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The infinitesimal Hopf algebra and the operads of planar forests

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ABSTRACT. We introduce two operads which own the set of planar forests as a basis. With its usual product and two other products defined by different types of graftings, the algebra of planar rooted trees $\mathcal{H}$ becomes an algebra over these operads. The compatibility with the infinitesimal coproduct of $\mathcal{H}$ and these structures is studied. As an application, an inductive way of computing the dual basis of $\mathcal{H}$ for its infinitesimal pairing is given. Moreover, three Cartier-Quillen-Milnor-Moore theorems are given for the operads of planar forests and a rigidity theorem for one of them.

KEYWORDS. Infinitesimal Hopf algebra, Planar rooted trees, Operads.

AMS CLASSIFICATION. 16W30, 05C05, 18D50.

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Introduction

The Connes-Kreimer Hopf algebra of rooted trees, introduced in [1, 6, 7, 8], is a commutative, non cocommutative Hopf algebra, its coproduct being given by admissible cuts of trees. A non commutative version, the Hopf algebra of planar rooted trees, is introduced in [3, 5]. We furthermore introduce in [4] an infinitesimal version of this object, replacing admissible cuts by left admissible cuts: this last object is here denoted by $\mathcal{H}$. Similarly with the Hopf case, $\mathcal{H}$ is a self-dual object and it owns a non-degenerate, symmetric Hopf pairing, denoted by $\langle -, - \rangle$. This pairing is related to a partial order on the set of planar rooted forests, making it isomorphic to the Tamari poset. As a consequence, $\mathcal{H}$ is given a dual basis denoted by $(f_F)_{F \in \mathbf{F}}$, indexed by the set $\mathbf{F}$ of planar forest. In particular, the sub-family $(f_t)_{t \in \mathbf{T}}$ indexed by the set of planar rooted trees $\mathbf{T}$ is a basis of the space of primitive elements of $\mathcal{H}$.

The aim of this text is to introduce two structures of operad on the space of planar forests. We introduce two (non-symmetric) operads $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$ defined in the following way:

1. $\mathbb{P}_{\searrow}$ is generated by m and $\searrow \in \mathbb{P}_{\searrow}(2)$, with relations:

$$\begin{cases} m \circ (\searrow, I) &= \searrow \circ (I, m), \\ m \circ (m, I) &= m \circ (I, m), \\ \searrow \circ (m, I) &= \searrow \circ (I, \searrow). \end{cases}$$

2. $\mathbb{P}_{\nearrow}$ is generated by m and $\nearrow \in \mathbb{P}_{\nearrow}(2)$, with relations:

$$\begin{cases} m \circ (\nearrow, I) &= \nearrow \circ (I, m), \\ m \circ (m, I) &= m \circ (I, m), \\ \nearrow \circ (\nearrow, I) &= \nearrow \circ (I, \nearrow). \end{cases}$$

We then introduce two products on $\mathcal{H}$ or on its augmentation ideal $\mathcal{M}$, denoted by $\nearrow$ and $\searrow$. The product $F \nearrow G$ consists of grafting F on the left leaf of G and the product $F \searrow G$ consists of grafting F on the left root of G . Together with its usual product m , $\mathcal{M}$ becomes both a $\mathbb{P}_{\searrow}$ - and a $\mathbb{P}_{\nearrow}$ -algebra. More precisely, $\mathcal{M}$ is the free $\mathbb{P}_{\searrow}$ - and $\mathbb{P}_{\nearrow}$ -algebra generated by a single element $\bullet$. As a consequence, $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$ inherits a combinatorial representation using planar forests, with composition iteratively described using the products $\searrow$ and $\nearrow$.

We then give several applications of these operadic structures. For example, the antipode of $\mathcal{H}$ is described in term of the operad $\mathbb{P}_{\searrow}$. We show how to compute elements f_t 's, with $t \in \mathbf{T}$, using the action of $\mathbb{P}_{\searrow}$, and the elements f_F 's, $F \in \mathbf{F}$ from the preceding ones using the action of $\mathbb{P}_{\nearrow}$. Combining all these results, it is possible to compute by induction the basis $(f_F)_{F \in \mathbf{F}}$.

We finally study the compatibilities of products m , $\nearrow$, $\searrow$, the coproduct $\tilde{\Delta}$ and the coproduct $\tilde{\Delta}_{\nearrow}$ dual of $\nearrow$. This leads to the definition of two types of $\mathbb{P}_{\nearrow}$ -bialgebras, and one type of $\mathbb{P}_{\searrow}$ -bialgebras. Each type then define a suboperad of $\mathbb{P}_{\nearrow}$ or $\mathbb{P}_{\searrow}$ corresponding to primitive elements of $\mathcal{M}$, which are explicatively described:

1. The first one is a free operad, generated by the element $\mathbf{1} - \bullet \in \mathbb{P}_{\nearrow}(2)$. As a consequence, the space of primitive elements of $\mathcal{H}$ admits a basis $(p_t)_{t \in \mathbf{T}_b}$ indexed by the set of planar binary trees. The link with the basis $(f_t)_{t \in \mathbf{T}}$ is given with the help of the Tamari order.
2. The second one admits a combinatorial representation in terms of planar rooted trees. It is generated by the corollas $c_n \in \mathbb{P}_{\nearrow}(n)$, $n \geq 2$, with the following relations: for all $k, l \geq 2$,

$$c_k \circ (c_l, \underbrace{I, \dots, I}_{k-1 \text{ times}}) = c_l \circ (\underbrace{I, \dots, I}_{l-1 \text{ times}}, c_k).$$

(b) If s and s' are not comparable for $\geq_{high}$, we shall say that $s \geq_{left} s'$ if one of these assertions is satisfied:

- i. s is a vertex of t_i and s' is a vertex of t_j , with $i < j$.
- ii. s and s' are vertices of the same t_i , and $s \geq_{left} s'$ in the forest obtained from t_i by deleting its root.

This defines the partial order $\geq_{left}$ for all forests F , by induction on the the weight.

(c) We shall say that $s \geq_{h,l} s'$ if $s \geq_{high} s'$ or $s \geq_{left} s'$. This defines a total order on the vertices of F .

1.2 Infinitesimal Hopf algebra of planar forests

1. Let $F \in \mathbf{F}$. An *admissible cut* is a non empty cut of certain edges and trees of F , such that each path in a non-cut tree of F meets at most one cut edge. The set of admissible cuts of F will be denoted by $Adm(F)$. If c is an admissible cut of F , the forest of the vertices which are over the cuts of c will be denoted by $P^c(t)$ (branch of the cut c), and the remaining forest will be denoted by $R^c(t)$ (trunk of the cut). An admissible cut of F will be said to be *left-admissible* if, for all vertices x and y of F , $x \in P^c(F)$ and $x \leq_{left} y$ imply that $y \in P^c(F)$. The set of left-admissible cuts of F will be denoted by $Adm^l(F)$.

2. $\mathcal{H}$ is given a coproduct by the following formula: for all $F \in \mathbf{F}$,

$$\Delta(F) = \sum_{c \in Adm^l(F)} P^c(F) \otimes R^c(F) + F \otimes 1 + 1 \otimes F.$$

Then $(\mathcal{H}, \Delta)$ is an infinitesimal bialgebra, that is to say: for all $x, y \in \mathcal{H}$,

$$\Delta(xy) = (x \otimes 1)\Delta(y) + \Delta(x)(1 \otimes y) - x \otimes y.$$

Examples.

