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Abstract: We consider a group or committee that faces a binary decision under uncertainty. Each member holds some private information. Members agree which decision should be taken in each state of nature, had this been known, but they may attach different values to the two types of mistake that may occur. Most voting rules have a plethora of uninformative equilibria, and informative voting may be incompatible with equilibrium. We analyze an anonymous randomized majority rule that has a unique equilibrium. This equilibrium is strict, votes are informative, and the equilibrium implements the optimal decision with probability one in the limit as the committee size goes to infinity. We show that this also holds for the usual majority rule under certain perturbations of the behavioral assumptions: (i) a slight preference for voting according to one's conviction, and (ii) transparency and a slight preference for esteem. We also show that a slight probability for voting mistakes strengthens the incentive for informative voting.

Key Words : time-consistency, hyperbolic discounting, stochastic dynamic programming, multiplicity, uniqueness.

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1 Introduction

Much attention has recently been paid in the economics literature to dynamic optimization problems with so-called hyperbolic or quasi-exponential time preferences. In such models, a decision-maker chooses a feasible action in each period $\tau = 0, 1, 2, 3, ...$ The decision-maker cannot pre-commit to future actions — she is free to re-optimize in every period. The decision-maker attaches discount weights to future periods. She thus chooses a current action so as to maximize the expected present value of the current and future instantaneous utilities, based on anticipations of the actions she will later choose.

A distinctive feature of such models is that the preferences the decision-maker holds, while in period τ, over future actions may differ from the preferences she will hold in period $\tau + 1$ of the same future actions. Therefore, the solutions to such models may exhibit dynamic inconsistency: the actions that the decision-maker would choose in period τ if she were able to pre-commit differ from those she will find optimal when she reaches those future periods. These models explain certain behavioral regularities, such as under-saving and procrastination, see Strotz (1956), Pollak (1968), Phelps and Pollak (1968), Peleg and Yaari (1973), Elster (1979), Goldman (1980), Asheim (1997), Laibson (1997), Bernheim et al. (1999), Barro (1999), Laibson and Harris (2001) and Krusell and Smith (2003a).[1]

In these studies, the time horizon is usually infinite and the decision-maker in each period is modelled as a distinct player in the sense of non-cooperative game theory. As a result, one obtains a sequential game with countably infinitely many players who each acts only once, but who all care not only about their instantaneous utility in their own period but also about the instantaneous utilities in subsequent periods, discounted by the given discount function. In some studies, notably Phelps and Pollak (1968), the decision-makers are successive generations in a dynasty, "selves".[2] The solution concept most commonly used is that of subgame perfect equilibrium. Each self (generation) then maximizes the conditionally expected future utility stream, as evaluated from (and including) the current period and state, and given the strategies of all future selves (generations).

A by-product of such modelling assumptions is that the solution may not be unique. This has been shown in Asilis et al. (1991), Laibson (1994), Kocherlakota (1996), Asheim (1997), and Krusell and Smith (2003a,b). We here provide simple, intuitive and robust examples with multiple subgame-perfect equilibria, within a canonical framework with a small number of states and simple random shocks. We also provide a sufficient condition for uniqueness of subgame-perfect equilibria in the special case of a single state, that is, in infinitely repeated decision problems.[3] This condition is met by the quasi-hyperbolic discount functions suggested by Phelps

[1] For a discussion of empirical evidence, see Eisenhauer and Ventura (2006) and the references therein.

[2] Saez-Marti and Weibull (2005) identify conditions under which discounting of future utilitites is consistent with *pure altruism*, that is, a concern for future selves' or generations' welfare, as defined in terms of their preferences.

[3] Harris and Laibson (2004) analyze a continuous-time model and show that stochasticity and an assumption

and Pollak (1968) and Laibson (1997). It is also met by the hyperbolic discount functions used in the psychology literature (see Mazur (1981), Herrnstein (1987) and Ainslie (1992)), as well as by decision-makers with limited foresight as modelled in Jehiel (2001).

Section 2 provides an open set of simple stochastic dynamic decision problems that admit two stationary subgame-perfect equilibria with distinct payoffs. Section 2.2 discusses a savings-consumption example of this variety. Section 3 gives an example of multiple subgame-perfect equilibria in a repeated decision problem faced by a consumer and it also provides our uniqueness result for repeated decision problems.

