Excursions of the integral of the Brownian motion

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Abstract

The integrated Brownian motion is sometimes known as the Langevin process. Lachal studied several excursion laws induced by the latter. Here we follow a different point of view developed by Pitman for general stationary processes. We first construct a stationary Langevin process and then determine explicitly its stationary excursion measure. This is then used to provide new descriptions of Itô's excursion measure of the Langevin process reflected at a completely inelastic boundary, which has been introduced recently by Bertoin.

Key words. Langevin process, stationary process, excursion measure, time-reversal, h-transform.

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1 Introduction

The Langevin process in a non-viscous fluid is simply defined as the integrated Brownian motion, that is:

$$Y_t = Y_0 + \int_0^t W_s ds,$$

where W is a Brownian motion started an arbitrary $v \in \mathbb{R}$ (so v is the initial velocity of Y). The Langevin process is not Markovian, but the pair $Z = (Y, W)$, which is sometimes known as the Kolmogorov process, enjoys the Markovian property. We refer to Lachal [7] for a rich source of information on this subject.

Lachal [7] has studied in depth both the "vertical" and "horizontal" excursions of the Brownian integral. The purpose of this work is to follow a different (though clearly related) point of view, which has been developed in a very general setting by Pitman [11]. Specifically, we start from the basic observation that the Lebesgue measure on $\mathbb{R}^2$

is invariant for the Kolmogorov process, so one can work with a stationary version of the latter. The set of times at which the stationary Kolmogorov process visits $\{0\} \times \mathbb{R}$ forms a random homogeneous set in the sense of Pitman, and we are interested in the excursion measure Q_{ex} that arises naturally in this setting. We shall show that Q_{ex} has a remarkably simple description and fulfills a useful invariance property under time-reversal. We then study the law of the excursions of the Langevin process away from 0 conditionally on its initial and final velocity, in the framework of Doob's h-transform. Finally, we apply our results to investigate the Langevin process reflected at a completely inelastic boundary, an intriguing process which has been studied recently by Bertoin [2, 3]. In particular we obtain new expressions for the Itô measure of its excursions away from 0.

2 Preliminaries

In this part we introduce some general or intuitive notations and recall some known results that we will use later on. We write Y for the Langevin process, W for its derivative, and Z for the Kolmogorov process (Y, W), which, unlike Y, is Markovian.

The law of the Kolmogorov process with initial condition (x, u) will be written $\mathbb{P}^+_{x,u}$, and the expectation under this measure $\mathbb{E}^+_{x,u}$. Here, the exponent $+$ refers to the fact that the time parameter t is nonnegative. We denote by $p_t(x, u; dy, dv)$ the probability transitions of Z, and by $p_t(x, u; y, v)$ their density. For $x, u, y, v \in \mathbb{R}$, we have:

$$p_t(x, u; y, v)dudv := p_t(x, u; dy, dv) := \mathbb{P}^+_{x,u}(Z_t \in dydv).$$

These densities are known explicitly and given by:

$$p_t(x, u; y, v) = \frac{\sqrt{3}}{\pi t^2} \exp\Big[-\frac{6}{t^3}(y - x - tu)^2 + \frac{6}{t^2}(y - x - tu)(v - u) - \frac{2}{t}(v - u)^2\Big]. \quad (2.1)$$

One can check from the formula that the following identities are satisfied:

$$\begin{aligned} p_t(x, u; y, v) &= p_t(0, 0; y - x - ut, v - u), && (2.2)\\ p_t(x, u; y, v) &= p_t(-x, -u; -y, -v), && (2.3)\\ p_t(x, u; y, v) &= p_t(x, v; y, u). && (2.4) \end{aligned}$$

A combination of these formulas gives

$$p_t(x, u; y, v) = p_t(y, -v; x, -u), \quad (2.5)$$

that we will use later on. See for example the equations (1.1), p 122, and (2.3), p 128, in [7], for references.

The semigroup of the Kolmogorov process will be written P_t. If f is a nonnegative measurable function, we have:

$$P_t f(x,u) := \mathbb{E}^+_{x,u}\big(f(Y_t,W_t)\big) = \int_{\mathbb{R}^2} dydvp_t(x,u;y,v)f(y,v).$$

The law of the Kolmogorov process with initial distribution given by the Lebesgue measure λ on $\mathbb{R}^2$ will be written $\mathbb{P}^+_\lambda$. It is given by the expression:

$$\mathbb{P}^+_\lambda = \int_{\mathbb{R}^2} \lambda(dx,du)\mathbb{P}^+_{x,u}.$$

Although λ is only a σ-finite measure, the expression above still defines what we call a stochastic process in a generalized sense (this is a common generalization, though). We still use all the usual vocabulary, such as the law of the process, the law of the process at the instant t, even though this laws are now σ-finite measures and not probabilities.

Eventually, we recall the scaling property of the Langevin process:

$$\mathbb{E}^+_{x,u}\Big(F\big((X_t)_{t\geq 0}\big)\Big) = \mathbb{E}^+_{k^3x,ku}\Big(F\big((k^{-3}X_{k^2t})_{t\geq 0}\big)\Big), \tag{2.6}$$

where F is any nonnegative measurable functional.

3 Stationary Kolmogorov process

The stationary Kolmogorov process is certainly not something new for the specialists, as it is known that λ is an invariant measure for the Kolmogorov process. This part still gives, for the interested reader, a rigorous introduction to the stationary Kolmogorov process, including a duality property that allows us to consider the effect of time-reversal, what will be a central point of this paper.

3.1 Stationarity and duality lemmas

We write λ for the Lebesgue measure on $\mathbb{R}^2$.

Lemma 1. *For any nonnegative measurable functions f, g on $\mathbb{R}^2$ and every $t \geq 0$, we have:*

$$\mathbb{E}^+_\lambda\big(f(Y_t,W_t)\big) = \mathbb{E}^+_\lambda\big(f(Y_0,W_0)\big),$$

and:

$$\mathbb{E}^+_\lambda\big(f(Y_0,W_0)g(Y_t,W_t)\big) = \mathbb{E}^+_\lambda\big(f(Y_t,-W_t)g(Y_0,-W_0)\big).$$

This lemma states the (weak) stationarity of the measure λ and a duality property of the process under this measure.

Proof. Let f be a nonnegative measurable function on $\mathbb{R}^2$, and t be a positive real number.

$$\begin{aligned}
\mathbb{E}^+_\lambda\big(f(Y_t,W_t)\big) &:= \int dxdu\mathbb{E}^+_{x,u}\big(f(Y_t,W_t)\big)\\
&= \int dxdu \int dydv\, p_t(x,u;y,v)f(y,v)\\
&= \int\int dxdudydv\, p_t(y,-v;x,-u)f(y,v) \qquad \text{by (2.5)}\\
&= \int dydv f(y,v)\int dxdu\, p_t(y,-v;x,u)\\
&= \int dydv f(y,v)\\
&= \mathbb{E}^+_\lambda\big(f(Y_0,W_0)\big),
\end{aligned}$$

where in the fourth line we made the simple change of variables $u \to -u$.

For the second part, let f and g be two nonnegative measurable functions, and t a positive real number.

$$\begin{aligned}
&\mathbb{E}^+_\lambda\big(f(Y_0,W_0)g(Y_t,W_t)\big)\\
&= \int dxduf(x,u)\int dydv\, p_t(x,u;y,v)g(y,v)\\
&= \int\int dxdudydv f(x,-u)g(y,-v)\, p_t(x,-u;y,-v)\\
&= \int dydvg(y,-v)\int dxduf(x,-u)p_t(y,v;x,u) \qquad \text{by (2.5) again}\\
&= \mathbb{E}^+_\lambda\big(g(Y_0,-W_0)f(Y_t,-W_t)\big).
\end{aligned}$$

The lemma is proved. □

We immediately deduce the following corollary.

Corollary 1. *For any $t > 0$, we have:*

1) Stationarity: The law of the process $(Y_{t+s},W_{t+s})_{s\geq 0}$ under $\mathbb{P}^+_\lambda$ is $\mathbb{P}^+_\lambda$.

2) Duality: the laws of the processes $(Y_{t-s},-W_{t-s})_{0\leq s\leq t}$ and $(Y_s,W_s)_{0\leq s\leq t}$ under $\mathbb{P}^+_\lambda$ are the same.

This corollary provides a probabilistic interpretation of the stationarity and the duality property, here stated in a strong sense. Strong sense means that we consider here the whole trajectory and not merely the two-dimensional time-marginals. We thus see that the stationarity is a property of invariance of the process by time-translation, and that the duality is a property of symmetry of the process by time-reversal.