$$\begin{aligned} \Delta(.) &= . \otimes 1 + 1 \otimes ., \\ \Delta(..) &= .. \otimes 1 + 1 \otimes .. + . \otimes ., \\ \Delta(\mathbf{I}) &= \mathbf{I} \otimes 1 + 1 \otimes \mathbf{I} + . \otimes ., \\ \Delta(\mathbf{I}.) &= \mathbf{I} . \otimes 1 + 1 \otimes \mathbf{I} . + . \otimes .. + \mathbf{I} \otimes ., \\ \Delta(\mathbf{V}) &= \mathbf{V} \otimes 1 + 1 \otimes \mathbf{V} + .. \otimes . + . \otimes \mathbf{I}, \\ \Delta(\mathbf{I}) &= \mathbf{I} \otimes 1 + 1 \otimes \mathbf{I} + \mathbf{I} \otimes . + . \otimes \mathbf{I}, \\ \Delta(....) &= \otimes 1 + 1 \otimes . \otimes ... + .. \otimes .. + ... \otimes ., \\ \Delta(\mathbf{I}..) &= \mathbf{I}.. \otimes 1 + 1 \otimes \mathbf{I}.. + . \otimes ... + \mathbf{I} \otimes .. + \mathbf{I} . \otimes ., \\ \Delta(. \mathbf{I}.) &= . \mathbf{I} . \otimes 1 + 1 \otimes . \mathbf{I} . + . \otimes \mathbf{I} . + .. \otimes .. + \mathbf{I} \otimes ., \\ \Delta(.. \mathbf{I}) &= .. \mathbf{I} \otimes 1 + 1 \otimes .. \mathbf{I} + . \otimes . \mathbf{I} + .. \otimes \mathbf{I} + ... \otimes ., \\ \Delta(. \mathbf{V}) &= . \mathbf{V} \otimes 1 + 1 \otimes . \mathbf{V} + . \otimes \mathbf{V} + .. \otimes \mathbf{I} + ... \otimes ., \\ \Delta(. \mathbf{I}) &= . \mathbf{I} \otimes 1 + 1 \otimes . \mathbf{I} + . \otimes \mathbf{I} + .. \otimes \mathbf{I} + \mathbf{I} \otimes ., \\ \Delta(\mathbf{V} .) &= \mathbf{V} . \otimes 1 + 1 \otimes \mathbf{V} . + . \otimes \mathbf{I} . + .. \otimes .. + \mathbf{V} \otimes ., \\ \Delta(\mathbf{I} .) &= \mathbf{I} . \otimes 1 + 1 \otimes \mathbf{I} . + . \otimes \mathbf{I} . + \mathbf{I} \otimes .. + \mathbf{I} \otimes ., \\ \Delta(\mathbf{I} \mathbf{I}) &= \mathbf{I} \mathbf{I} \otimes 1 + 1 \otimes \mathbf{I} \mathbf{I} + . \otimes . \mathbf{I} + \mathbf{I} \otimes \mathbf{I} + \mathbf{I} . \otimes ., \end{aligned}$$

$$\begin{aligned}
\Delta(\Psi) &= \Psi \otimes 1 + 1 \otimes \Psi + . \otimes \mathbf{V} + .. \otimes \mathbf{I} + ... \otimes ., \\
\Delta(\downarrow \mathbf{V}) &= \downarrow \mathbf{V} \otimes 1 + 1 \otimes \downarrow \mathbf{V} + . \otimes \mathbf{V} + \mathbf{I} \otimes \mathbf{I} + \mathbf{I} . \otimes ., \\
\Delta(\mathbf{V} \downarrow) &= \mathbf{V} \downarrow \otimes 1 + 1 \otimes \mathbf{V} \downarrow + . \otimes \mathbf{I} + .. \otimes \mathbf{I} + . \mathbf{I} \otimes ., \\
\Delta(\mathbf{Y}) &= \mathbf{Y} \otimes 1 + 1 \otimes \mathbf{Y} . \otimes \mathbf{I} + .. \otimes \mathbf{I} + \mathbf{V} \otimes ., \\
\Delta(\mathbf{I}) &= \mathbf{I} \otimes 1 + 1 \otimes \mathbf{I} + . \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{I} + \mathbf{I} \otimes ..
\end{aligned}$$

We proved in [4] that $\mathcal{H}$ is an infinitesimal Hopf algebra, that is to say has an antipode S . This antipode satisfies $S(1) = 1$, $S(t) \in \text{Prim}(\mathcal{H})$ for all $t \in \mathbf{T}$, and $S(F) = 0$ for all $F \in \mathbf{F} - (\mathbf{T} \cup \{1\})$.

1.3 Pairing on $\mathcal{H}$

1. We define the operator $B^+ : \mathcal{H} \longrightarrow \mathcal{H}$, which associates, to a forest $F \in \mathbf{F}$, the tree obtained by grafting the roots of the trees of F on a common root. For example, $B^+(\mathbf{V}.) = \mathbf{V} \downarrow$, and $B^+(\mathbf{V} \downarrow) = \mathbf{V} \downarrow$.
2. The application γ is defined by:

$$\gamma : \begin{cases} \mathcal{H} & \longrightarrow \mathcal{H} \\ t_1 \dots t_n \in \mathbf{F} & \longrightarrow \delta_{t_1, .} t_2 \dots t_n. \end{cases}$$

3. There exists a unique pairing $\langle -, - \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow K$, satisfying:
 - i. $\langle 1, x \rangle = \varepsilon(x)$ for all $x \in \mathcal{H}$.
 - ii. $\langle xy, z \rangle = \langle y \otimes x, \Delta(z) \rangle$ for all $x, y, z \in \mathcal{H}$.
 - iii. $\langle B^+(x), y \rangle = \langle x, \gamma(y) \rangle$ for all $x, y \in \mathcal{H}$.

Moreover:

- iv. $\langle -, - \rangle$ is symmetric and non-degenerate.
- v. If x and y are homogeneous of different weights, $\langle x, y \rangle = 0$.
- vi. $\langle S(x), y \rangle = \langle x, S(y) \rangle$ for all $x, y \in \mathcal{H}$.

This pairing admits a combinatorial interpretation using the partial orders $\geq_{\text{left}}$ and $\geq_{\text{high}}$ and is related to the Tamari order on planar binary trees, see [4].

4. We denote by $(f_F)_{F \in \mathbf{F}}$ the dual basis of the basis of forests for the pairing $\langle -, - \rangle$. In other terms, for all $F \in \mathbf{F}$, f_F is defined by $\langle f_F, G \rangle = \delta_{F, G}$, for all forest $G \in \mathbf{F}$. The family $(f_t)_{t \in \mathbf{T}}$ is a basis of the space $\text{Prim}(\mathcal{H})$ of primitive elements of $\mathcal{H}$.

2 The operads of forests and graftings

2.1 A few recalls on non- Σ -operads

1. We shall work here with non- Σ -operads [11]. Recall that such an object is a family $\mathbb{P} = (\mathbb{P}(n))_{n \in \mathbb{N}}$ of vector spaces, together with a composition for all $n, k_1, \dots, k_n \in \mathbb{N}$:

$$\begin{cases} \mathbb{P}(n) \otimes \mathbb{P}(k_1) \otimes \dots \otimes \mathbb{P}(k_n) & \longrightarrow \mathbb{P}(k_1 + \dots + k_n) \\ p \otimes p_1 \otimes \dots \otimes p_n & \longrightarrow p \circ (p_1, \dots, p_n). \end{cases}$$

The following associativity condition is satisfied: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, $\dots$, $p_n \in \mathbb{P}(k_n)$, $p_{1,1}, \dots, p_{n,k_n} \in \mathbb{P}$,

$$\begin{aligned} & (p \circ (p_1, \dots, p_n)) \circ (p_{1,1}, \dots, p_{1,k_1}, \dots, p_{n,1}, \dots, p_{n,k_n}) \\ &= p \circ (p_1 \circ (p_{1,1}, \dots, p_{1,k_1}), \dots, p_n \circ (p_{n,1}, \dots, p_{n,k_n})). \end{aligned}$$

Moreover, there exists a unit element $I \in \mathbb{P}(1)$, satisfying: for all $p \in \mathbb{P}(n)$,

$$\begin{cases} p \circ (I, \dots, I) &= p, \\ I \circ p &= p. \end{cases}$$

An operad is a non- Σ -operad $\mathbb{P}$ with a right action of the symmetric group S_n on $\mathbb{P}(n)$ for all n , satisfying a certain compatibility with the composition.