2 Multiplicity

2.1 A class of Markov-equilibrium examples

We here demonstrate the possibility of multiple and distinct solutions to a class of dynamic decision problems with non-exponential discounting of the frequently used (β, δ)-variety. The setup is as follows. Time is discrete, $t = 0, 1, 2, ...$, and there are two states, A and B, and two actions, a and b. In each period t and state $\omega_t \in \Omega = \{A, B\}$ the current decision maker, player t, must choose an action, $x_t \in X = \{a, b\}$. The state-action pair (ω_t, x_t) determines both the current payoff or utility, $u(\omega_t, x_t)$, and the transition probability $p(\cdot \mid \omega_t, x_t)$ to the state ω_{t+1} in the next period. Each decision maker knows the current state and strives to maximize the conditionally expected value of the sum of current and future payoffs,

$$u(\omega_t, x_t) + \beta \sum_{k>t} \delta^{k-t} u(\omega_k, x_k), \tag{1}$$

for $\beta, \delta \in (0, 1)$. As shown by Saez-Marti and Weibull (2005), this is equivalent to exponentially decaying true altruism towards future decision makers.

Given $\beta, \delta \in (0, 1)$, such a decision problem is fully specified by eight numbers: four utilities and four probabilities. Hence, we may think of these decision problems as points in $\mathbb{R}^4 \times [0, 1]^4$.[4] For each action $x \in X$, let $p_x := p(B \mid A, x)$ and $q_x := p(A \mid B, x)$. In other words, p_x is the probability of moving out of state A and q_x the probability of moving out of state B. We will focus on the full-dimensional class $\mathcal{P}$ of such decision problems in which **A1**–**3** below hold:

A1 $p_b - p_a \geq q_a - q_b > 0$: Action a is more likely to lead out of state B than action b, and action b is even more likely to lead out of state A than action a,

A2 $\min\{u(A, a), u(A, b)\} > \max\{u(B, a), u(B, b)\}$: The lowest payoff in state A is higher than the highest payoff in state B,

of "instantaneous gratification" together imply uniqueness.

[4] Obviously, the dimension may be reduced by observing that any positive affine transformation of the payoff function u yields a strategically equivalent decision problem.

A3 $u(\omega, b) > u(\omega, a) \quad \forall \omega \in \Omega$: Action b gives higher instantaneous payoff than action a, in both states.

In other words, state A is the "good" state and B the "bad" state, and action a may be described as "virtuous" and action b as "hedonic."

Consider the decision problem $P^* \in \mathcal{P}$ defined by the table of instantaneous payoff values

	A	B
a	1	0
b	$1+\beta\delta$	$\beta\delta$

(2)

and transition probabilities $p_a = q_b = 0$ and $p_b = q_a = 1$; that is, action a leads to A for sure and action b leads to B for sure, irrespective of the current state.[5]

For each action $x \in X$, let σ_x denote the strategy profile that always plays x, irrespective of past play. Irrespective of the initial state, the profile σ_a yields a higher payoff than the profile σ_b:

	A	B
σ_a	$1+\beta\frac{\delta}{1-\delta}$	$\beta\frac{\delta}{1-\delta}$
σ_b	$1+\beta\delta+\beta^2\frac{\delta^2}{1-\delta}$	$\beta\delta\left(1+\beta\frac{\delta}{1-\delta}\right)$

(3)

Despite the fact that they yield distinct payoffs, each of these two strategy profiles constitutes a subgame perfect equilibrium (SPE) in P^*. To see this, consider an arbitrary time period t and first assume that player t expects a to be played in all future periods, irrespective of the current action. If player t would play b instead of a, the effect would be two-fold. On the one hand, this would increase the instantaneous payoff by $\beta\delta$. On the other hand, the state in the next period, $t+1$, would then be B rather than A, so the instantaneous payoff in that period would become one unit lower. Since player t attaches the discount factor $\beta\delta$ to next period, these two effects cancel out. Hence σ_a is an SPE. Likewise, if player t would expect b to be played in all future periods, irrespective of his current action, then playing a rather than b would decrease his instantaneous payoff by $\beta\delta$ and increase the next instantaneous payoff by 1. Again player t is indifferent, so also σ_b is an SPE.