Proof. As the processes we consider are continuous, their laws are determined by their finite-dimensional marginals. The strong stationarity is a simple consequence from the weak stationarity and the Markov property, while the strong duality needs a bit more work. Let $n \in \mathbb{N}$, let $0 = t_0 \leq t_1 \leq ... \leq t_n$ be real numbers and let f_0, f_1, ..., f_n be $n+1$ nonnegative measurable functions. We have to prove that the following equality is satisfied (recall $Z = (Y, W)$):

$$\mathbb{E}_\lambda^+\big[f_0(Z_0)f_1(Z_{t_1})...f_n(Z_{t_n})\big] \quad = \quad \mathbb{E}_\lambda^+\big[f_n(Z_0)f_{n-1}(Z_{t_n-t_{n-1}})...f_1(Z_{t_n-t_1})f_0(Z_{t_n}))\big]. \qquad (3.1)$$

This is checked by induction on n. For $n = 1$, this is nothing else than the weak duality. We suppose now that the identity (3.1) is true for any integer strictly smaller than n. We have:

$$\begin{aligned}
& \mathbb{E}_\lambda^+\big[f_0(Z_0)f_1(Z_{t_1})...f_n(Z_{t_n})\big] \\
= \;& \mathbb{E}_\lambda^+\big[f_0(Z_0)...f_{n-1}(Z_{t_{n-1}})\mathbb{E}^+_{Z_{t_{n-1}}}[f_n(Z_{t_n-t_{n-1}})]\big] \\
= \;& \mathbb{E}_\lambda^+\big[\mathbb{E}^+_{Z_0}[f_n(Z_{t_n-t_{n-1}})]f_{n-1}(Z_0)f_{n-2}(Z_{t_{n-1}-t_{n-2}})...f_0(Z_{t_{n-1}})\big] \\
= \;& \mathbb{E}_\lambda^+\Big[f_n(Z_{t_n-t_{n-1}})\mathbb{E}^+_{Z_0}\big[f_{n-1}(Z_0)f_{n-2}(Z_{t_{n-1}-t_{n-2}})...f_0(Z_{t_{n-1}})\big]\Big] \\
= \;& \mathbb{E}_\lambda^+\Big[f_n(Z_0)\mathbb{E}^+_{Z_{t_n-t_{n-1}}}\big[f_{n-1}(Z_0)f_{n-2}(Z_{t_{n-1}-t_{n-2}})...f_0(Z_{t_{n-1}})\big]\Big] \\
= \;& \mathbb{E}_\lambda^+\big[f_n(Z_0)f_{n-1}(Z_{t_{n-1}})...f_0(Z_{t_n})\big].
\end{aligned}$$

To get the second equality, we used (3.1) with the functions f_0, ..., f_{n-2} and $\tilde{f}_{n-1} : (x, u) \rightarrow f_{n-1}(x, u)\mathbb{E}^+_{x,u}\big[f_n(Z_{t_n-t_{n-1}})\big]$. To get the fourth equality, we use the weak duality with times 0 and $t_n - t_{n-1}$.

This completes our proof. □

3.2 Construction of the stationary Kolmogorov process

We are ready to construct the stationary Kolmogorov process with time parameter $t \in \mathbb{R}$. First, we construct a process indexed by $\mathbb{R}$ with a position (x, u) at time 0. The process $(Z_t)_{t\in\mathbb{R}} = (Y_t, W_t)_{t\in\mathbb{R}}$ is such that $(Y_t, W_t)_{t\in\mathbb{R}_+}$ has the law $\mathbb{P}^+_{x,u}$ and $(Y_{-t}, -W_{-t})_{t\in\mathbb{R}_+}$ is an independent process and of law $\mathbb{P}^+_{x,-u}$. The law of the process $(Z_t)_{t\in\mathbb{R}}$ will be denoted by $\mathbb{P}_{x,u}$.

Definition 1. *The stationary Kolmogorov process is the generalized process of law $\mathbb{P}_\lambda$ given by:*

$$\mathbb{P}_\lambda = \int dxdu\mathbb{P}_{x,u}. \qquad (3.2)$$

Lemma 1 and Corollary 1 still hold if we drop the superscript +. We stress that the stationary Kolmogorov process has a natural filtration given by $\mathcal{F}_t = \sigma(\{Z_s\}_{-\infty<s\leq t}) = \sigma(\{Y_s\}_{-\infty<s\leq t})$. If $(Z_t)_{t\in\mathbb{R}} = (Y_t, W_t)_{t\in\mathbb{R}}$, we call conjugate of Z and write $\overline{Z}$ for the process $\big(\overline{Z}\big)_{t\in\mathbb{R}} = (Y_t, -W_t)_{t\in\mathbb{R}}$.

Lemma 2. *The stationary Kolmogorov process has the following properties:*

1. *Under $\mathbb{P}_\lambda$, The processes Z and $\left(\overline{Z}_{-t}\right)_{t\in\mathbb{R}}$ have the same law. That is, the law $\mathbb{P}_\lambda$ is invariant by time-reversal and conjugation.*

2. *Under $\mathbb{P}_\lambda$, the processes $(Y_t, W_t)_{t\in\mathbb{R}}$ and $(Y_{t_0+t}, W_{t_0+t})_{t\in\mathbb{R}}$ have the same law for any $t_0 \in \mathbb{R}$. That is, the law $\mathbb{P}_\lambda$ is invariant by time-translation.*

3. *The process Z is a stationary Markov process under $\mathbb{P}_\lambda$.*

Proof. (1) Let us consider Z a process of law $\mathbb{P}_{x,u}$. It is immediate from the definition that the conjugate of the time-reversed process, that is $\left(\overline{Z}_{-t}\right)_{t\in\mathbb{R}}$, is a process of law $\mathbb{P}_{x,-u}$. The result follows.

(2) Let us write $\mathbb{P}_\lambda^{t_0}$ for the law of the process $(Y_{t_0+t}, W_{t_0+t})_{t\in\mathbb{R}}$ under $\mathbb{P}_\lambda$, and let us suppose in this proof that t_0 is positive. We want to prove that $\mathbb{P}_\lambda^{t_0}$ and $\mathbb{P}_\lambda$ are equal. It is enough to prove that for any suitable functional f, g and h, the expectations of the variable

$$f\big((Y_t)_{t\le -t_0}\big)\ g\big((Y_t)_{-t_0\le t\le 0}\big)\ h\big((Y_t)_{0\le t}\big)$$

under these two measures are equal [1]. On the one hand, we have:

$$\begin{aligned}
&\mathbb{E}_\lambda^{t_0}\big[f((Y_t)_{t\le -t_0})g((Y_t)_{-t_0\le t\le 0})h((Y_t)_{0\le t})\big]\\
=\ &\mathbb{E}_\lambda\big[f((Y_t)_{t\le 0})g((Y_t)_{0\le t\le t_0})h((Y_t)_{t_0\le t})\big]\\
=\ &\mathbb{E}_\lambda\Big[\mathbb{E}_{Y_0,W_0}\big[f((Y_t)_{t\le 0})\big]\ \mathbb{E}_{Y_0,W_0}\big[g((Y_t)_{0\le t\le t_0})\mathbb{E}_{Y_{t_0},W_{t_0}}[h((Y_t)_{t_0\le t})]\big]\Big]\\
=\ &\mathbb{E}_\lambda\Big[\mathbb{E}_{Y_0,W_0}\big[f((Y_t)_{t\le 0})\big]\ g\big((Y_t)_{0\le t\le t_0}\big)\ \mathbb{E}_{Y_{t_0},W_{t_0}}\big[h((Y_t)_{t_0\le t})\big]\Big]\\
=\ &\mathbb{E}_\lambda\big[F(Y_0,W_0)\ g((Y_t)_{0\le t\le t_0})\ H(Y_{t_0},W_{t_0})\big],
\end{aligned}$$

where we wrote $F(x,u) = \mathbb{E}_{x,u}\big[f((Y_t)_{t\le 0})\big]$ and $H(x,u) = \mathbb{E}_{x,u}\big[h((Y_t)_{t_0\le t})\big]$. To get the third line we use the independence of $(Y_t)_{t\le 0}$ and $(Y_t)_{t\ge 0}$ conditionally on (Y_0, W_0) and the Markov property of $(Y_t)_{t\ge 0}$ at time t_0.

On the other hand, we have:

$$\begin{aligned}
&\mathbb{E}_\lambda\big[f((Y_t)_{t\le -t_0})g((Y_t)_{-t_0\le t\le 0})h((Y_t)_{0\le t})\big]\\
=\ &\mathbb{E}_\lambda\big[\mathbb{E}_{Y_{-t_0},W_{-t_0}}[f((Y_t)_{t\le 0})]\ g((Y_t)_{-t_0\le t\le 0})\ \mathbb{E}_{Y_0,W_0}[h((Y_t)_{0\le t})]\big]\\
=\ &\mathbb{E}_\lambda\big[F(Y_{-t_0},W_{-t_0})\ g((Y_t)_{-t_0\le t\le 0})\ H(Y_0,W_0)\big]\\
=\ &\mathbb{E}_\lambda\big[H(Y_0,W_0)\ g((Y_{t_0-t})_{0\le t\le t_0})\ F(Y_{t_0},W_{t_0})\big],
\end{aligned}$$

where F and H are defined above and we used the time-reversal invariance property for $\mathbb{P}_\lambda$ to get the last line.

[1] We take only functionals of Y and not of W. This is in order to make the notations simpler and has no incidence, as W can be recovered from Y by taking derivatives.