2. Let $\mathbb{P}$ be a non- Σ -operad. A $\mathbb{P}$ -algebra is a vector space A , together with an action of $\mathbb{P}$:

$$\begin{cases} \mathbb{P}(n) \otimes A^{\otimes n} &\longrightarrow A \\ p \otimes a_1 \otimes \dots \otimes a_n &\longrightarrow p.(a_1, \dots, a_n), \end{cases}$$

satisfying the following compatibility: for all $p \in \mathbb{P}(n)$, $p_1 \in \mathbb{P}(k_1)$, $\dots$, $p_n \in \mathbb{P}(k_n)$, for all $a_{1,1}, \dots, a_{n,k_n} \in A$,

$$\begin{aligned} & (p \circ (p_1, \dots, p_n)).(a_{1,1}, \dots, a_{1,k_1}, \dots, a_{n,1}, \dots, a_{n,k_n}) \\ &= p.(p_1.(a_{1,1}, \dots, a_{1,k_1}), \dots, p_n.(a_{n,1}, \dots, a_{n,k_n})). \end{aligned}$$

Moreover, $I.a = a$ for all $a \in A$.

In particular, if V is a vector space, the free $\mathbb{P}$ -algebra generated by V is:

$$F_{\mathbb{P}}(V) = \bigoplus_{n \in \mathbb{N}} \mathbb{P}(n) \otimes V^{\otimes n},$$

with the action of $\mathbb{P}$ given by:

$$\begin{aligned} & p.((p_1 \otimes a_{1,1} \otimes \dots \otimes a_{1,k_1}), \dots, (p_n \otimes a_{n,1} \otimes \dots \otimes a_{n,k_n})) \\ &= (p \circ (p_1, \dots, p_n)) \otimes a_{1,1} \otimes \dots \otimes a_{1,k_1} \otimes \dots \otimes a_{n,1} \otimes \dots \otimes a_{n,k_n}. \end{aligned}$$

3. Let $\mathbf{T}_b$ be the set of planar binary trees:

$$\mathbf{T}_b = \left\{ \mathbf{1}, \text{Y}, \text{Y}, \text{Y}, \text{Y}, \text{Y}, \text{Y}, \text{Y}, \text{Y}, \text{Y}, \text{Y} \dots \right\}.$$

For all $n \in \mathbb{N}$, $\mathbb{T}_b(n)$ is the vector space generated by the elements of $\mathbf{T}_b$ with n leaves:

$$\begin{aligned} \mathbb{T}_b(0) &= (0), \\ \mathbb{T}_b(1) &= Vect(\mathbf{1}), \\ \mathbb{T}_b(2) &= Vect\left(\text{Y}\right), \\ \mathbb{T}_b(3) &= Vect\left(\text{Y}, \text{Y}\right), \\ \mathbb{T}_b(4) &= Vect\left(\text{Y}, \text{Y}, \text{Y}, \text{Y}, \text{Y}\right). \end{aligned}$$

The family of vector spaces $\mathbb{T}_b$ is given a structure of non- Σ -operad by graftings on the leaves. More precisely, if $t, t_1, \dots, t_n \in \mathbf{T}_b$, t with n leaves, then $t \circ (t_1, \dots, t_n)$ is the binary tree obtained by grafting t_1 on the first leave of t , t_2 on the second leave of t , and so on (note that the leaves of t are ordered from left to right). The unit is $\mathbf{1}$.

It is known that $\mathbb{T}_b$ is the free non- Σ -operad generated by $\mathbf{Y} \in \mathbb{T}_b(2)$. Similarly, given elements $m_1, \dots, m_k$ in $\mathbb{P}(2)$, it is possible to describe the free non- Σ -operad $\mathbb{P}$ generated by these elements in terms of planar binary trees whose internal vertices are decorated by $m_1, \dots, m_k$.

2.2 Presentations of the operads of forests

Definition 1

1. $\mathbb{P}_{\searrow}$ is the non- Σ -operad generated by m and $\searrow \in \mathbb{P}_{\searrow}(2)$, with relations:

$$\begin{cases} m \circ (\searrow, I) &= \searrow \circ (I, m), \\ m \circ (m, I) &= m \circ (I, m), \\ \searrow \circ (m, I) &= \searrow \circ (I, \searrow). \end{cases}$$

2. $\mathbb{P}_{\nearrow}$ is the non- Σ -operad generated by m and $\nearrow \in \mathbb{P}_{\nearrow}(2)$, with relations:

$$\begin{cases} m \circ (\nearrow, I) &= \nearrow \circ (I, m), \\ m \circ (m, I) &= m \circ (I, m), \\ \nearrow \circ (\nearrow, I) &= \nearrow \circ (I, \nearrow). \end{cases}$$

Remark. We shall prove in [2] that these quadratic operads are Koszul.

2.3 Grafting on the root

Let $F, G \in \mathbf{F} - \{1\}$. We put $G = t_1 \dots t_n$ and $t_1 = B^+(G_1)$. We define:

$$F \searrow G = B^+(FG_1)t_2 \dots t_n.$$

In other terms, F is grafted on the root of the first tree of G , on the left. In particular, $F \searrow \cdot = B^+(F)$.

Examples.

$$\begin{array}{c|c|c|c|c} \dots \searrow \mathbf{1} &= \text{diagram} & \mathbf{1} \searrow \dots &= \text{diagram} & \dots \searrow \dots &= \text{diagram} \\ \mathbf{1} \searrow \mathbf{1} &= \text{diagram} & \mathbf{1} \searrow \mathbf{1} &= \text{diagram} & \dots \searrow \mathbf{1} &= \text{diagram} \\ \mathbf{1} \searrow \mathbf{1} &= \text{diagram} & \mathbf{1} \searrow \mathbf{1} &= \text{diagram} & \mathbf{1} \searrow \mathbf{1} &= \text{diagram} \\ \mathbf{V} \searrow \mathbf{1} &= \text{diagram} & \mathbf{1} \searrow \mathbf{V} &= \text{diagram} & \dots \searrow \mathbf{V} &= \text{diagram} \\ \mathbf{1} \searrow \mathbf{1} &= \text{diagram} & \mathbf{1} \searrow \mathbf{1} &= \text{diagram} & \dots \searrow \mathbf{1} &= \text{diagram} \end{array}$$

Obviously, $\searrow$ can be inductively defined in the following way: for $F, G, H \in \mathbf{F} - \{1\}$,

$$\begin{cases} F \searrow \cdot &= B^+(F), \\ F \searrow (GH) &= (F \searrow G)H \\ F \searrow B^+(G) &= B^+(FG). \end{cases}$$

We denote by $\mathcal{M}$ the augmentation ideal of $\mathcal{H}$, that is to say the vector space generated by the elements of $\mathbf{F} - \{1\}$. We extend $\searrow: \mathcal{M} \otimes \mathcal{M} \longrightarrow \mathcal{M}$ by linearity.

Proposition 2 For all $x, y, z \in \mathcal{M}$:

$$x \searrow (yz) = (x \searrow y)z, \quad (1)$$

$$x \searrow (y \searrow z) = (xy) \searrow z. \quad (2)$$

Proof. We can restrict ourselves to $x, y, z \in \mathbf{F} - \{1\}$. Then (1) is immediate. In order to prove (2), we put $z = B^+(z_1)z_2$, $z_1, z_2 \in \mathbf{F}$. Then:

$$x \searrow (y \searrow z) = x \searrow (B^+(yz_1)z_2) = B^+(xyz_1)z_2 = (xy) \searrow (B^+(z_1)z_2) = (xy) \searrow z,$$

which proves (2). $\square$

Corollary 3 $\mathcal{M}$ is given a graded $\mathbb{P}_{\searrow}$ -algebra structure by its products m and by $\searrow$.

Proof. Immediate, by proposition 2. $\square$

2.4 Grafting on the left leaf

Let $F, G \in \mathbf{F}$. Suppose that $G \neq 1$. Then $F \nearrow G$ is the planar forest obtained by grafting F on the leaf of G which is at most on the left. For $G = 1$, we put $F \nearrow 1 = F$. In particular, $F \nearrow \cdot = B^+(F)$.