This multiplicity is non-generic, however, in the sense that the payoffs are such that each player is indifferent between the two actions, a and b. One may thus wonder whether there are decision problems P near P^* in which the players are not indifferent between the two actions, while still both σ_a and σ_b are SPE. We proceed to answer this question in the affirmative and will show that the claim is robust in the sense that it holds for an open set of decision problems close to P^*. Formally:

Proposition 2.1 *Any neighborhood of P^* (in $\mathbb{R}^4 \times [0,1]^4$) contains an open set of decision problems $P \in \mathcal{P}$ in which σ_a and σ_b are subgame perfect equilibria, yield different payoffs, and in which no player is indifferent between actions a and b.*

[5] This decision problem is similar in character to that in Example 3 in Asheim (1997), but simpler.

We prove this claim in two steps. First, by identifying necessary and sufficient conditions on a decision problem $P \in \mathcal{P}$ for σ_a and σ_b to be subgame perfect. Second, we perturb slightly the instantaneous payoffs and transition probabilities in P^* in such a way that the new decision problem still belongs to $\mathcal{P}$, σ_a and σ_b remain subgame perfect with distinct payoffs, but each player now has a strict preference for action x in the profile σ_x, for $x = a, b$.

Let P be a decision problem with instantaneous utility function u and transition probabilities p and q satisfying **A1**–**3**.

Lemma 2.2 *The strategy profile σ_a is a SPE in P if and only if*

$$\frac{u(A,a) - u(B,a)}{1 - (1 - p_a - q_a)\delta} \geq \frac{1}{\beta\delta} \max \left\{ \frac{u(A,b) - u(A,a)}{p_b - p_a}, \frac{u(B,b) - u(B,a)}{q_a - q_b} \right\}. \tag{4}$$

The strategy profile σ_b is a SPE in P if and only if

$$\frac{u(A,b) - u(B,b)}{1 - (1 - p_b - q_b)\delta} \leq \frac{1}{\beta\delta} \min \left\{ \frac{u(A,b) - u(A,a)}{p_b - p_a}, \frac{u(B,b) - u(B,a)}{q_a - q_b} \right\}. \tag{5}$$

(See appendix for a proof.) Note that denominators on the right hand sides are positive by **A1** and that all payoff differences are positive by **A2** and **A3**. Moreover, it is easily verified that (4) and (5) hold with equality for P^*.

We now take the second step by way of constructing a family of decision problems near P^*. For $\varepsilon, \lambda \geq 0$, let $\mathcal{P}_{\varepsilon,\lambda}$ be those decision decision problems in $\mathcal{P}$ in which

$$p_b + q_b = 1 \quad \text{and} \quad p_a + q_a = 1 - \varepsilon \tag{6}$$

$$u(A,a) - u(B,a) = 1 \tag{7}$$

and

$$\frac{u(A,b) - u(A,a)}{p_b - p_a} = \frac{u(B,b) - u(B,a)}{q_a - q_b} = \lambda. \tag{8}$$

Observe that $P^* \in \mathcal{P}_{\varepsilon^*,\lambda^*}$ for $(\varepsilon^*, \lambda^*) := (0, \beta\delta)$ and that any neighborhood of $(\varepsilon^*, \lambda^*)$ contains points $(\varepsilon, \lambda) \in \mathbb{R}^2_+$ satisfying

$$\frac{1}{1 - \varepsilon\beta\delta} < \frac{\lambda}{\beta\delta} < \frac{1}{1 - \varepsilon\delta}. \tag{9}$$

It is easily verified that the two conditions in the lemma, (4) and (5), hold with strict inequality for any $P \in \mathcal{P}_{\varepsilon,\lambda}$ satisfying (9) (see appendix for a proof). Since the correspondence $(\varepsilon, \lambda) \mapsto \mathcal{P}_{\varepsilon,\lambda}$ is lower hemi-continuous, any neighborhood $\mathcal{O}$ of P^* satisfies $\mathcal{O} \cap \mathcal{P}_{\varepsilon,\lambda} \neq \varnothing$ for all (ε, λ) close enough to $(\varepsilon^*, \lambda^*)$. Therefore, the neighborhood $\mathcal{O}$ of P^* contains at least one decision problem $P \in \mathcal{P}$ that satisfy both (4) and (5) with strict inequality. By continuity, this still holds in a neighborhood of P. Finally, since σ^a and σ^b induce different payoffs in P^*, they also induce different payoffs in any decision problem close enough to P^*. This proves the proposition.

