nu)} \frac{n^e_i e_k - n^e_k e_i}{r^2}
\]

(31)

where \( e_i = (y_i - x_i)/r \). A trivial integration of the above expression shows that:

\[
\int_{(E_e - e^\varepsilon)} \frac{n^e_i e_k - n^e_k e_i}{r^2} dS_x = \int_0^{\theta^e} (n^e_i e_k(\alpha) - n^e_k e_i(\alpha)) \ln L(\alpha) d\alpha
\]

\[
-2\sin \left( \frac{\theta^e}{2} \right) (n^e_i b_k^e - n^e_k b_i^e) \ln \varepsilon
\]

(32)

where \( L(\alpha) \) is the length of the segment defined in the figure (10). Thanks to identity (20), integral \( I^k_i(y) \) is then the sum of element integrals defined by:

\[
I^k_i(y; E_e) = \frac{1 - 2\nu}{8\pi(1-\nu)} \int_0^{\theta^e} (n^e_i e_k(\alpha) - n^e_k e_i(\alpha)) \ln L(\alpha) d\alpha
\]

(33)

Figure 10: Distance \( L(\alpha) \) on \( E_e \)

Calculation of \( J^k_i(y; E_e) \)

The integral \( J^k_i(y) \) can be written in the form:

\[
J^k_i(y) = \lim_{\varepsilon \to 0} \sum_{e \in I(y)} \int_{(E_e - e^\varepsilon)} U^k_i(x, y) dS_x
\]

Substituting expression (11) for \( U^k_i(x, y) \) in (34) gives the expression of \( J^k_i(y; E_e) \):

\[
J^k_i(x; E_e) = \frac{1}{16\pi\mu(1-\nu)} \int_0^{\theta^e} (e_i e_k + (3 - 4\nu)\delta_{ik}) L(\alpha) d\alpha
\]

(35)
References