Examples.

$$\begin{array}{c|c|c|c} \dots \nearrow \mathbf{1} = \mathbb{Y} & \mathbf{1} \nearrow \dots = \mathbf{1}.. & .. \nearrow \dots = \mathbb{V}.. & \dots \nearrow .. = \mathbb{V}. \\ \cdot \mathbf{1} \nearrow \mathbf{1} = \mathbb{Y}^{\downarrow} & \mathbf{1} \nearrow \cdot \mathbf{1} = \mathbf{1}\mathbf{1} & .. \nearrow \cdot \mathbf{1} = \mathbb{V}\mathbf{1} & \cdot \mathbf{1} \nearrow .. = \mathbb{V}^{\downarrow}. \\ \mathbf{1} \cdot \nearrow \mathbf{1} = \mathbb{Y}^{\downarrow} & \mathbf{1} \nearrow \mathbf{1} \cdot = \mathbf{1} \cdot & .. \nearrow \mathbf{1} \cdot = \mathbb{Y} \cdot & \mathbf{1} \cdot \nearrow .. = \mathbb{V}^{\downarrow}. \\ \mathbb{V} \nearrow \mathbf{1} = \mathbb{Y}^{\downarrow} & \mathbf{1} \nearrow \mathbb{V} = \mathbb{V}^{\downarrow} & .. \nearrow \mathbb{V} = \mathbb{V}^{\downarrow} & \mathbb{V} \nearrow .. = \mathbb{Y} \cdot \\ \mathbf{1} \nearrow \mathbf{1} = \mathbf{1} & \mathbf{1} \nearrow \mathbf{1} = \mathbf{1} & .. \nearrow \mathbf{1} = \mathbf{1} & \mathbf{1} \nearrow .. = \mathbf{1}.. \end{array}$$

In an obvious way, $\nearrow$ can be inductively defined in the following way: for $F, G, H \in \mathbf{F}$,

$$\begin{cases} F \nearrow 1 = F, \\ F \nearrow (GH) = (F \nearrow G)H \text{ if } G \neq 1, \\ F \nearrow B^+(G) = B^+(F \nearrow G). \end{cases}$$

We extend $\nearrow: \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$ by linearity.

Proposition 4 1. For all $x, z \in \mathcal{H}$, $y \in \mathcal{M}$:

$$x \nearrow (yz) = (x \nearrow y)z. \quad (3)$$

2. For all $x, y, z \in \mathcal{H}$:

$$x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z.$$

So $(\mathcal{H}, \nearrow)$ is an associative algebra, with unitary element 1.

Proof. Note that (3) is immediate for $x, y, z \in \mathbf{F}$, with $y \neq 1$. This implies the first point. In order to prove the second point, we consider:

$$Z = \{z \in \mathcal{H} \mid \forall x, y \in \mathcal{H}, x \nearrow (y \nearrow z) = (x \nearrow y) \nearrow z\}.$$

Let us first prove that $1 \in Z$: for all $x, y \in \mathcal{H}$,

$$x \nearrow (y \nearrow 1) = x \nearrow y = (x \nearrow y) \nearrow 1.$$

Let $z_1, z_2 \in Z$. Let us show that $z_1 z_2 \in Z$. By linearity, we can separate the proof into two cases:

1. $z_1 = 1$. Then it is obvious.
2. $\varepsilon(z_1) = 0$. Let $x, y \in \mathcal{H}$. By the first point:

$$\begin{aligned} x \nearrow (y \nearrow (z_1 z_2)) &= x \nearrow ((y \nearrow z_1) z_2) \\ &= (x \nearrow (y \nearrow z_1)) z_2 \\ &= ((x \nearrow y) \nearrow z_1) z_2 \\ &= (x \nearrow y) \nearrow (z_1 z_2). \end{aligned}$$

So Z is a subalgebra of $\mathcal{H}$. Let us show that it is stable by B^+ . Let $z \in Z$, $x, y \in \mathcal{H}$. Then:

$$\begin{aligned} x \nearrow (y \nearrow B^+(z)) &= x \nearrow B^+(y \nearrow z) \\ &= B^+(x \nearrow (y \nearrow z)) \\ &= B^+((x \nearrow y) \nearrow z) \\ &= (x \nearrow y) \nearrow B^+(z). \end{aligned}$$

So Z is a subalgebra of $\mathcal{H}$, stable by B^+ . Hence, $Z = \mathcal{H}$. □

Remarks.

1. (3) is equivalent to: for any $x, y, z \in \mathcal{H}$,

$$x \nearrow (yz) - \varepsilon(y)x \nearrow z = (x \nearrow y)z - \varepsilon(y)xz.$$

2. Let $F \in \mathbf{F} - \{1\}$. There exists a unique family $(\cdot F_1, \dots, \cdot F_n)$ of elements of $\mathbf{F}$ such that:

$$F = (\cdot F_1) \nearrow \dots \nearrow (\cdot F_n).$$

For example, $\bigvee \mathbf{1} \cdot = (\cdot \cdot) \nearrow (\cdot \cdot) \nearrow (\cdot \mathbf{1} \cdot)$. As a consequence, $(\mathcal{H}, \nearrow)$ is freely generated by $\cdot \mathbf{F}$ as an associative algebra.

Corollary 5 $\mathcal{M}$ is given a graded $\mathbb{P}_{\nearrow}$ -algebra structure by its product m and by $\nearrow$.

Proof. Immediate, by proposition 4. □

2.5 Dimensions of $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$

We now compute the dimensions of $\mathbb{P}_{\searrow}(n)$ and $\mathbb{P}_{\nearrow}(n)$ for all n and deduce that $\mathcal{M}$ is the free $\mathbb{P}_{\searrow}$ - and $\mathbb{P}_{\nearrow}$ -algebra generated by $\cdot \cdot$.

Notation. We denote by r_n the number of planar rooted forests and we put $R(X) = \sum_{n=1}^{+\infty} r_n X^n$. It is well-known (see [3, 13]) that $R(X) = \frac{1 - 2X - \sqrt{1 - 4X}}{2X}$.

Proposition 6 For $\xrightarrow{?} \in \{\searrow, \nearrow\}$ and all $n \in \mathbb{N}^*$, in the $\mathbb{P}_{\xrightarrow{?}}$ -algebra $\mathcal{M}$:

$$\mathbb{P}_{\xrightarrow{?}}(n).(\cdot, \dots, \cdot) = \text{Vect}(\text{planar forests of weight } n).$$

As a consequence, $\mathcal{M}$ is generated as a $\mathbb{P}_{\xrightarrow{?}}$ -algebra by $\cdot$.

Proof. $\subseteq$. Immediate, as $\mathcal{M}$ is a graded $\mathbb{P}_{\xrightarrow{?}}$ -algebra.

$\supseteq$. Induction on n . For $n = 1$, $I(\cdot) = \cdot$. For $n \geq 2$, two cases are possible.

1. $F = F_1 F_2$, $\text{weight}(F_i) = n_i < n$. By the induction hypothesis, there exists $p_1, p_2 \in \mathbb{P}_{\xrightarrow{?}}$, such that $F_1 = p_1(\cdot, \dots, \cdot)$ and $F_2 = p_2(\cdot, \dots, \cdot)$. Then $(m \circ (p_1, p_2)).(\cdot, \dots, \cdot) = m.(F_1, F_2) = F_1 F_2$.
2. $F \in \mathbf{T}$. Let us put $F = B^+(G)$. Then there exists $p \in \mathbb{P}_{\xrightarrow{?}}$, such that $p(\cdot, \dots, \cdot) = G$. Then:

$$\begin{cases} (\searrow \circ (p, I)).(\cdot, \dots, \cdot) = G \searrow \cdot = F, \\ (\nearrow \circ (p, I)).(\cdot, \dots, \cdot) = G \nearrow \cdot = F. \end{cases}$$

Hence, in both cases, $F \in \mathbb{P}_{\xrightarrow{?}}(n).(\cdot, \dots, \cdot)$. □

Corollary 7 For all $\xrightarrow{?} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\xrightarrow{?}}(n)) \geq r_n$.

Proof. Because we proved the surjectivity of the following application:

$$ev_{\xrightarrow{?}} : \begin{cases} \mathbb{P}_{\xrightarrow{?}}(n) & \longrightarrow & \text{Vect}(\text{planar forests of weight } n) \\ p & \longrightarrow & p(\cdot, \dots, \cdot). \end{cases}$$

□

Lemma 8 For all $\xrightarrow{?} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\xrightarrow{?}}(n)) \leq r_n$.