2.2 A savings/addiction example

Consider a decision-maker, with given $\beta, \delta \in (0,1)$, facing a decision problem $P^0 \in \mathcal{P}$ defined by the table of instantaneous utilities

	A	B
a	1	0
b	x_1	x_0

(10)

for $0 < x_0 < 1 < x_1$ and transition probabilities $p_a = q_b = 0$ and $p_b = q_a = 1$; that is, action a leads to A for sure and action b leads to B for sure, irrespective of the current state. Action a could, for example, be for a consumer to "consume prudently" and action b could be to "squander," with state A representing "being solvent" and state B "being broke." Alternatively, action a could be to abstain from smoking and action b to smoke, with state A representing good health and state B less good health.

As before, let σ_b denote the strategy profile that always plays b, irrespective of past play — the permanent smoker. Let σ_a' denote the strategy profile that takes action a if action b was never taken before and otherwise takes action b. In other words, under strategy profile σ_a' the decision-maker believes that once taken, action b will always be taken, "If I take one cigarette, then I will become a permanent smoker." We note that these two strategy profiles agree on the set of histories in which action b was taken in some earlier period. The payoffs to the two profiles, starting from each of the two states, are

	A	B
σ_a'	$1+\frac{\beta\delta}{1-\delta}$	$\frac{\beta\delta}{1-\delta}$
σ_b	$x_1+\frac{\beta\delta}{1-\delta}x_0$	$\left(1+\frac{\beta\delta}{1-\delta}\right)x_0$

(11)

Suppose that

$$x_1 < \frac{1-\delta+\beta\delta}{1-\delta} - \frac{\beta\delta}{1-\delta}x_0 \tag{12}$$

and

$$x_0 < \frac{\beta\delta}{1-\delta+\beta\delta} \tag{13}$$

Then σ_a' earns more than σ_b in each state. Arguing in the same way as in decision problem P^* above, it is easily verified that strategy profile σ_b is subgame perfect if and only if the following condition holds:

$$\frac{1-\beta\delta x_0}{1-\beta\delta} \le x_1 \le \frac{1+\beta\delta}{\beta\delta}x_0 \tag{14}$$

We note that the lower bound on x_1 is indeed lower than the upper bound if and only if $\beta\delta < x_0$. Moreover, the upper bound then exceeds unity. Hence, σ_b is then subgame perfect in an open set of decision problems P^0. This set is illustrated in the diagram below, for $\beta = 0.7$

and $\delta = 0.8$. This is the area above the downward-sloping straight line (the lower bound on x_1) and below the upward-sloping straight line (the upper bound on x_1).

(Figure 1 here)

The strategy profile σ'_a is subgame perfect if it does not pay to deviate from it in any of the two states (i) if action b was *never* taken before, and (ii) if action b *was* taken before. As noted above, the incentive condition in (ii) is met if and only if σ_b is subgame perfect, that is, if and only if condition (14) is met. So it only remains to check the incentive condition in (i), which amounts to the following two inequalities:

$$1 + \frac{\beta\delta}{1-\delta} \geq x_1 + \frac{\beta\delta}{1-\delta} x_0 \tag{15}$$

and

$$\frac{\beta\delta}{1-\delta} \geq x_0 + \frac{\beta\delta}{1-\delta} x_0. \tag{16}$$

We note that (15) follows from (12) and (16) from (13). Hence, if conditions (12), (13) and (14) are met, then not only do the two strategy profiles σ'_a and σ_b yield distinct payoffs, but they are also both subgame perfect. In the numerical example above, conditions (12) and (13) are met by all points (x_0, x_1) to the left of the two dashed lines. For example, all points near $(0.7, 1.7)$ meet all these conditions. Since payoffs are continuous in the transition probabilities, all of the above qualitative conclusions hold for all transition probabilities in a neighborhood of $p_a = q_b = 0$ and $p_b = q_a = 1$.

2.3 A generalized savings/addiction example

The reader may feel some concern that the above analysis may depend on specific features of the two-state case. In order to relieve this concern, we here briefly sketch a generalization of the preceding example. Let the state space Ω now consist of $K+1$ states $\Omega = \{0, 1, \ldots, K\}$. For simplicity, suppose that there are only two actions, a and b. As before, the action a can be interpreted as "prudent"and the action b as "imprudent". When in state k, action a leads for sure to state $k+1$, while action b leads for sure to state $k-1$.[6] Action a (resp. b) yields a payoff of y_k (resp. of x_k) when taken in state k. Suppose that:

- Both sequences (x_k) and (y_k) are non-decreasing: the higher the state, the higher the instantaneous utility from both a and b.