Now, the fact that the two expressions we get are equal is a direct consequence of the duality property stated in a strong sense.

(3) In this third statement the important word is the word *Markov*, not the word *stationary*. Indeed the Markov property for negative times is not immediate in the definition of $\mathbb{P}_\lambda$. But the Markov property for positive times is, and this combined with the stationarity immediately gives the Markov property for any time. $\square$

In the following, we will speak about the stationary Kolmogorov process for the process (Y, W) under $\mathbb{P}_\lambda$, and about the stationary Langevin process for the process Y under $\mathbb{P}_\lambda$.

Before speaking about excursions of these processes, let us notice that we could have constructed the stationary Kolmogorov process starting from time $-\infty$ with using just the stationarity (and not the duality). The way to do it is to consider the family of measures $({}^t\mathbb{P}^+_\lambda)_{t\le 0}$, where ${}^t\mathbb{P}^+_\lambda$ is the measure of the Kolmogorov process starting from the measure λ at time t . The stationarity gives us that these measures are compatible. We thus can use Kolmogorov extension theorem and construct the measure starting from time $-\infty$.

In this construction, though, the nontrivial fact is that the process is invariant by time-reversal, and we need the duality property to prove it.

4 Excursions of the stationary Langevin process

4.1 Stationary excursion measure

We will now study the *stationary excursion measure* for a stationary process given by Pitman in [11].

If t is a time such that $Y_t = 0$ and $W_t \neq 0$, we will write e^t or $(e^t_s)_{0\le s\le\zeta}$ for the excursion of Y away from 0 started at time t, and ζ for its lifetime, that is, $\zeta(Y) := \inf\{s > 0 : Y_{t+s} = 0\}$ and $e^t_s := Y_{t+s}$ for $0 \le s \le \zeta$.

It belongs to the *set of vertical excursions* $\mathcal{E}$, that is, the set of continuous functions $t \to X_t$, defined on $\mathbb{R}_+$, that have a càdlàg right-derivative $\dot{X}$, such that X starts from zero ($X_0 = 0$), X leaves immediately zero (X has a strictly positive lifetime $\zeta(X)$), and dies after its first return to 0. This definition is inspired by the terminology of Lachal [7], except that he considers the set of vertical excursions for the two-dimensional process.

We write $\mathbb{P}^\partial_{x,u}$ for the law of the Langevin process starting with position x and velocity $u \neq 0$, and killed at its first return-time to 0. So it is a law on the set of vertical excursions, and under $\mathbb{P}^+_{0,u}$, the excursion starting at time 0 is written e^0 and has law $\mathbb{P}^\partial_{0,u}$.

Considering the stationary Langevin process and the homogeneous set $\{t, Y_t = 0\}$, we define in the sense of Pitman [11] the stationary excursion measure:

Definition 2. *We call* stationary excursion measure *of the stationary Langevin process, and we write Q_{ex}, the measure given by:*

$$Q_{ex}(\bullet) = \mathbb{E}_\lambda\big[\#\{0 < t < 1, Y_t = 0, e^t \in \bullet\}\big]. \tag{4.1}$$

We stress that this measure does not give a finite mass to the set of excursions with lifetime greater than 1, contrarily to the Itô excursion measure of a Markov process. By a slight abuse of notation, when A is an event, we will write $Q_{ex}(\mathbf{1}_A)$ for $Q_{ex}(A)$.

We stress that for convenience we focus here and thereafter on the Langevin process; clearly this induces no loss of generality as the Kolmogorov process can be recovered from the Langevin process by taking derivatives. For instance, if we use the definition of a stationary excursion measure by Pitman for the stationary Kolmogorov process and the homogeneous set $\{t, (Y_t, W_t) \in \{0\} \times \mathbb{R}\}$, the measure we get is nothing else that the Q_{ex}-measure of $(X, \dot{X})$, where X denotes the canonical process and $\dot{X}$ the velocity of X. Our main result is the following:

Theorem 1. *1) There is the identity:*

$$Q_{ex}(de) = \int_{u=-\infty}^{+\infty} |u| \mathbb{P}^{\partial}_{0,u}\big(e^0 \in de\big) du. \tag{4.2}$$

2) The measure Q_{ex} is invariant by time-reversal (at the lifetime): Namely, the measure of Y under Q_{ex} is the same as that of $\widehat{Y}$ under Q_{ex}, where $\widehat{Y}$ is defined by

$$\widehat{Y}_s = Y_{\zeta - s} \text{ for } 0 \le s \le \zeta.$$

Let us adopt the notation $\widehat{Q}_{ex}$ for the law of $\widehat{Y}$ under Q_{ex}. The second part of the theorem can be written $\widehat{Q}_{ex} = Q_{ex}$.

Let a Langevin process start from location 0 and have initial velocity distributed according to $|u|du$. Then the distribution of its velocity at the first instant when it returns to 0 is again $|u|du$.

This remarkable fact can be proved directly as follows. We use the formula found by McKean [10], which gives, under $\mathbb{P}_{0,u}$, the joint density of ζ and $W_{\zeta-}$, and which specifies the density of $W_{\zeta-}$. For $u > 0$ and $v \ge 0$, we have:

$$\mathbb{P}_{0,u}(\zeta \in ds, |W_{\zeta-}| \in dv) = \frac{3u}{\pi\sqrt{2}s^2} \exp\Big(- 2\frac{v^2 - uv + u^2}{s}\Big) \int_0^{\frac{4uv}{s}} \mathrm{e}^{-\frac{3\theta}{2}} \frac{d\theta}{\sqrt{\pi\theta}}, \tag{4.3}$$

and in particular:

$$\mathbb{P}_{0,u}(-W_{\zeta-} \in dv) = \frac{3}{2\pi} \frac{u^{\frac{1}{2}} v^{\frac{3}{2}}}{u^3 + v^3} dv. \tag{4.4}$$

In the calculation, we just need the second formula.

The velocity of the canonical process X at the first time it returns to zero is equal to $V_{\zeta-}$. Let v be any positive real number. We have:

$$
\begin{aligned}
Q_{ex}(V_{\zeta^-} \in dv) &= \int_{u=-\infty}^{+\infty} |u| \mathbb{P}^{\partial}_{0,u}(V_{\zeta^-} \in dv) du \\
&= \int_{u=-\infty}^{0} |u| \mathbb{P}_{0,u}(W_{\zeta^-} \in dv) du \\
&= \int_{u=0}^{\infty} |u| \mathbb{P}_{0,u}(-W_{\zeta^-} \in dv) du \\
&= vdv \int_{u=-\infty}^{+\infty} \frac{3}{2\pi} \frac{u^{\frac{3}{2}} v^{\frac{1}{2}}}{u^3+v^3} du \\
&= vdv.
\end{aligned}
$$

The integral gives one as it is the integral of the density of $-W_{\zeta^-}$ under $\mathbb{P}_{0,u}$, thanks to (4.4). The case v negative is similar and gives us $Q_{ex}(V_{\zeta^-} \in dv) = -vdv$, as claimed.

Proof of Theorem 1. 1) This proof is mainly a combination of the work of Pitman [11] translated to the Langevin process, and of known results on the Langevin process, results that we can find in [7].

We recall and adapt some of their notations.

In [7], we consider the Langevin process on positive times, and the last instant that the process crosses zero before a fixed time T is written τ_T^-. In [11], we write G_u for the last instant before u that the stationary process crosses zero. The variable G_u can take finite strictly negative values, while the variable τ_T^- cannot. If T is a positive time, then we can write $\tau_T^- = \mathbf{1}_{G_T \geq 0} G_T$.

In [11], the part (iv) of the Theorem (p 291), rewritten with our notations, states [2] :

$$
\mathbb{P}_\lambda(-\infty < G_0 < 0, e^{G_0} \in de) = Q_{ex}(de)\zeta(e) \tag{4.5}
$$

In [7], the Lemma 2.5, p 129, states an important and simple relation, that can be written

$$
\mathbb{P}^{\partial}_{0,v}\big((X_t, \dot{X}_t) \in dxdu\big)|v|dvdt = \mathbb{P}_{x,-u}(\zeta \in dt, -V_{\zeta^-} \in dv)dxdu, \tag{4.6}
$$

and that is a main tool used to prove the Theorem 2.6. The points 1) and 4) of this Theorem state:

$$
\mathbb{P}^+_{x,u}\big\{(\tau_T^-, W_{\tau_T^-}) \in dsdv\big\}/dsdv = |v| p_s(x,u,0,v) \mathbb{P}^+_{0,v}\{\zeta > T-s\}, \tag{4.7}
$$

$$
\mathbb{E}^+_{x,u}\big[F(\tau_T^-, e_Z^{\tau_T^-}) | (\tau_T^-, W_{\tau_T^-}) = (s,v)\big] = \mathbb{E}^+_{0,v}[F(s, e_Z^0) | \zeta > T-s], \tag{4.8}
$$

where F is any suitable functional, and e_Z^t denotes the excursion of the two-dimensional process started at a time t such that $Y_t = 0$.