Proof. We prove it for $\xrightarrow{?} = \nearrow$. Let us fix $n \in \mathbb{N}^*$. Then $\mathbb{P}_{\nearrow}(n)$ is linearly generated by planar binary trees whose internal vertices are decorated by m and $\nearrow$. The following relations hold:

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}, \quad \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m, \quad \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m.$$

In the sequel of the proof, we shall say that such a tree is *admissible* if it satisfies the following conditions:

1. For each internal vertex s decorated by m , the left child of s is a leaf.
2. For each internal vertex s decorated by $\nearrow$, the left child of s is a leaf or is decorated by m .

For example, here are the admissible trees with 1, 2 or 3 leaves:

$$1, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}^m.$$

The preceding relations imply that $\mathbb{P}_{\nearrow}(n)$ is linearly generated by admissible trees with n leaves. So $\dim(\mathbb{P}_{\nearrow}(n))$ is smaller than a_n , the number of admissible trees with n leaves. For $n \geq 2$, we denote by b_n the number of admissible trees with n leaves whose root is decorated by

m , and by c_n the number of admissible trees with n leaves whose root is decorated by $\nearrow$. We also put $b_1 = 1$ and $c_1 = 0$. Finally, we define:

$$A(X) = \sum_{n \geq 1} a_n X^n, \quad B(X) = \sum_{n \geq 1} b_n X^n, \quad C(X) = \sum_{n \geq 1} c_n X^n.$$

Immediately, $A(X) = B(X) + C(X)$. Every admissible tree with $n \geq 2$ leaves whose root is decorated by m is of the form $m \circ (I, t)$, where t is an admissible tree with $n - 1$ leaves. Hence, $B(X) = XA(X) + X$. Moreover, every admissible tree with $n \geq 2$ leaves whose root is decorated by $\nearrow$ is of the form $\nearrow \circ (t_1, t_2)$, where t_1 is an admissible tree with k leaves whose eventual root is decorated by m and t_2 an admissible tree with $n - k$ leaves ($1 \leq k \leq n - 1$). Hence, for all $n \geq 2$, $c_n = \sum_{k=1}^{n-1} b_k a_{n-k}$, so $C(X) = B(X)A(X)$. As a conclusion:

$$\begin{cases} A(X) &= B(X) + C(X), \\ B(X) &= XA(X) + X, \\ C(X) &= B(X)A(X). \end{cases}$$

So $A(X) = XA(X) + X + B(X)A(X) = XA(X) + X + XA(X)^2 + XA(X)$, and:

$$XA(X)^2 + (2X - 1)A(X) + X = 0.$$

As $a_1 = 1$:

$$A(X) = \frac{1 - 2X - \sqrt{1 - 4X}}{2X} = R(X).$$

So, for all $n \geq 1$, $\dim(\mathbb{P}_{\nearrow}(n)) \leq a_n = r_n$. The proof is similar for $\mathbb{P}_{\searrow}$. $\square$

As immediate consequences:

Theorem 9 For $\xrightarrow{?} \in \{\searrow, \nearrow\}$, $n \in \mathbb{N}^*$, $\dim(\mathbb{P}_{\xrightarrow{?}}(n)) = r_n$. Moreover, the following application is bijective:

$$ev_{\xrightarrow{?}} : \begin{cases} \mathbb{P}_{\xrightarrow{?}}(n) &\longrightarrow Vect(\text{planar forests of weight } n) \subseteq \mathcal{M} \\ p &\longrightarrow p.(., \dots, .). \end{cases}$$

Corollary 10 1. $(\mathcal{M}, m, \searrow)$ is the free $\mathbb{P}_{\searrow}$ -algebra generated by $..$

2. $(\mathcal{M}, m, \nearrow)$ is the free $\mathbb{P}_{\nearrow}$ -algebra generated by $..$

2.6 A combinatorial description of the composition

Let $\xrightarrow{?} \in \{\searrow, \nearrow\}$. We identify $\mathbb{P}_{\xrightarrow{?}}$ and the vector space of non-empty planar forests via theorem 9. In other terms, we identify $F \in \mathbf{F}(n)$ and $ev_{\xrightarrow{?}}^{-1}(F) \in \mathbb{P}_{\xrightarrow{?}}(n)$.

Notations.

1. In order to distinguish the compositions in $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$, we now denote:

- (a) $F \searrow_{\mathbb{P}} (F_1, \dots, F_n)$ the composition of $\mathbb{P}_{\searrow}$,
- (b) $F \nearrow_{\mathbb{P}} (F_1, \dots, F_n)$ the composition of $\mathbb{P}_{\nearrow}$.

2. In order to distinguish the action of the operads $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$ on $\mathcal{M}$, we now denote:

- (a) $F \searrow_{\bullet} (x_1, \dots, x_n)$ the action of $\mathbb{P}_{\searrow}$ on $\mathcal{M}$,
- (b) $F \nearrow_{\bullet} (x_1, \dots, x_n)$ the action of $\mathbb{P}_{\nearrow}$ on $\mathcal{M}$.

Our aim in this paragraph is to describe the compositions of $\mathbb{P}_{\searrow}$ and $\mathbb{P}_{\nearrow}$ in term of forests. We shall prove the following result:

Theorem 11 1. The composition of $\mathbb{P}_{\searrow}$ in the basis of planar forests can be inductively defined in this way:

$$\begin{cases} \bullet \searrow (H) &= H, \\ B^+(F) \searrow (H_1, \dots, H_{n+1}) &= (F \searrow (H_1, \dots, H_n)) \searrow H_{n+1}, \\ FG \searrow (H_1, \dots, H_{n_1+n_2}) &= F \searrow (H_1, \dots, H_{n_1}) G \searrow (H_{n_1+1}, \dots, H_{n_1+n_2}). \end{cases}$$

2. The composition of $\mathbb{P}_{\nearrow}$ in the basis of planar forests can be inductively defined in this way:

$$\begin{cases} \bullet \nearrow (H) &= H, \\ B^+(F) \nearrow (H_1, \dots, H_{n+1}) &= (F \nearrow (H_1, \dots, H_n)) \nearrow H_{n+1}, \\ FG \nearrow (H_1, \dots, H_{n_1+n_2}) &= F \nearrow (H_1, \dots, H_{n_1}) G \nearrow (H_{n_1+1}, \dots, H_{n_1+n_2}). \end{cases}$$

Examples. Let $F_1, F_2, F_3 \in \mathbf{F} - \{1\}$.

$$\begin{array}{l|l} \bullet \searrow (F_1, F_2) &= F_1 F_2, \\ \mathbf{!} \searrow (F_1, F_2) &= F_1 \nearrow F_2, \\ \dots \searrow (F_1, F_2, F_3) &= F_1 F_2 F_3, \\ \bullet \mathbf{!} \searrow (F_1, F_2, F_3) &= F_1 (F_2 \nearrow F_3), \\ \mathbf{!} \bullet \searrow (F_1, F_2, F_3) &= (F_1 \nearrow F_2) F_3, \\ \mathbf{V} \searrow (F_1, F_2, F_3) &= (F_1 F_2) \nearrow F_3, \\ \mathbf{!} \searrow (F_1, F_2, F_3) &= (F_1 \nearrow F_2) \nearrow F_3, \end{array} \quad \begin{array}{l} \bullet \searrow (F_1, F_2) &= F_1 F_2, \\ \mathbf{!} \searrow (F_1, F_2) &= F_1 \searrow F_2, \\ \dots \searrow (F_1, F_2, F_3) &= F_1 F_2 F_3, \\ \bullet \mathbf{!} \searrow (F_1, F_2, F_3) &= F_1 (F_2 \searrow F_3), \\ \mathbf{!} \bullet \searrow (F_1, F_2, F_3) &= (F_1 \searrow F_2) F_3, \\ \mathbf{V} \searrow (F_1, F_2, F_3) &= (F_1 F_2) \searrow F_3, \\ \mathbf{!} \searrow (F_1, F_2, F_3) &= (F_1 \searrow F_2) \searrow F_3. \end{array}$$

Proposition 12 Let $\overset{?}{\rightarrow} \in \{\searrow, \nearrow\}$.