[6] With the boundary conditions that when a (resp. b) is played in state K (resp. 0), the state does not change.

- The marginal utility gain from switching from a to b, $x_k - y_k$ is positive and decreasing with k.

We will prove that, for every K and $\beta < 1$, and for δ close enough to one, there is an open set of such values for x_k and y_k for which both strategy profiles σ'_a and σ_b defined above in the special case of two states) are subgame-perfect equilibria of the corresponding decision problem.[7] We first state necessary and sufficient conditions under which both profiles are subgame perfect, and then proceed to deduce transparent sufficient conditions under which both profiles are subgame perfect for all δ close enough to 1. We finally argue that these conditions are met on a open of values of the parameters.

Let us fix x_k, y_k ($k \in \Omega$), and denote by $\mathcal{P}$ the corresponding decision problem. The strategy profile σ'_a is a SPE of $\mathcal{P}$ if for each k, the overall utility of the decision-maker when playing b in state k, and then sticking to σ'_a does not exceed the overall utility when playing a in state k, and then sticking to σ'_a.

When $0 < k < K$, the former utility is equal to

$$x_k + \beta \left\{ \delta x_{k-1} + \cdots + \delta^{k-1} x_1 + \frac{\delta^k}{1-\delta} x_0 \right\} \tag{17}$$

while the latter is[8]

$$y_k + \beta \left\{ \delta y_{k+1} + \cdots + \delta^{K-k-1} y_{K-1} + \frac{\delta^{K-k}}{1-\delta} y_K \right\}. \tag{18}$$

If moreover the inequality is strict for each k, the profile σ'_a remains a SPE in all decision problems in a neighborhood of $\mathcal{P}$.

Observe that, as $\delta \to 1$, all these incentive conditions boil down to

$$y_K > x_0. \tag{19}$$

Similarly, the strategy profile σ_b is a SPE of the decision problem $\mathcal{P}$ if for each k, the overall utility obtained when playing a in state k, and then sticking to σ_b does not exceed the overall utility when playing b in state k, and then sticking to σ_b.

When $0 < k < K$, the former utility is equal to

$$y_k + \beta \left\{ \delta x_{k+1} + \delta^2 x_k + \cdots + \delta^{k+1} x_1 + \frac{\delta^{k+2}}{1-\delta} x_0 \right\} \tag{20}$$

[7] In addition, since all incentive inequalities will be strict and by continuity, both σ'_a and σ_b remain SPE if transitions are also perturbed.

[8] We omit the corresponding formulas for $k = 0$ and $k = K$, as these are quite similar, with obvious and minor adjustments.

while the latter utility is[9]

$$x_k + \beta \left\{ \delta x_{k-1} + \delta^2 x_{k-2} + \cdots + \delta^{k-1} x_1 + \frac{\delta^k}{1-\delta} x_0 \right\}. \tag{21}$$

Again, if the corresponding inequality is strict for each k, the strategy profile σ_b remains a SPE in some neighborhood of the decision problem $\mathcal{P}$.

When subtracting (21) from (20), the necessary and sufficient condition for $k \neq 0, K$ boils down to

$$(y_k - x_k) + \beta \left\{ \delta (x_{k+1} - x_{k-1}) + \cdots \delta^k (x_2 - x_0) + \delta^{k+1} (x_1 - x_0) \right\} \leq 0. \tag{22}$$

Letting now δ go to 1, it appears that (22) holds with strict inequality for all δ sufficiently close to one, whenever

$$x_k - y_k > \beta \{x_k + x_{k+1} - 2x_0\}, \text{for all } 0 < k < K. \tag{23}$$

It is readily checked that the corresponding conditions, for $k = K$ and $k = 0$, boil down to

$$x_K - y_K > \beta \{x_K - x_0\} \tag{24}$$

and

$$x_0 - y_0 > \beta \{x_1 - x_0\} \tag{25}$$

respectively.