[2] Actually, the article of Pitman states $\mathbb{P}_\lambda(-\infty < G_u < u, e^{G_u} \in de) = Q_{ex}(de)\zeta(e)$ for any $u \in \mathbb{R}$.

Let us now begin. From (4.5), it is sufficient to prove the following:

$$\mathbb{P}_\lambda(-\infty < G_0 < 0, e^{G_0} \in de) = \zeta(e) \int_{u=-\infty}^{+\infty} |u| \mathbb{P}^\partial_{0,u}(X \in de) du. \tag{4.9}$$

We start from:

$$\begin{aligned}
\mathbb{P}_\lambda(-\infty < G_0 < 0, e^{G_0} \in de) &= \lim_{T\to\infty} \mathbb{P}_\lambda(-T < G_0 < 0, e^{G_0} \in de), \\
&= \lim_{T\to\infty} \mathbb{P}_\lambda(0 < G_T < T, e^{G_T} \in de), \\
&= \lim_{T\to\infty} \int dxdu \mathbb{P}_{x,u}(0 < G_T < T, e^{G_T} \in de),
\end{aligned}$$

where the second line follows from monotone convergence. Hence we have:

$$\mathbb{P}_\lambda(-\infty < G_0 < 0, e^{G_0} \in de) = \lim_{T\to\infty} \int dxdu \mathbb{P}_{x,u}(0 < \tau_T^- < T, e^{\tau_T^-} \in de).$$

Let us write the term in the limit.

$$\begin{aligned}
&\int dxdu \;\; \mathbb{P}_{x,u}(0 < \tau_T^- < T, e^{\tau_T^-} \in de) \\
= &\int dxdu \int \mathbb{P}_{x,u}\big((\tau_T^-, W_{\tau_T^-}) \in dsdv\big) \quad \mathbb{P}_{x,u}\big(e^{\tau_T^-} \in de | (\tau_T^-, W_{\tau_T^-}) = (s,v)\big), \\
= &\int dxdu \int \mathbb{P}^+_{x,u}\big((\tau_T^-, W_{\tau_T^-}) \in dsdv\big) \quad \mathbb{P}^+_{x,u}\big(e^{\tau_T^-} \in de | (\tau_T^-, W_{\tau_T^-}) = (s,v)\big), \\
= &\int dxdu \int dsdv |v| p_s(x,u,0,v) \mathbb{P}^+_{0,v}\{\zeta > T-s\} \mathbb{P}^+_{0,v}(e^0 \in de | \zeta > T-s),
\end{aligned}$$

where the integrals cover $(x,u) \in \mathbb{R}^2$, $(s,v) \in [0,T] \times \mathbb{R}$. In the last line we used (4.7), and (4.8) with the simple function $F\big(s,(Y,W)\big) = \mathbf{1}_{Y \in de}$.

By Fubini, the last expression is also equal to

$$\begin{aligned}
&\int dv|v| \int ds \Big(\int dxdu \; p_s(x,u,0,v) \Big) \mathbb{P}^+_{0,v}(e^0 \in de, \zeta > T-s). \\
= &\int dv|v| \int_0^T ds \mathbb{P}^\partial_{0,v}(X \in de, s > T - \zeta(e)) \\
= &\int dv|v| \mathbb{P}^\partial_{0,v}(X \in de)(\zeta(e) \wedge T),
\end{aligned}$$

where we get the second line because

$$\int dxdu \; p_s(x,u,0,v) = \int dxdu \; p_s(0,-v;x,-u) = 1.$$

Now, letting T go to ∞ gives us (4.9) and completes our proof.

2) We use the definition of Q_{ex} by the equation (4.1). The time-translation and time-reversal invariance of $\mathbb{E}_\lambda$ gives us the time-reversal invariance of Q_{ex}. □

We point out that the measure Q_{ex} has a remarkably simple potential, given by:

$$\int_{\mathbb{R}_+} Q_{ex}\big((X_t,\dot{X}_t)\in\bullet\big)dt=\lambda(\bullet). \tag{4.10}$$

Proof. This is a consequence of (4.2) and (4.6), that gives:

$$\begin{aligned}\int_{\mathbb{R}_+} Q_{ex}\big((X_t,\dot{X}_t)\in dxdu\big)dt &= \int_{\mathbb{R}_+} dt\int_{-\infty}^{+\infty}|v|dv\mathbb{P}^{\partial}_{0,v}\big((X_t,\dot{X}_t)\in dxdu\big)\\ &= dxdu\int_{\mathbb{R}_+}\int_{-\infty}^{+\infty}\mathbb{P}_{x,-u}(\zeta\in dt,-V_{\zeta^-}\in dv)\\ &= dxdu.\end{aligned}$$

□

Eventually, let us notice that we get a scaling property for the stationary excursion measure, which is a simple consequence from (2.6) and (4.2):

$$Q_{ex}\Big(F\big((X_t)_{t\geq 0}\big)\Big)=k^{-2}Q_{ex}\Big(F\big((k^{-3}X_{k^2t})_{t\geq 0}\big)\Big), \tag{4.11}$$

where F is any nonnegative measurable functional.

4.2 Conditioning and h-transform

In the preceding section we defined the stationary excursion measure, we described it with an easy formula and we proved its invariance by time-reversal. This is a global result for this measure. Now we would like to provide a more specific description according to the starting and ending velocities of the excursions. That is, we would like to define and understand the excursion measure conditioned to start with a velocity u and end with a velocity $-v$, that would be a probability measure written $Q_{u;v}$.

Let us first notice that the measure $Q_{ex}(V_0\in du,-V_{\zeta^-}\in dv)$ has a density with respect to the Lebesgue measure on $\mathbb{R}^2$, that we know explicitly from (4.4):

$$\begin{aligned}Q_{ex}(V_0\in du,-V_{\zeta^-}\in dv) &= |u|\mathbb{P}^{\partial}_{0,u}(-V_{\zeta^-}\in dv)du\\ &= \frac{3}{2\pi}\frac{|u|^{\frac{3}{2}}|v|^{\frac{3}{2}}}{|u|^3+|v|^3}\mathbf{1}_{uv>0}dudv\\ &= \varphi(u,v)dudv,\end{aligned}$$

where we have written $\varphi(u,v)=\dfrac{3}{2\pi}\dfrac{|u|^{\frac{3}{2}}|v|^{\frac{3}{2}}}{|u|^3+|v|^3}$. The support of this measure is $\{(u,v)\in\mathbb{R}^2,uv\geq 0\}$.

Definition 3. *We write $(Q_{u;v})_{uv>0}$ for a version of the conditional law of Q_{ex} given the initial speed is u and the final speed $-v$. That is, for $f : \mathbb{R}^2 \to \mathbb{R}$ and $\phi : \mathcal{E} \to \mathbb{R}$ nonnegative measurable functionals, we have:*

$$Q_{ex}\big(f(V_0, -V_{\zeta^-})\phi\big) = \int Q_{u;v}(\phi) f(u,v)\varphi(u,v)dudv. \tag{4.12}$$

It is clear that $Q_{-u;-v}$ is the image of $Q_{u;v}$ by the symmetry $X \to -X$, for almost all (u,v), so that in the following we will only be interested in $Q_{u;v}$ for $u > 0$, $v > 0$.

From the time-reversal invariance of the stationary excursion measure, i.e $\widehat{Q}_{ex} = Q_{ex}$, we deduce immediately the following time-reversal property of the conditioned measures:

$$\widehat{Q}_{u;v} = Q_{v;u} \qquad \text{for a. a. } (u,v) \in (\mathbb{R}_+)^2. \tag{4.13}$$

Recall from the formula (4.2) that $|u|\mathbb{P}^\partial_{0,u}$ is a version of the conditional law of Q_{ex} given the initial speed u. It follows that we have the following formula:

$$\mathbb{P}^\partial_{0,u} = |u|^{-1} \int Q_{u;v}\varphi(u,v)dv \quad \text{ for almost all u¿0} \tag{4.14}$$

The measure $|u|^{-1}\varphi(u,v)dv$ is the law of $-V_{\zeta^-}$ under $\mathbb{P}^\partial_{0,u}$. Hence $Q_{u;v}$ is a version of the conditional law of $\mathbb{P}^\partial_{0,u}$ given $-V_{\zeta^-} = -v$. Before going on, we need precise informations on the variable $-V_{\zeta^-}$ and its law, under different initial conditions. The results we need are gathered in the following lemma. We take the notations $\mathbb{R}^*_+$ for $\mathbb{R}_+\backslash\{0\}$, and D for the domain $\big((\mathbb{R}^*_+) \times \mathbb{R}\big) \bigcup \big(\{0\} \times (\mathbb{R}^*_+)\big)$.