1. $\bullet$ is the unit element of $\mathbb{P}_{\overset{?}{\rightarrow}}$.
2. $\bullet \bullet = m$ in $\mathbb{P}_{\overset{?}{\rightarrow}}(2)$. Consequently, in $\mathbb{P}_{\overset{?}{\rightarrow}}$, $\bullet \bullet \circ (F, G) = FG$ for all $F, G \in \mathbf{F} - \{1\}$.
3. Let $F, G \in \mathbf{F}$. In $\mathbb{P}_{\overset{?}{\rightarrow}}$, $\mathbf{!} = \overset{?}{\rightarrow}$. Consequently, $\mathbf{!} \overset{?}{\rightarrow} (F, G) = F \overset{?}{\rightarrow} G$ for all $F, G \in \mathbf{F} - \{1\}$.

Proof.

1. Indeed, $ev_{\overset{?}{\rightarrow}}(\bullet) = \bullet = ev_{\overset{?}{\rightarrow}}(I)$. Hence, $\bullet = I$.
2. By definition, $ev_{\overset{?}{\rightarrow}}(\bullet \bullet) = \bullet \bullet = ev_{\overset{?}{\rightarrow}}(m)$. So $\bullet \bullet = m$ in $\mathbb{P}_{\overset{?}{\rightarrow}}(2)$. Moreover, for all $F, G \in \mathbf{F} - \{1\}$:

$$\begin{aligned} ev_{\overset{?}{\rightarrow}}(FG) &= FG \\ &= m_{\bullet}^{\overset{?}{\rightarrow}}(F, G) \\ &= m_{\bullet}^{\overset{?}{\rightarrow}}(F \overset{?}{\rightarrow} (\bullet, \dots, \bullet), G \overset{?}{\rightarrow} (\bullet, \dots, \bullet)) \\ &= \left(m_{\bullet}^{\overset{?}{\rightarrow}}(F, G) \right) \overset{?}{\rightarrow} (\bullet, \dots, \bullet) \\ &= ev_{\overset{?}{\rightarrow}}(m_{\bullet}^{\overset{?}{\rightarrow}}(F, G)). \end{aligned}$$

So $FG = m_{\bullet}^{\overset{?}{\rightarrow}}(F, G) = \bullet \bullet \overset{?}{\rightarrow} (F, G)$.

3. Indeed, $ev_{\rightarrow}(\mathbf{1}) = \bullet \xrightarrow{\cdot} \bullet = ev_{\rightarrow}(\xrightarrow{\cdot})$. So $\mathbf{1} = \xrightarrow{\cdot}$ in $\mathbb{P}_{\rightarrow}(2)$. Moreover:

$$\begin{aligned} ev_{\rightarrow}(F \xrightarrow{\cdot} G) &= F \xrightarrow{\cdot} G \\ &= \xrightarrow{\cdot} \bullet (F, G) \\ &= \xrightarrow{\cdot} \bullet (F \xrightarrow{\cdot} (\cdot, \dots, \cdot), G \xrightarrow{\cdot} (\cdot, \dots, \cdot)) \\ &= (\xrightarrow{\cdot} \bullet (F, G)) \cdot (\cdot, \dots, \cdot) \\ &= ev_{\rightarrow}(\xrightarrow{\cdot} \bullet (F, G)). \end{aligned}$$

So, $F \xrightarrow{\cdot} G = \xrightarrow{\cdot} \bullet (F, G) = \mathbf{1} \xrightarrow{\cdot} (F, G)$.

□

Proposition 13 1. Let $F, G \in \mathbf{F}$, different from 1, of respective weights n_1 and n_2 . Let $H_{1,1}, \dots, H_{1,n_1}$ and $H_{2,1}, \dots, H_{2,n_2} \in \mathbf{F} - \{1\}$. Let $\xrightarrow{\cdot} \in \{\searrow, \nearrow\}$. Then, in $\mathbb{P}_{\rightarrow}$:

$$(FG) \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}, H_{2,1}, \dots, H_{2,n_2}) = F \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}) G \xrightarrow{\cdot} (H_{2,1}, \dots, H_{2,n_2}).$$

2. Let $F \in \mathbf{F}$, of weight $n \geq 1$. Let $H_1, \dots, H_{n+1} \in \mathbf{F}$. In $\mathbb{P}_{\rightarrow}$:

$$B^+(F) \xrightarrow{\cdot} (H_1, \dots, H_{n+1}) = (F \xrightarrow{\cdot} (H_1, \dots, H_n)) \xrightarrow{\cdot} H_{n+1}.$$

Proof.

1. Indeed, in $\mathbb{P}_{\rightarrow}$:

$$\begin{aligned} &(FG) \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}, H_{2,1}, \dots, H_{2,n_2}) \\ &= (m \xrightarrow{\cdot} (F, G)) \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}, H_{2,1}, \dots, H_{2,n_2}) \\ &= m \xrightarrow{\cdot} (F \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}), G \xrightarrow{\cdot} (H_{2,1}, \dots, H_{2,n_2})) \\ &= F \xrightarrow{\cdot} (H_{1,1}, \dots, H_{1,n_1}) G \xrightarrow{\cdot} (H_{2,1}, \dots, H_{2,n_2}). \end{aligned}$$

2. In $\mathbb{P}_{\rightarrow}$:

$$\begin{aligned} B^+(F) \xrightarrow{\cdot} (H_1, \dots, H_{n+1}) &= (F \xrightarrow{\cdot} \bullet) \xrightarrow{\cdot} (H_1, \dots, H_{n+1}) \\ &= (\mathbf{1} \xrightarrow{\cdot} (F, \bullet)) \xrightarrow{\cdot} (H_1, \dots, H_{n+1}) \\ &= \mathbf{1} \xrightarrow{\cdot} (F \xrightarrow{\cdot} (H_1, \dots, H_n), \bullet \xrightarrow{\cdot} (H_{n+1})) \\ &= \mathbf{1} \xrightarrow{\cdot} (F \xrightarrow{\cdot} (H_1, \dots, H_n), H_{n+1}) \\ &= (F \xrightarrow{\cdot} (H_1, \dots, H_n)) \xrightarrow{\cdot} H_{n+1}. \end{aligned}$$

□

Combining propositions 12 and 13, we obtain theorem 11.

3 Applications to the infinitesimal Hopf algebra $\mathcal{H}$

3.1 Antipode of $\mathcal{H}$

We here give a description of the antipode of $\mathcal{H}$ in terms of the action $\searrow_{\bullet}$ of the operad $\mathbb{P}_{\searrow}$.

Notations. For all $n \in \mathbb{N}^*$, we denote $l_n = (B^+)^n(1) \in \mathbf{F}(n)$. For example:

$$l_1 = \bullet, l_2 = \mathbf{1}, l_3 = \begin{array}{c} \mathbf{1} \\ | \\ \bullet \end{array}, l_4 = \begin{array}{c} \mathbf{1} \\ | \\ \mathbf{1} \\ | \\ \bullet \end{array}, l_5 = \begin{array}{c} \mathbf{1} \\ | \\ \mathbf{1} \\ | \\ \mathbf{1} \\ | \\ \bullet \end{array} \dots$$

Lemma 14 *Let $t \in \mathbf{T}$. There exists a unique $k \in \mathbb{N}^*$, and a unique family $(t_2, \dots, t_k) \in \mathbf{T}^{k-1}$ such that:*

$$t = l_k \searrow_{\mathbf{q}} (\bullet, t_2, \dots, t_k).$$

Proof. Induction on the weight n of t . If $n = 1$, then $t = \bullet$, so $k = 1$ and the family is empty. We suppose the result at all rank $< n$. We put $t = B^+(s_1 \dots s_m)$. Necessarily, $t_k = B^+(s_2 \dots s_m)$ and $l_{n-1} \searrow_{\mathbf{q}} (\bullet, t_2, \dots, t_{k-1}) = s_1$. We conclude with the induction hypothesis on s_1 . $\square$

Example.

$$\begin{array}{c} \mathbf{1} \\ \swarrow \searrow \\ \mathbf{1} \end{array} \searrow_{\mathbf{q}} (\bullet, \mathbf{1}, \mathbf{1}, \mathbf{V}) = l_4 \searrow_{\mathbf{q}} (\bullet, \mathbf{1}, \mathbf{1}, \mathbf{V}).$$

Definition 15 For all $n \in \mathbb{N}^*$, we put $p_n = \sum_{k=1}^n \sum_{\substack{a_1 + \dots + a_k = n \\ \forall i, a_i > 0}} (-1)^k l_{a_1} \dots l_{a_k}$.