To summarize: both profiles σ'_a and σ_a are subgame perfect for all δ sufficiently close to 1, if all four inequalities (19), (23), (24) and (25) hold. It is straight-forward to verify that this set of inequalities admits a solution if and only if $\beta < 1$.[10]

3 Uniqueness

Our proof of multiplicity hinges on the assumption that $\beta < 1$, since inequality (9) is violated when $\beta = 1$. Indeed, when $\beta = 1$, each decision problem $P \in \mathcal{P}$ has a unique subgame perfect equilibrium payoff, a fact that can be established by standard arguments for stochastic dynamic programming under exponential discounting (see Vieille and Weibull, 2003). Another crucial assumption behind the multiplicity result is that there are at least two distinct states. One may thus ask about multiplicity/uniqueness of solutions when there is only one state, that is, for infinitely repeated static decision problems. It turns out that uniqueness holds under fairly general conditions, but not always.

In order to shed light on this issue, consider *repeated* decision problems of the following kind: in each time period $t = 0, 1, 2...$ a decision maker, player t, must choose an action x_t from a set

[9] Again, the expressions are quite similar when $k = 0$ and $k = K$, and are omitted.

[10] As a simple solution, set $y_k = 0$ for all $k < K$, $y_K = 1$, $x_0 = 1 - \varepsilon^2$, $x_k = 1$ for all $0 < k < K$, and $x_K = 1 + \varepsilon$, where $0 < \varepsilon < (1 - \beta)/\beta$.

X. This player receives the instantaneous payoff or utility $u(x_t)$. For the sake of brevity and transparency, we focus on cases when maximization of the instantaneous utility has a unique solution. Formally, $X^* = \arg\max_{x \in X} u(x)$ is a singleton set with unique element x^*. Each player t strives to maximize the expected value of the discounted sum of payoffs

$$f(0)u(x_t) + \sum_{k>t}^{\infty} f(k-t)\, u(x_k), \tag{26}$$

where $f : \mathbb{N} \to \mathbb{R}$ is nonnegative, non-increasing and summable: $\sum_{s=0}^{\infty} f(s) < +\infty$. A special case of such discounting is evidently the quasi-exponential discounting analyzed in the preceding section.

To always take the optimal action x^* is clearly a SPE. The following example exhibit multiplicity of SPE outcomes in such repeated decision problems.[11]

Example: Consider a consumer with instantaneous Cobb-Douglas utility

$$u(x_{t1}, x_{t2}) = \alpha_1 \ln x_{t1} + \alpha_2 \ln x_{t2} \tag{27}$$

from positive consumption $x_t = (x_{t1}, x_{t2})$ in any period t, where $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$. Let f be the consumer's discount function, which we take to be nonnegative, non-increasing and summable. Suppose that the consumer in each period faces the same prices $p_1 > 0$ and $p_2 > 0$ and receives the same fixed income $y > 0$, without any possibility of saving or transferring goods from one period to another. This seemingly trivial decision problem seems to have a unique solution, namely, to consume all income in each period according to $x_{ti} = x_i^* = \alpha_i y / p_i$. Indeed, this is a subgame perfect strategy profile. However, there may exist others as well.

First, suppose that the consumer is indifferent between consuming now or in the next period: $f(1) = f(0) > 0$. Let $\hat{x} = (\hat{x}_1, \hat{x}_2)$ be an *arbitrary* positive consumption vector in the the consumer's budget set: $p_1\hat{x}_1 + p_2\hat{x}_2 \leq y$. We claim that the consumption sequence $(\hat{x}, x^*, x^*, \cdots)$ is consistent with subgame perfection. To see this, define a strategy profile $\sigma = (\sigma_t)_{t \in \mathbb{N}}$ inductively as follows. Strategy σ_0 prescribes consumption $\hat{x}$. For $t > 0$, the strategy σ_t prescribes $x_t = x^*$ if consumption in period $t-1$ was consistent with σ_{t-1}. If not, player t "punishes" player $t-1$ by choosing $\hat{x}$. It is readily verified that σ constitutes a subgame perfect equilibrium that implements $(\hat{x}, x^*, x^*, \cdots)$. The strategy profile is subgame perfect because a deviator's punishment, occurring in the subsequent period, is not discounted by the deviator, and so the deviator's potential utility gain never exceeds the subsequent utility loss.