Lemma 3. • *For any (x,u) in D, the density of the law of the variable $-V_{\zeta^-}$ under $\mathbb{P}^\partial_{x,u}$ with respect to the Lebesgue measure on $(0,\infty)$ exists and is written $h_v(x,u)$ for $v > 0$. We have:*

$$h_v(x,u) \quad = \quad v\Big[\Phi_0(x,u;-v) - \frac{3}{2\pi}\int_0^\infty \frac{\mu^{\frac{3}{2}}}{\mu^3+1}\Phi_0(x,u;\mu v)d\mu\Big], \tag{4.15}$$

where $\Phi_0(x,u;v) := \Phi(x,u;0,v)$ and

$$\Phi(x,u;y,v) := \int_0^\infty p_t(x,u;y,v)dt.$$

For $x = 0$, this formula can be simplified as:

$$h_v(0,u) \quad = \quad \frac{3}{2\pi}\frac{u^{\frac{1}{2}}v^{\frac{3}{2}}}{u^3+v^3}. \tag{4.16}$$

• *The function $(v,x,u) \to h_v(x,u)$ is continuous on $E := \mathbb{R}^*_+ \times D$. The function Φ_0 is continuous and differentiable on $D \times \mathbb{R}$. Moreover, we have the following equivalence for v in the neighborhood of zero:*

$$h_v(x,u) \sim \overline{h_0}(x,u)v^{\frac{3}{2}}, \tag{4.17}$$

where $\overline{h_0}(x,u)$ *is given by*

$$\overline{h_0}(x,u) = \frac{3}{\pi}\int \alpha^{-\frac{1}{2}}\frac{\partial \Phi_0}{\partial v}(x,u;\alpha)d\alpha.$$

For $x = 0$*, this formula can be simplified as*

$$\overline{h_0}(0,u) = \frac{3u^{\frac{1}{2}}}{2\pi}.$$

This is a technical lemma, with a long proof that we report in the Appendix.

The idea is now, thanks to this lemma, to prove that the law $\mathbb{P}^{\partial}_{0,u}$ conditioned on the event $-V_{\zeta^-} \in [v, v+\eta]$, has a limit when η goes to zero. This limit is necessarily $Q_{u;v}$ a. s. Hence we get an expression for $Q_{u;v}$, that will happen to be a bi-continuous version.

Let us fix $u, v, t > 0$, and let ϕ_t be an $\mathcal{F}_t$-measurable nonnegative functional. We have:

$$\begin{aligned}\lim_{\eta\to 0}\mathbb{E}^{\partial}_{0,u}\big(\phi_t \mathbf{1}_{\zeta>t} | -V_{\zeta^-} \in [v, v+\eta]\big) &= \lim_{\eta\to 0}\frac{\mathbb{E}^{\partial}_{0,u}\big(\phi_t \mathbf{1}_{\zeta>t, -V_{\zeta^-}\in[v,v+\eta]}\big)}{\mathbb{P}^{\partial}_{0,u}(-V_{\zeta^-} \in [v,v+\eta])} \\ &= \mathbb{E}^{\partial}_{0,u}\left(\phi_t \mathbf{1}_{\zeta>t}\lim_{\eta\to 0}\frac{\mathbb{P}^{\partial}_{X_t,V_t}(-V_{\zeta^-}\in[v,v+\eta])}{\mathbb{P}^{\partial}_{0,u}(-V_{\zeta^-}\in[v,v+\eta])}\right).\end{aligned}$$

The limit exists and is equal to the quotient of $h_v(X_t, V_t)$ by $h_v(0, u)$. Hence, we get:

$$Q_{u;v}(\phi_t \mathbf{1}_{\zeta>t}) = \mathbb{E}^{\partial}_{0,u}\left(\phi_t \mathbf{1}_{\zeta>t}\frac{h_v(X_t,V_t)}{h_v(0,u)}\right). \tag{4.18}$$

for any $t > 0$, any $\mathcal{F}_t$-measurable functional ϕ_t.

From the continuity of h we deduce that $Q_{u;v}$ is jointly continuous in u, v, $(u, v) \in (\mathbb{R}^*_+)^2$. Furthermore, thanks to (4.17), when v goes to zero, the quotient goes to $\frac{\overline{h_0}(X_t,V_t)}{\overline{h_0}(0,u)}$. We deduce that the measures $Q_{u;v}$ have a weak limit when v goes to zero, that we write $Q_{u;0}$. We have

$$Q_{u;0}(\phi_t \mathbf{1}_{\zeta>t}) = \mathbb{E}^{\partial}_{0,u}\left(\phi_t \mathbf{1}_{\zeta>t}\frac{\overline{h_0}(X_t,V_t)}{\overline{h_0}(0,u)}\right). \tag{4.19}$$

This shows that these measures $Q_{u;v}$ make appear h-transforms of the usual probability transitions of the Langevin process $\mathbb{E}_{0,u}$. The h-transforms are common when dealing with conditioned Markov process, see for example [1], and in particular the chapters *4.7.* and *6.4.* for the connexion with time-reversal.

Informally, in the case of two processes in duality, changing the initial condition for one process corresponds to changing the probability transitions of the second process into an h-transform of these probability transitions. The h-transform means the measure "conditioned" with using a certain harmonic function h, that we can write explicitly.

We finish this section with giving the scaling property of the measures $Q_{u;v}$, that follows for example from (2.6) and (4.12):

Proposition 1. *For any $u > 0$, $v \geq 0$, we have:*

$$Q_{u;v}\Big(F\big((X_t)_{t\geq 0}\big)\Big) = Q_{ku;kv}\Big(F\big((k^{-3}X_{k^2t})_{t\geq 0}\big)\Big), \tag{4.20}$$

where F is any nonnegative measurable functional.

5 Reflected Kolmogorov process

The question of the existence of the Langevin process reflected at a completely inelastic boundary was raised by B.Maury in 2004 in [9]. J.Bertoin answered to this question: In [2] he defines the reflected Langevin process, proves its existence and the uniqueness in law, and gets some other results. We also mention another paper [3] that studies the problem of the reflected Langevin process from the point of view of the stochastic differential equations.

We recall the definition of this reflected process:

Definition 4. *We say that (X, V) is a Kolmogorov process reflected at a completely inelastic boundary (or just reflected Kolmogorov process) if it is a càdlàg strong Markov process with values in $\mathbb{R}_+ \times \mathbb{R}$ which starts from $(0, 0)$, such that:*

$$dX_t = V_t dt\ ,\ \int_0^\infty \mathbf{1}_{\{X_t=0\}}dt = 0 \textit{ and } \big(X_t = 0\ \Rightarrow\ V_t = 0\big) a.s.,$$

and which "evolves as a Kolmogorov process when $X > 0$", in the following sense:

> For every stopping time S in the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of X, conditionally on $X_S = x > 0$ and $V_S = v$, the process $(X_{S+t})_{t\geq 0}$ stopped at 0 is independent of $\mathcal{F}_S$, and has the distribution of a Langevin process started with velocity v from the location x and stopped at 0.

We say that X is a Langevin process reflected at a completely inelastic boundary (or just reflected Langevin process) if $(X, \dot{X})$ is a reflected Kolmogorov process.

In the following we choose the vocabulary and the notations of the one-dimensional process, that is the Langevin process, to state our results.

In his paper Bertoin gives an explicit construction of a reflected Langevin process, combining Skorokod's reflection and time-substitution. He also proves the uniqueness of the law of a reflected Langevin process, so that we will speak about *the* reflected Langevin process. In the rest of the paper, we will concentrate our attention on what is one of the first steps in the study of this process, that is to say its Itô excursion measure. We recall that it is unique up to a multiplicative constant.

The set of vertical excursions $\mathcal{E}$, endowed with the supremum norm of the process and its derivative, is a metric space. In the following, we write $F : \mathcal{E} \to \mathbb{R}$ for a general continuous bounded functional which is identically 0 on some neighborhood of the path $e \equiv 0$.

We are ready to state a first formula, given[3] by Bertoin [2]:

Proposition 2. *The following limit*

$$\mathbf{n}_0\big(F(e)\big) := \lim_{x\to 0+} x^{-\frac{1}{6}}\mathbb{E}^{\partial}_{x,0}\big(F(e)\big),$$

exists and defines uniquely a measure on $\mathcal{E}$ with $\mathbf{n}_0(0) = 0$. The measure $\mathbf{n}_0$ is an Itô excursion measure of the reflected Langevin process.

This is to say, we get an expression for the Itô excursion measure of the reflected Langevin process as a limit of known measures.

This result resembles the classical approximation of the Itô measure of the absolute value of the Brownian motion by $x^{-1}\mathbb{P}^{\partial}_x$, where $\mathbb{P}^{\partial}_x$ is the law of the Brownian motion starting from x and killed when hitting 0.

As a consequence of this expression, we can give the scaling property of this measure, also mentioned in [2], Proposition 2:

Corollary 2. *We have:*

$$\mathbf{n}_0\Big(F\big((X_t)_{t\geq 0}\big)\Big) = k^{\frac{1}{2}}\mathbf{n}_0\Big(F\big((k^{-3}X_{k^2t})_{t\geq 0}\big)\Big),$$

for any nonnegative measurable functional F.