Examples.

$$\begin{aligned} p_1 &= \bullet, \\ p_2 &= -\mathbf{1} + \bullet\bullet, \\ p_3 &= -\begin{array}{c} \mathbf{1} \\ | \\ \bullet \end{array} + \mathbf{1}\bullet + \bullet\mathbf{1} - \dots, \\ p_4 &= -\begin{array}{c} \mathbf{1} \\ | \\ \mathbf{1} \\ | \\ \bullet \end{array} + \begin{array}{c} \mathbf{1} \\ | \\ \bullet\bullet \end{array} + \begin{array}{c} \mathbf{1} \mathbf{1} \end{array} + \begin{array}{c} \mathbf{1} \\ | \\ \bullet\mathbf{1} \end{array} - \mathbf{1}\dots - \bullet\mathbf{1}\bullet - \dots\mathbf{1} + \dots \end{aligned}$$

Remark that p_n is in fact the antipode of l_n in $\mathcal{H}$. It is also the antipode of l_n in the non commutative Connes-Kreimer Hopf algebra of planar trees [3].

Corollary 16 *Let $t \in \mathbf{T}$, written under the form $t = l_k \searrow_{\mathbf{q}} (t_1, \dots, t_k)$, with $t_1 = \bullet$. Then:*

$$S(t) = p_k \searrow_{\mathbf{q}} (t_1, \dots, t_k).$$

Proof. Corollary of proposition 15 of [4], observing that left cuts are cuts on edges from the root of t_i to the root of t_{i+1} in t , for $i = 1, \dots, n-1$. $\square$

3.2 Inverse of the application γ

Proposition 17 *The restriction $\gamma : \text{Prim}(\mathcal{H}) \longrightarrow \mathcal{H}$ is bijective.*

Proof. By proposition 21 of [4]:

$$\gamma|_{\text{Prim}(\mathcal{H})} : \begin{cases} \text{Prim}(\mathcal{H}) & \longrightarrow \mathcal{H} \\ f_{B^+(F)} (F \in \mathbf{F}) & \longrightarrow f_F. \end{cases}$$

So this restriction is clearly bijective. $\square$

We shall denote $\gamma_{|\text{Prim}(\mathcal{H})}^{-1} : \mathcal{H} \longrightarrow \text{Prim}(\mathcal{H})$ the inverse of this restriction. Then, for all $F \in \mathbf{F}$, $\gamma_{|\text{Prim}(\mathcal{H})}^{-1}(f_F) = f_{B^+(F)}$. Our aim is to express $\gamma_{|\text{Prim}(\mathcal{H})}^{-1}$ in the basis of forests.

by the first step. As the tree $\bullet$ is primitive, we deduce that, for all forest $F = t_1 \dots t_n \in \mathbf{F}$, $q_{n+1} \searrow_{\mathbf{q}} (\bullet, t_1, \dots, t_n) \in \text{Prim}(\mathcal{H})$.

Third step. Let us show that for all $x, y \in \mathcal{M}$, $\gamma(q_2 \searrow_{\mathbf{q}} (x, y)) = \gamma(x)y$. We can limit ourselves to $x, y \in \mathbf{F} - \{1\}$. Then $q_2 \searrow_{\mathbf{q}} (x, y) = xy - x \searrow y$. Moreover, by definition of $\searrow$, $x \searrow y$ is a forest whose first tree is not equal to $\bullet$. Hence, $\gamma(q_2 \searrow_{\mathbf{q}} (x, y)) = \gamma(xy) - 0 = \gamma(x)y$.

Last step. Let us show by induction on n that $\gamma(q_{n+1} \searrow_{\mathbf{q}} (\bullet, t_1, \dots, t_n)) = t_1 \dots t_n$. As $q_1 \searrow_{\mathbf{q}} (\bullet) = \bullet$, this is obvious if $n = 0$. Let us suppose the result at rank $n - 1$. By the third step:

$$\begin{aligned} \gamma(q_{n+1} \searrow_{\mathbf{q}} (\bullet, t_1, \dots, t_n)) &= \gamma(q_2 \searrow_{\mathbf{q}} (q_n \searrow_{\mathbf{q}} (\bullet, t_1, \dots, t_{n-1}), t_n)) \\ &= \gamma(q_n \searrow_{\mathbf{q}} (\bullet, t_1, \dots, t_{n-1})) t_n \\ &= t_1 \dots t_n. \end{aligned}$$

Consequently, $x = q_{n+1} \searrow_{\mathbf{q}} (\bullet, t_1, \dots, t_n) \in \text{Prim}(\mathcal{H})$, and satisfies $\gamma(x) = t_1 \dots t_n$, which proves proposition 19. $\square$

Examples. Let $t_1, t_2, t_3 \in \mathbf{T}$.

$$\begin{aligned} \gamma_{|\text{Prim}(\mathcal{H})}^{-1}(t_1) &= \bullet t_1 - \bullet \searrow t_1, \\ \gamma_{|\text{Prim}(\mathcal{H})}^{-1}(t_1 t_2) &= \bullet t_1 t_2 - (\bullet \searrow t_1) t_2 - (\bullet t_1) \searrow t_2 + (\bullet \searrow t_1) \searrow t_2, \\ \gamma_{|\text{Prim}(\mathcal{H})}^{-1}(t_1 t_2 t_3) &= \bullet t_1 t_2 t_3 - (\bullet \searrow t_1) t_2 t_3 - (\bullet t_1) \searrow t_2 t_3 + (\bullet \searrow t_1) \searrow t_2 t_3 - (\bullet t_1 t_2) \searrow t_3 \\ &\quad + (\bullet \searrow t_1 t_2) \searrow t_3 + ((\bullet t_1) \searrow t_2) \searrow t_3 - ((\bullet \searrow t_1) \searrow t_2) \searrow t_3. \end{aligned}$$

3.3 Elements of the dual basis

Lemma 20 For all $x, y \in \mathcal{H}$, $\Delta(x \nearrow y) = x \nearrow y^{(1)} \otimes y^{(2)} + x^{(1)} \otimes x^{(2)} \nearrow y - x \otimes y$. In other terms, $(\mathcal{H}, \nearrow, \Delta)$ is an infinitesimal Hopf algebra.

Proof. We restrict to $x = F \in \mathbf{F} - \{1\}$, $y = G \in \mathbf{F} - \{1\}$. The non-empty and non-total left-admissible cuts of the tree $F \nearrow G$ are:

- The cut on the edges relating F to G . For this cut c , $P^c(F \nearrow G) = F$ and $R^c(F \nearrow G) = G$.
- Cuts acting only on edges of F or on edges relating F to G , at the exception of the preceding case. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of F , such that $P^c(F \nearrow G) = P^{c'}(F)$ and $R^c(F \nearrow G) = R^{c'}(F) \nearrow G$.
- Cuts acting on edges of G . Then necessarily $F \subseteq P^c(F \nearrow G)$. For such a cut, there exists a unique non-empty, non-total left-admissible cut c' of t , such that $P^c(F \nearrow G) = F \nearrow P^{c'}(G)$ and $R^c(F \nearrow G) = R^{c'}(G)$.

Summing these cuts, we obtain, denoting $\Delta(F) = F \otimes 1 + 1 \otimes F + F' \otimes F''$ and $\Delta(G) = G \otimes 1 + 1 \otimes G + G' \otimes G''$:

$$\begin{aligned} \tilde{\Delta}(F \nearrow G) &= (F \nearrow G) \otimes 1 + 1 \otimes (F \nearrow G) + F \otimes G + F' \otimes F'' \nearrow G + F \nearrow G' \otimes G'' \\ &= (F \otimes 1) \nearrow \Delta(G) + \Delta(F) \nearrow (1 \otimes G) - F \otimes G. \end{aligned}$$

So $(\mathcal{H}, \nearrow, \Delta)$ is an infinitesimal bialgebra. As it is graded and connected, it has an antipode. $\square$

Proposition 21 Let $F = t_1 \dots t_n \in \mathbf{F}$. Then $f_F = f_{t_n} \nearrow \dots \nearrow f_{t_1}$.