Secondly, suppose that the consumer is a "classical" exponential discounter: $f(t) = \delta^t$ for all t, where $1/2 < \delta < 1$. Let $v^* = u(\hat{x}) / (1-\delta)$, the present value of the optimal consumption stream $(x^*, x^*, \cdots)$. Let $\hat{v} < v^*$ be arbitrary. We claim that there exists a subgame perfect equilibrium with present value $\hat{v}$. Let $\lambda = \exp(\hat{v} - v^*)$, and let $\hat{x}_0^* = \lambda x^*$. Then $0 < \lambda < 1$

[11] See Vieille and Weibull (2003) for a detailed analysis of both examples. The second example is close in spirit to examples in Laibson (1994).

and the consumption sequence $(\hat{x}_0, x^*, x^*, \cdots)$ has present value $\hat{v}$. Moreover, it is induced by a strategy profile $\sigma = (\sigma_t)_{t\geq 0}$ defined inductively as follows. Strategy σ_0 prescribes consumption $\hat{x}_0$. For $t > 0$, strategy σ_t prescribes consumption $x_t = x^*$ if the consumption in period $t-1$ is consistent with σ_{t-1}. If not, player t "punishes" player $t-1$ by consuming $x_t = \hat{x}_t$, where

$$\hat{x}_t = \lambda^{2^t} \hat{x}_0 \quad \text{for} \quad t = 1, 2, ... \tag{28}$$

In other words, as t increases $\hat{x}_t$ is an ever shrinking fraction of $\hat{x}_0$. It is not difficult to verify that σ is subgame perfect if $\delta > 1/2$. To see this, consider any period $t \geq 0$. If $t > 0$ and player t deviates when she should play x^*, she can only loose. If she deviates when she should punish her predecessor, then she can at most make the payoff gain

$$u(x^*) - u(\hat{x}_t) + \delta [u(\hat{x}_{t+1}) - u(x^*)] = -2^{t+1} \ln \lambda + \delta 2^{t+2} \ln \lambda \tag{29}$$

an amount that is non-positive for all $t \geq 0$ iff $\delta > 1/2$.

This example shows that in order to restore uniqueness, one needs to assume a lower bound on instantaneous payoffs (a familiar condition in dynamic programming). Moreover, the decision maker should not be so patient that he or she is indifferent concerning postponement from the current period to the next. The following result establishes that if the discount function is strictly decreasing wherever it is positive, then uniqueness holds under the mentioned boundedness condition:

Proposition 3.1 *If* $\inf_{x\in X} u(x) > -\infty$ *and* $f(t) > f(t+1)$ *for all* t *such that* $f(t) > 0$*, then the unique pure subgame perfect equilibrium is to take the optimal action* $x = x^*$ *in each period.*

See Appendix for a proof and a discussion of mixed SPEs. Note that the result applies to any finite set X and quasi-exponentinal discounting of the (β, δ) variety studied in section 2.

4 Appendix

4.1 Proof of Lemma 2.2

We prove only the first claim, since the second one follows along similar lines. Denote by

$$W(\omega) := \mathbb{E}_{\omega,\sigma_a}\left[\sum_{t=0}^{+\infty} \delta^t u(\omega_t, a_t)\right]$$

the expected, *exponentially* discounted payoff, induced by σ_a when starting from state ω.

Since $W(\omega) = u(\omega, a) + \delta \mathbb{E}_{p(\cdot|\omega,a)}[W(\cdot)]$ for $\omega = A, B$, one has

$$W(A) - W(B) = \frac{u(A,a) - u(B,a)}{1 - \delta(1 - p_a - q_a)} b, \tag{30}$$

which is the left-hand side of (4).

The profile σ_a is a SPE if and only if the decision maker, in period 0, (weakly) prefers to play a rather than b in both states, when expecting the continuation payoff to be given by $W(\cdot)$. More precisely,$\forall\omega \in \{A, B\}$:

$$u(\omega, a) + \beta\delta\mathbb{E}_{p(\cdot|\omega,a)}[W(\cdot)] \geq u(\omega, b) + \beta\delta\mathbb{E}_{p(\cdot|\omega,b)}[W(\cdot)] \tag{31}$$

Expanding and rewriting (31) for $\omega = A, B$ leads to the following equivalent condition:

$$\begin{cases} \beta\delta(p_a - p_b)\,[W(B) - W(A)] \geq u(A, b) - u(A, a) \\ \beta\delta(q_a - q_b)\,[W(A) - W(B)] \geq u(B, b) - u(B, a). \end{cases} \tag{32}$$

The claim then follows by use of (30). For σ_b, similar inequalities are obtained, and (5) follows by finally taking the negative of both sides of the resulting inequality.