Proof. Let F be a general continuous bounded functional which is identically 0 on some neighborhood of the path $e \equiv 0$. The proposition gives us:

$$\begin{aligned}
\mathbf{n}_0\big(F((X_t)_{t\geq 0})\big) &= \lim_{x\to 0+} x^{-\frac{1}{6}}\mathbb{E}^{\partial}_{x,0}\big(F((X_t)_{t\geq 0})\big) \\
&= k^{\frac{1}{2}}\lim_{x\to 0+}(k^3x)^{-\frac{1}{6}}\mathbb{E}^{\partial}_{k^3x,0}\big(F((k^{-3}X_{k^2t})_{t\geq 0})\big) \qquad \text{by (2.6)} \\
&= k^{\frac{1}{2}}\mathbf{n}_0\big(F((k^{-3}X_{k^2t})_{t\geq 0})\big).
\end{aligned}$$

The result follows. □

We give here two new expressions of the Itô excursion measure of the reflected process. The first one is similar to the one above, expressed as a limit. But it is a limit of laws of the process starting with a zero position and a small speed, instead of a zero speed and a small position.

Theorem 2. *The following limit*

$$\mathbf{n}_1\big(F(e)\big) = \lim_{u\to 0+} u^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,u}\big(F(e)\big),$$

exists and defines uniquely a measure on $\mathcal{E}$ with $\mathbf{n}_1(0) = 0$. We have:

$$\mathbf{n}_1 = \left(\frac{3}{2}\right)^{\frac{1}{6}}\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{3}\right)\mathbf{n}_0.$$

[3] Actually Bertoin states this result in a slightly different form, as the set of excursions he considers is not the exactly same as the one we consider here. Nevertheless, his argument still works in our settings.

This formula is useful because we have more explicit densities for the law $\mathbb{P}_{0,u}$ than for the law $\mathbb{P}_{x,0}$ (cf (4.3) and (4.4)). For example, we can easily infer the following corollaries:

Corollary 3. *The joint density of ζ and $V_{\zeta-}$ under $\mathbf{n}_1$ is given by:*

$$\mathbf{n}_1(\zeta \in ds, |V_{\zeta-}| \in dv) = 6\sqrt{\frac{2v^3}{\pi^3 s^5}} \exp\Big(- 2\frac{v^2}{s}\Big) ds dv.$$

Remark. Taking the second marginal of this density, this gives the $\mathbf{n}_1$-density of $-V_{\zeta-}$,

$$\mathbf{n}_1(|V_{\zeta-}| \in dv) = \frac{45}{8\pi} v^{-\frac{3}{2}} dv.$$

This improves Corollary 2 (ii) in [2].

Proof. It is easy to check, for example from the corresponding property for the free Langevin process, that $|V_{\zeta-}| \neq 0$ $\mathbf{n}_1$-almost surely. But $e \to (\zeta(e), |V_{\zeta-}|(e))$ is continuous on $|V_{\zeta-}| \neq 0$ thus we can use the limit formula to get the density:

$$\mathbf{n}_1(\zeta \in ds, |V_{\zeta-}| \in dv) = \lim_{u \to 0} u^{-\frac{1}{2}} \mathbb{P}^{\partial}_{0,u}(\zeta \in ds, |V_{\zeta-}| \in dv).$$

Now, using (4.3), we can calculate:

$$\begin{aligned}
& \frac{u^{-\frac{1}{2}}}{ds dv} \mathbb{P}^{\partial}_{0,u}(\zeta \in ds, |V_{\zeta-}| \in dv) \\
= \;& u^{-\frac{1}{2}} \frac{3v}{\pi\sqrt{2}s^2} \exp\Big(- 2\frac{u^2 - vu + v^2}{s}\Big) \int_0^{\frac{4uv}{s}} \mathrm{e}^{-\frac{3\theta}{2}} \frac{d\theta}{\sqrt{\pi\theta}} \\
\sim \;& \frac{3vu^{-\frac{1}{2}}}{\pi\sqrt{2}s^2} \exp\Big(- 2\frac{v^2}{s}\Big) \int_0^{\frac{4uv}{s}} \frac{d\theta}{\sqrt{\pi\theta}} \\
\sim \;& \frac{6\sqrt{2}}{\pi^{\frac{3}{2}}} \sqrt{\frac{v^3}{s^5}} \exp\Big(- 2\frac{v^2}{s}\Big),
\end{aligned}$$

so that we have, as stated:

$$\mathbf{n}_1(\zeta \in ds, |V_{\zeta-}| \in dv) = c\sqrt{\frac{v^3}{s^5}} \exp\Big(- 2\frac{v^2}{s}\Big) ds dv.$$

□

Corollary 4. *The measure $\overline{h}_0(x, u) dx du$, $x \geq 0, u \in \mathbb{R}$, is invariant for the reflected Kolmogorov process.*

Proof. It is well-known that the occupation measure under the Itô's excursion measure

$$\mu(dx, du) = \mathbf{n}_1\left(\int_{[0,\zeta]} \mathbf{1}_{Z_t \in (dx,du)} dt \right),$$

is an invariant measure for the underlying Markov process (cf Theorem 8.1 in [4])

This enables us to calculate:

$$\begin{aligned}
\mu(dx,du) &= \lim_{v\to 0} v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}\left(\int_{[0,\zeta]} \mathbf{1}_{Z_t\in(dx,du)}dt\right). \\
&= \lim_{v\to 0} v^{-\frac{1}{2}}\int_{\mathbb{R}_+} \mathbb{P}^{\partial}_{0,v}\big(Z_t\in(dx,du)\big)dt \\
&= dxdu\lim_{v\to 0} v^{-\frac{3}{2}}\frac{\mathbb{P}_{x,-u}(-V_{\zeta^-}\in dv)}{dv} && \text{by (4.6)} \\
&= \overline{h}_0(x,u)dxdu && \text{by Lemma 3.}
\end{aligned}$$

□

Proof of Theorem 2. In order to prove $\mathbf{n}_1 = c_1\mathbf{n}_0$, it is enough to prove that $\mathbf{n}_1(F(e)) = c_1\mathbf{n}_0(F(e))$, for F a Lipschitz bounded functional. The idea of this proof will be to compare the quantities

$$v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}\big(F(e)\big) \quad \text{and} \quad v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}\big(F\circ\Theta_{\tau_0}(e)\big),$$

where Θ is the usual translation operator, defined by

$$\Theta_t((X_s)_{s\geq 0}) := (X_{t+s})_{s\geq 0},$$

and τ_x is the hitting time of x for the velocity process.

First we will control the difference, cutting the space on two events, the event that τ_0 is "small", on which we will use that F is Lipschitz, and the event that τ_0 is "big", that has a small probability. Next we will use a Markov property to see that the quantity $v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}\big(F\circ\Theta_{\tau_0}(e)\big)$ can be compared to $\mathbf{n}_0(F)$.

As a preliminary we prove some estimates:

• We write $\mathbf{P}_u$ for the law of the Brownian motion started from u. We write τ_x for both the hitting time of x for the velocity process under $\mathbb{P}^{\partial}_{0,u}$, and the hitting time of x for the Brownian motion under $\mathbf{P}_u$. Let a be a constant. A simple calculation based on the scaling property of the Brownian motion and on the reflection principle gives:

$$\begin{aligned}
u^{-\frac{1}{2}}\mathbb{P}^{\partial}_{0,u}(\tau_0 > au) &= u^{-\frac{1}{2}}\mathbf{P}_u(\tau_0 > au) \\
&= u^{-\frac{1}{2}}\mathbf{P}_0(\tau_{a^{-\frac{1}{2}}u^{\frac{1}{2}}} > 1) \\
&= u^{-\frac{1}{2}}\mathbf{P}\big(\mathcal{N}(0,1)\in[-a^{-\frac{1}{2}}u^{\frac{1}{2}}, a^{-\frac{1}{2}}u^{\frac{1}{2}}]\big) \\
&\leq a^{-\frac{1}{2}}\sqrt{\frac{2}{\pi}},
\end{aligned}$$

where $\mathcal{N}(0,1)$ is a Gaussian variable with mean zero and variance 1.

• Let us write h for the supremum of the absolute value of the velocity process of an excursion. Let b be a constant. We have:

$$\begin{aligned} u^{-\frac{1}{2}}\mathbb{P}^{\partial}_{0,u}(h \geq b) &\leq u^{-\frac{1}{2}}\mathbb{P}^{\partial}_{0,u}(\tau_b < \tau_0) + u^{-\frac{1}{2}}\mathbb{P}^{\partial}_{0,u}(h \circ \Theta_{\tau_0} \geq b) \\ &\leq u^{-\frac{1}{2}}\mathbb{P}_{0,u}(\tau_b < \tau_0) + \int_{\mathbb{R}_+} \mathbb{P}_{0,u}(Y_{\tau_0} \in dx)\mathbb{P}^{\partial}_{x,0}(h \geq b) \\ &\leq \frac{u^{\frac{1}{2}}}{b} + u^{-\frac{1}{2}} \int \mathbb{P}_{0,u}(Y_{\tau_0} \in dx) x^{\frac{1}{6}} f(x), \end{aligned}$$

where the function $f : x \to x^{-\frac{1}{6}}\mathbb{P}^{\partial}_{x,0}(h \geq b)$ is bounded and has limit $f(0) = \mathbf{n}_0(h \geq b)$ at zero, thanks to Proposition 2. In the sum, the second term is thus equal to:

$$\begin{aligned} u^{-\frac{1}{2}}\mathbb{E}_{0,u}\big(Y_{\tau_0}^{\frac{1}{6}} f(Y_{\tau_0})\big) &= u^{-\frac{1}{2}}\mathbb{E}_{0,1}\big((u^3 Y_{\tau_0})^{\frac{1}{6}} f(u^3 Y_{\tau_0})\big) \\ &= \mathbb{E}_{0,1}\big(Y_{\tau_0}^{\frac{1}{6}} f(u^3 Y_{\tau_0})\big) \\ &\to_{u\to 0} \mathbb{E}_{0,1}\big(Y_{\tau_0}^{\frac{1}{6}}\big) f(0), \end{aligned}$$

where in the second line we used the usual scaling property for the Langevin process.

We write $c_1 = \mathbb{E}_{0,1}\big(Y_{\tau_0}^{\frac{1}{6}}\big)$, so that we have the bound:

$$u^{-\frac{1}{2}}\mathbb{P}^{\partial}_{0,u}(h > b) \leq \frac{u^{\frac{1}{2}}}{b} + c_1 \mathbf{n}_0(h > b).$$

We would like to prove that c_1 is finite. We can actually calculate it explicitly. Indeed, thanks to Lefebvre [8] we know that the density of the variable Y_{τ_0} under $\mathbb{P}_{0,1}$ is given by:

$$\mathbb{P}_{0,1}(Y_{\tau_0} \in d\xi) = \frac{\Gamma(\frac{2}{3})}{3^{\frac{1}{6}} 2^{\frac{2}{3}} \pi} \xi^{-\frac{4}{3}} \mathrm{e}^{-\frac{2}{9\xi}} d\xi,$$

so that we can calculate:

$$\begin{aligned} c_1 &= \int_{\mathbb{R}_+} \xi^{\frac{1}{6}} \mathbb{P}_{0,1}(Y_{\tau_0} \in d\xi) \\ &= \frac{\Gamma(\frac{2}{3})}{2^{\frac{2}{3}} 3^{\frac{1}{6}} \pi} \int_{\mathbb{R}_+} \xi^{-\frac{7}{6}} \mathrm{e}^{-\frac{2}{9\xi}} d\xi \\ &= \frac{\Gamma(\frac{2}{3})}{2\pi 3^{\frac{1}{6}}} \int_{\mathbb{R}_+} \left(\frac{9}{2}\right)^{\frac{1}{6}} x^{-\frac{5}{6}} \mathrm{e}^{-x} dx \\ &= \frac{3^{\frac{1}{6}}}{2^{\frac{5}{6}} \pi} \Gamma\Big(\frac{2}{3}\Big)\Gamma\Big(\frac{1}{6}\Big) \\ &= \left(\frac{3}{2}\right)^{\frac{1}{6}} \frac{1}{\sqrt{\pi}} \Gamma\Big(\frac{1}{3}\Big), \end{aligned}$$

Let us notice that this is the constant that appears in the theorem.

• We are ready to tackle the proof of this theorem. We write l for the Lipschitz constant of F. We have:

$$\begin{aligned} v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}\big(|F(e)-F\circ\Theta_{\tau_0}(e)|\mathbf{1}_{\tau_0\leq av,h\leq b}\big) &\leq v^{-\frac{1}{2}}l.(av).b \\ &\leq abv^{\frac{1}{2}}l, \end{aligned}$$

and

$$\begin{aligned} & v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}\big(|F(e)-F\circ\Theta_{\tau_0}(e)|\mathbf{1}_{\tau_0>av \text{ or } h>b}\big) \\ \leq\ & \big(2\sup(F)\big)\left(\sqrt{\frac{2}{\pi}}a^{-\frac{1}{2}}+\frac{v^{\frac{1}{2}}}{b}+c_1\mathbf{n}_0(h>b)\right), \end{aligned}$$

thus we deduce

$$\limsup_{v\to 0} v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}\big(|F(e)-F\circ\Theta_{\tau_0}(e)|\big) \leq \big(2\sup(F)\big)\left(\sqrt{\frac{2}{\pi}}a^{-\frac{1}{2}}+c_1\mathbf{n}_0(h>b)\right).$$

The $\limsup$ is bounded by this expression, a and b being any positive constant. Letting a and b go to infinity shows that

$$\lim_{v\to 0} v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}\big(|F(e)-F\circ\Theta_{\tau_0}(e)|\big)=0.$$

• Next, we just need to prove that $v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}(F\circ\Theta_{\tau_0}(e))$ has a limit when v goes to zero, and that this limit is $c_1\mathbf{n}_0(F(e))$, in order to get that $\mathbf{n}_1$ is well-defined and equal to $c_1\mathbf{n}_0$.

The calculation is similar to the one above, that we did with $\mathbf{1}_{h>b}$ instead of F. Here again, the Markov property gives us:

$$v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}(F\circ\Theta_{\tau}(e)) = v^{-\frac{1}{2}}\int \mathbb{P}_{0,v}(Y_\tau\in dx)x^{\frac{1}{6}}f_F(x),$$

where the function $f_F : x\to x^{-\frac{1}{6}}\mathbb{E}^{\partial}_{x,0}(F(e))$ is bounded and has limit $f_F(0)=\mathbf{n}_0(F(e))$ at zero. We thus have:

$$v^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,v}(F\circ\Theta_{\tau}(e)) \to_{v\to 0} \mathbb{E}_{0,1}\big(Y_\tau^{\frac{1}{6}}\big)f_F(0)=c_1\mathbf{n}_0(F(e)),$$

and the theorem is proved. □

The second new expression we get is different, this time the measure is given directly and not as a limit.

Proposition 3. *The measure $\mathbf{n}_1$ is also given by the expression:*

$$\mathbf{n}_1\big(F(e)\big)=\frac{3}{2\pi}\int_{\mathbb{R}_+}u^{-\frac{3}{2}}Q_{u;0}\big(F(\hat{e})\big)du, \tag{5.1}$$

where $(\hat{e}_t)_{0\leq t\leq\zeta}$ is the time-reversed excursion, defined by $\hat{e}_t=e_{\zeta-t}$.

The price to pay is that we need to consider the time-reversed excursions and to use the laws $Q_{u;0}$ instead of $\mathbb{P}^{\partial}_{0,u}$. That is, the probability transitions of the excursions are no more the ones of the Langevin process, killed at zero, they become the $\overline{h}_0$-transforms of these, as written in (4.19).

Proof. This proposition is a consequence of the material developed in Section 4.2. Indeed, we have:

$$\begin{aligned}
\mathbf{n}_1(F(e)) &= \lim_{u\to 0} u^{-\frac{1}{2}}\mathbb{E}^{\partial}_{0,u}(F(e)) \\
&= \lim_{u\to 0} u^{-\frac{3}{2}}\int_{\mathbb{R}_+} Q_{u;v}(F(e))\varphi(u,v)dv \qquad \text{from (4.14)} \\
&= \lim_{u\to 0}\int_{\mathbb{R}_+} \frac{3}{2\pi}\frac{v^{\frac{3}{2}}}{u^3+v^3}Q_{v;u}(F(\hat{e}))dv \\
&= \int_{\mathbb{R}_+} \frac{3}{2\pi}v^{-\frac{3}{2}}Q_{v;0}(F(\hat{e}))dv,
\end{aligned}$$

where in the third line, we wrote the expression of φ and used (4.13). □

6 Appendix

Proof of Lemma 3. The law of $-V_{\zeta^-}$ under $\mathbb{P}^{\partial}_{x,u}$ is equal to the law of $-W_{\zeta^-}$ under $\mathbb{P}_{x,u}$, and the density of that variable is known. The first part of the lemma is thus just a summary of known results, the case $x=0$ is nothing else that the equation (4.4) we mentioned before, and the general case given by Gor'kov in [5] and Lachal in [6]. In this article Lachal also underlines that taking $x=0$ in (4.15) does yield (4.16).

For the second part we first prove that Φ_0 and h are well-defined and continuous [4]. For this we just give rough bounds and use the theorem of dominated convergence and the theorem of derivation under the integral. The main technical difficulty stems from the number of variables.

We have

$$p_t(x,u;0,v) = \frac{\sqrt{3}}{\pi t^2}\exp\big(-R(x,u,v,t)\big),$$

where $R(x,u,v,t)$ is the quotient:

$$\begin{aligned}
R(x,u,v,t) &= \frac{6}{t^3}(x+tu)^2 + \frac{6}{t^2}(x+tu)(v-u) + \frac{2}{t}(v-u)^2 \\
&= \frac{1}{t^3}\Big[\frac{1}{2}(3x+tu+2tv)^2 + \frac{3}{2}(x+tu)^2\Big].
\end{aligned}$$

The quotient R is nonnegative.

[4] Note that $\Phi(x,u;y,v) = \Phi_0(x-y,u;v)$.