4.2 Derivation of (9)

Observe first that (4) reduces to

$$\beta\delta > \lambda\left((1-\delta) + \delta\left(p_a + q_a\right)\right). \tag{33}$$

Secondly, we note that the right-hand side of (5) equals $-\lambda/\beta\delta$, and that

$$\begin{aligned} u(B, b) - u(A, b) &= u(B, b) - u(B, a) + u(B, a) - u(A, a) + u(A, a) - u(A, b) \\ &= \lambda\left((p_a + q_a) - 1 - (p_b + q_b)\right). \end{aligned}$$

Algebraic manipulations show that (5) thus reduces to

$$\beta\delta < \lambda\left[(1-\delta) + \delta\left((1-\beta)\left(p_b + q_b\right) + \beta\left(p_a + q_a\right)\right)\right]. \tag{34}$$

Since $p_b + q_b > p_a + q_a$, conditions (33) and (34) define a non-empty, open interval of λ-values. In the special case when $p_b + q_b = 1$, and $p_a + q_a = 1 - \varepsilon$, these conditions are together equivalent to (9).

4.3 Proof of Proposition 2.1

In the absence of any measurable structure on X, and any regularity on u, behavior strategies involving randomization are not even defined. To allow for such strategies, we assume that X is a measurable set and that $u : X \to \mathbb{R}$ is measurable. Moreover, without loss of generality, we assume that $u(x^*) = 0$. Let m be the infimum (overall) payoff to player 0, taken over all subgame perfect equilibria. We need to prove that $m = 0$. Let $\pi_t(\sigma)$ be the overall payoff to player t when strategy profile σ is played. We normalize the discount factors so that $\sum_{t=0}^{+\infty} f(t) = 1$.

Step 1: Let $\varepsilon > 0$ be given (conditions on ε will be imposed below). Choose a subgame perfect equilibrium σ with $\pi_0(\sigma) < m + \varepsilon$. Let σ' be the strategy profile induced by σ from period $t = 1$ on, after player $t = 0$ has chosen x^* in period 0.

By the equilibrium condition, choosing x^* in period 0 does not increase 0's overall payoff:

$$\sum_{t=0}^{+\infty} f(t+1)\,\mathbb{E}_{\sigma'}[u(x_t)] \leq \pi_0(\sigma) < m + \varepsilon. \tag{35}$$

Since σ' is a subgame perfect equilibrium, one has

$$\pi_0(\sigma') = \sum_{t=0}^{+\infty} f(t)\,\mathbb{E}_{\sigma'}[u(x_t)] \geq m.$$

Hence,

$$\sum_{t=0}^{+\infty} (f(t) - f(t+1))\mathbb{E}_{\sigma'}[u(x_{t+1})] > -\varepsilon. \tag{36}$$

Step 2: We proceed to prove, by way of contradiction, that $m = 0$. Suppose $m < 0$, and let $\mu = \inf_X u < 0$. Then $x = m/\mu \in (0, 1]$. Define

$$T_0 = \inf \left\{ T : \sum_{t=T}^{+\infty} f(t) < \frac{x}{3} \right\}. \tag{37}$$

Observe that $f(t) > 0$ for each $t < T_0$, and hence $f(t) - f(t+1) > 0$ for each $t < T_0$. We also note that T_0 is finite since $\sum_t f(t) < +\infty$.

Step 3: Let ε in Step 1 be such that $\varepsilon < (f(t) - f(t+1))\,|m|/3$ for all $t < T_0$. In particular, $\varepsilon < |m|/3$. By (36), this implies $\mathbb{E}_{\sigma'}[u(x_{t+1})] > m/3$, for each $t < T_0$. Therefore,

$$\sum_{t=0}^{+\infty} f(t+1)\,\mathbb{E}_{\sigma'}[u(x_t)] > \frac{1}{3}m \sum_{t=0}^{T_0-1} f(t+1) + \mu \sum_{t=T_0}^{+\infty} f(t+1) > \frac{2}{3}m$$

a contradiction to (35). Thus, $m = 0$.

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