Let (x_0, u_0, v_0) be in $D \times \mathbb{R}$. We search for a neighborhood of (x_0, u_0, v_0) (in $D \times \mathbb{R}$) on which the integrand is bounded by an integrable function (of t). This will prove that Φ_0 is well-defined on this neighborhood and continuous at (x_0, u_0, v_0). We distinguish two cases:

1) $x_0 \neq 0$: Then $R(x, u, v, t)$ is equivalent to $\frac{6x_0^2}{t^3}$ in the neighborhood of $(x_0, u_0, v_0, 0)$, thus it is bounded below by $\frac{5x_0^2}{t^3}$ on a $V \times]0, \varepsilon]$, where V is a neighborhood of (x_0, u_0, v_0) and ε a strictly positive number.

On V, $p_t(x, u; 0, v)$ is bounded above by the function

$$\mathbf{1}_{]0,\varepsilon]}(t) \frac{\sqrt{3}}{\pi t^2} \exp\Big(- \frac{5x^2}{t^3} \Big) + \mathbf{1}_{]\varepsilon,\infty[}(t) \frac{\sqrt{3}}{\pi t^2},$$

which is integrable.

2) $x_0 = 0$: Then $u_0 > 0$. On a neighborhood V of $(0, u_0, v_0)$ we have $u > \frac{2u_0}{3}$, thus we have

$$R(x, u, v, t) \geq \frac{3}{2t^3}(x + tu)^2 \geq \frac{u_0^2}{t},$$

and thus the function $p_t(x, u; 0, v)$ is bounded above by

$$\frac{\sqrt{3}}{\pi t^2} \exp\big(- \frac{u_0}{t} \big),$$

which is integrable.

We thus proved the continuity of Φ_0. A similar method proves that Φ_0 is infinitely differentiable. To get a continuity result on h, we will need some bounds for $\Phi_0(x, u, v)$, but only for $v > 0$.

For $v > 0$, we have $R(x, u, v, t) \geq \frac{3v^2}{2t}$, thus we have:

$$\begin{aligned} \Phi_0(x, u, v) &\leq \int_0^\infty \frac{\sqrt{3}}{\pi t^2} \exp\Big(- \frac{3v^2}{2t} \Big) dt \\ &\leq \frac{2\sqrt{3}}{3\pi} v^{-2}. \end{aligned}$$

If (x_0, u_0, v_0) is a given point in $E = \mathbb{R}_+^* \times D$, then in the neighborhood of this point we have $v > \frac{v_0}{2}$ and we deduce:

$$\frac{\mu^{\frac{3}{2}}}{\mu^3 + 1} \Phi_0(x, u, \mu v) \leq \frac{8\sqrt{3}}{3\pi} \frac{\mu^{-\frac{1}{2}} v^{-2}}{\mu^3 + 1},$$

which, considered as a function of μ, is integrable on $\mathbb{R}_+$.

The function h is thus well-defined and continuous.

We now study the behavior of h when v is small.

$$
\begin{aligned}
\frac{1}{v}h_v(x,u) &= \Phi_0(x,u,v) - \frac{3}{2\pi}\int_0^\infty \frac{\mu^{\frac{3}{2}}}{\mu^3+1}\Phi_0(x,u,\mu v)d\mu \\
&= \Big[\Phi_0(x,u,0) - \frac{3}{2\pi}\int_0^\infty \frac{\mu^{\frac{3}{2}}}{\mu^3+1}\Phi_0(x,u,\mu v)d\mu\Big] + O(v) \\
&= \mathbb{E}\big[\Phi_0(x,u,0) - \Phi_0(x,u,v\xi)\big] + O(v),
\end{aligned}
$$

where ξ is a random variable with law the probability measure $\dfrac{3}{2\pi}\dfrac{\mu^{\frac{3}{2}}}{\mu^3+1}d\mu$. We next observe that:

$$
\begin{aligned}
\mathbb{E}\big[\Phi_0(x,u,0) - \Phi_0(x,u,v\xi)\big] &= -\int_{\mathbb{R}_+} \mathbb{P}(v\xi \geq \mu)\frac{\partial \Phi_0}{\partial v}(x,u,\mu)d\mu \\
&= v^{\frac{1}{2}}\int_{\mathbb{R}_+} f_v(\mu)d\mu,
\end{aligned}
$$

where we have written $f_v(\mu) = -v^{-\frac{1}{2}}\mathbb{P}(\xi \geq \mu v^{-1})\frac{\partial \Phi_0}{\partial v}(x,u,\mu)$.

But the probability $\mathbb{P}(\xi \geq a)$ is equivalent to $\frac{3}{\pi}a^{-\frac{1}{2}}$ when a goes to infinity, and bounded by the same $\frac{3}{\pi}a^{-\frac{1}{2}}$ for any a. On the one hand we deduce that the continuous functions f_v converge weakly to the function $f_0 : \mu \to -\dfrac{3}{\pi}\mu^{-\frac{1}{2}}\dfrac{\partial \Phi_0}{\partial v}(x,u,\mu)$ when v goes to zero, on the other hand that $|f_v| \leq |f_0|$. We just need to prove that f_0 is integrable with respect to the Lebesgue measure. We have:

$$
\begin{aligned}
-\frac{\partial \Phi_0}{\partial v}(x,u,v) &= -\int_{\mathbb{R}_+} \frac{\partial p_t}{\partial v}(x,u;0,v)dt \\
&= \int_{\mathbb{R}_+} \Big(\frac{6x}{t^2} + \frac{2u}{t} + \frac{4v}{t}\Big)p_t(x,u;0,v)dt \\
&= \int_{\mathbb{R}_+} \frac{2\sqrt{3}}{\pi t^4}(3x + tu + 2tv)\exp\big(-R(x,u,v,t)\big)dt.
\end{aligned}
$$

On the one hand, we have :

$$
\begin{aligned}
&\Big|\frac{\partial \Phi_0}{\partial v}(x,u,v)\Big| \\
\leq\ & 3x\int_{\mathbb{R}_+} \frac{2\sqrt{3}}{\pi t^4}\exp\Big(-\frac{3}{2t^3}(x+tu)^2\Big)|u+2v|\int_{\mathbb{R}_+} \frac{2\sqrt{3}}{\pi t^3}\exp\Big(-\frac{3}{2t^3}(x+tu)^2\Big) \\
\leq\ & (A + Bv),
\end{aligned}
$$

where A and B depend only on x and u.

On the other hand, we have:

$$\begin{aligned}
&\Big|\frac{\partial \Phi_0}{\partial v}(x,u,v)\Big| \\
\leq\ & \frac{6\sqrt{3}x}{\pi}\int_{\mathbb{R}_+}\frac{1}{t^4}\exp\Big(-\frac{3v^2}{2t}\Big)+\frac{(2u+4v)\sqrt{3}}{\pi}\int_{\mathbb{R}_+}\frac{1}{t^3}\exp\Big(-\frac{3v^2}{2t}\Big) \\
\leq\ & Cv^{-7}+D(u+2v)v^{-5},
\end{aligned}$$

where C and D are constants.

Let us gather the results. The function $|f_0|$ is bounded by a $O(\mu^{-\frac{1}{2}})$ in the neighborhood of zero and bounded by a $O(\mu^{-3})$ in the neighborhood of infinity, thus it is integrable. □

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References

[1] J. Azéma. Théorie générale des processus et retournement du temps. *Ann. Sci. École Norm. Sup. (4)*, 6:459–519 (1974), 1973.

[2] J. Bertoin. Reflecting a Langevin process at an absorbing boundary. *Ann. Probab.*, 35(6):2021–2037, 2007.

[3] J. Bertoin. A second order SDE for the Langevin process reflected at a completely inelastic boundary. *J. Eur. Math. Soc. (JEMS)*, 10(3):625–639, 2008.

[4] R. K. Getoor. Excursions of a Markov process. *Ann. Probab.*, 7(2):244–266, 1979.

[5] J. P. Gor′kov. A formula for the solution of a certain boundary value problem for the stationary equation of Brownian motion. *Dokl. Akad. Nauk SSSR*, 223(3):525–528, 1975.

[6] A. Lachal. Sur le premier instant de passage de l'intégrale du mouvement brownien. *Ann. Inst. H. Poincaré Probab. Statist.*, 27(3):385–405, 1991.

[7] A. Lachal. Application de la théorie des excursions à l'intégrale du mouvement brownien. In *Séminaire de Probabilités XXXVII*, volume 1832 of *Lecture Notes in Math.*, pages 109–195. Springer, Berlin, 2003.

[8] M. Lefebvre. First-passage densities of a two-dimensional process. *SIAM J. Appl. Math.*, 49(5):1514–1523, 1989.

[9] B. Maury. Direct simulation of aggregation phenomena. *Commun. Math. Sci.*, 2(suppl. 1):1–11, 2004.

[10] H. P. McKean, Jr. A winding problem for a resonator driven by a white noise. *J. Math. Kyoto Univ.*, 2:227–235, 1963.

[11] J. Pitman. Stationary excursions. In *Séminaire de Probabilités, XXI*, volume 1247 of *Lecture Notes in Math.*, pages 289–302. Springer, Berlin, 1987.