Proof. *First step.* We show the following result: for all $F \in \mathbf{F}$, $t \in \mathbf{T}$, $f_F \nearrow f_t = f_{tF}$. We proceed by induction on the weight n of F . If $n = 0$, then $F = 1$ and the result is obvious. We now suppose that the result is true at all rank $< n$. Let be $G \in \mathbf{F}$, and let us prove that $\langle f_F \nearrow f_t, G \rangle = \delta_{tF, G}$. Three cases are possible.

1. $G = 1$. Then $\langle f_F \nearrow f_t, G \rangle = \langle f_F \nearrow f_t, 1 \rangle = \varepsilon(f_F \nearrow f_t) = 0 = \delta_{tF, G}$.

2. $G = G_1 G_2$, $G_i \neq 1$. Then, by lemma 20:

$$\begin{aligned}
\langle f_F \nearrow f_t, G \rangle &= \langle \Delta(f_F \nearrow f_t), G_2 \otimes G_1 \rangle \\
&= \sum_{F_1 F_2 = F} \langle f_{F_2} \otimes f_{F_1} \nearrow f_t, G_2 \otimes G_1 \rangle \\
&\quad + \langle f_F \nearrow f_t \otimes 1 + f_F \nearrow 1 \otimes f_t, G_2 \otimes G_1 \rangle - \langle f_F \otimes f_t, G_2 \otimes G_1 \rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \text{weight}(F_1) < n}} \langle f_{F_2} \otimes f_{F_1} \nearrow f_t, G_2 \otimes G_1 \rangle + \langle 1 \otimes f_F \nearrow f_t, G_2 \otimes G_1 \rangle \\
&\quad + \langle f_F \nearrow f_t \otimes 1, G_2 \otimes G_1 \rangle + \langle f_F \otimes f_t, G_2 \otimes G_1 \rangle - \langle f_F \otimes f_t, G_2 \otimes G_1 \rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \text{weight}(F_1) < n}} \langle f_{F_2} \otimes f_{t_{F_1}}, G_2 \otimes G_1 \rangle \\
&= \sum_{\substack{F_1 F_2 = F, \\ \text{weight}(F_1) < n}} \delta_{F_2, G_2} \delta_{t_{F_1}, G_1} \\
&= \delta_{tF, G}.
\end{aligned}$$

3. $G = B^+(G_1)$. Note that $f_F \nearrow f_t$ is a linear span of forests $H_1 \nearrow H_2$, with $H_1, H_2 \neq 1$. By definition of $\nearrow$, the first tree of such a forest is not $\bullet$. Hence, $\gamma(f_F \nearrow f_t) = 0$ and:

$$\langle f_F \otimes f_t, G \rangle = \langle \gamma(f_F \otimes f_t), G_1 \rangle = 0 = \delta_{tF, G},$$

as $tF \notin \mathbf{T}$ because $F \neq 1$.

Second step. We now prove proposition 21 by induction on n . It is obvious for $n = 1$. Suppose the result at rank $n - 1$. By the first step:

$$f_{t_1 \dots t_n} = f_{t_2 \dots t_n} \nearrow f_{t_1} = (f_{t_n} \nearrow \dots \nearrow f_{t_2}) \nearrow f_{t_1} = f_{t_n} \nearrow \dots \nearrow f_{t_2} \nearrow f_{t_1},$$

using the induction hypothesis for the second equality. $\square$

Remarks.

1. As an immediate corollary, because $\nearrow$ is associative, for all forests $F_1, \dots, F_k \in \mathbf{F}$, $f_{F_1 \dots F_k} = f_{F_k} \nearrow \dots \nearrow f_{F_1}$.
2. In term of operads, proposition 21 can be rewritten in the following way:

Corollary 22 *Let $F_1, \dots, F_n \in \mathbf{F}$. Then $f_{F_1 \dots F_n} = l_n \nearrow (f_{F_n}, \dots, f_{F_1})$.*

Remark. Hence, the dual basis $(f_F)_{F \in \mathbf{F}}$ can be inductively computed, using proposition 21 of [4], together with propositions 19 and 21 of the present text:

$$\begin{cases} f_1 &= 1, \\ f_{t_1 \dots t_n} &= f_{t_n} \nearrow \dots \nearrow f_{t_1}, \\ f_{B^+(t_1 \dots t_n)} &= \gamma_{|Prim(\mathcal{H})}^{-1}(f_{t_1 \dots t_n}). \end{cases}$$

For example:

$f_1 = 1$	$f_{\cdot} = \cdot$
$f_{\cdot\cdot} = \mathbf{1}$	$f_{\mathbf{1}} = -\mathbf{1} + \cdot\cdot$
$f_{\cdot\cdot\cdot} = \mathbf{1}$	$f_{\cdot\mathbf{1}} = -\mathbf{1} + \mathbf{V}$
$f_{\mathbf{1}\cdot} = -\mathbf{1} + \mathbf{1}\cdot$	$f_{\mathbf{V}} = -\mathbf{V} + \cdot\mathbf{1}$
$f_{\mathbf{1}} = \mathbf{1} - \mathbf{1}\cdot - \cdot\mathbf{1} + \dots$	$f_{\cdot\cdot\cdot\cdot} = \mathbf{1}$
$f_{\cdot\cdot\mathbf{1}} = -\mathbf{1} + \mathbf{V}$	$f_{\cdot\mathbf{1}\cdot} = -\mathbf{1} + \mathbf{V}$
$f_{\cdot\mathbf{V}} = -\mathbf{V} + \mathbf{V}$	$f_{\mathbf{1}\cdot\mathbf{1}} = \mathbf{1} - \mathbf{V} - \mathbf{V} + \mathbf{V}$
$f_{\mathbf{1}\cdot\cdot} = -\mathbf{1} + \mathbf{1}\cdot$	$f_{\mathbf{1}\mathbf{1}} = \mathbf{1} - \mathbf{V} - \mathbf{1}\cdot + \mathbf{V}\cdot$
$f_{\mathbf{V}\cdot} = -\mathbf{V} + \mathbf{1}\mathbf{1}$	$f_{\mathbf{1}\cdot\cdot} = \mathbf{1} - \mathbf{1}\cdot - \mathbf{1}\mathbf{1} + \mathbf{1}\cdot\cdot$
$f_{\mathbf{V}} = -\mathbf{V} + \cdot\mathbf{1}$	$f_{\cdot\mathbf{V}} = \mathbf{V} - \mathbf{V} - \cdot\mathbf{1} + \cdot\mathbf{V}$
$f_{\cdot\mathbf{V}} = \mathbf{V} - \mathbf{V}\cdot - \cdot\mathbf{1} + \cdot\mathbf{1}\cdot$	$f_{\mathbf{V}\cdot} = \mathbf{V} - \mathbf{1}\mathbf{1} - \cdot\mathbf{V} + \cdot\mathbf{1}\mathbf{1}$
$f_{\cdot\cdot\cdot\cdot} = -\mathbf{1} + \mathbf{1}\cdot + \mathbf{1}\mathbf{1} - \mathbf{1}\cdot\cdot + \cdot\mathbf{1} - \cdot\mathbf{1}\cdot - \dots + \dots$	

4 Primitive suboperads

4.1 Compatibilities between products and coproducts

We define another coproduct $\Delta_{\nearrow}$ on $\mathcal{H}$ in the following way: for all $x, y, z \in \mathcal{H}$,

$$\langle \Delta_{\nearrow}(x), y \otimes z \rangle = \langle x, z \nearrow y \rangle.$$

Lemma 23 For all forest $F \in \mathbf{F}$, $\Delta_{\nearrow}(F) = \sum_{\substack{F_1, F_2 \in \mathbf{F} \\ F_1 F_2 = F}} F_1 \otimes F_2$.

Proof. Let $F, G, H \in \mathbf{F}$. Then:

$$\begin{aligned} \langle \Delta_{\nearrow}(F), f_G \otimes f_H \rangle &= \langle F, f_H \nearrow f_G \rangle \\ &= \langle F, f_{GH} \rangle \\ &= \delta_{F, GH} \\ &= \sum_{\substack{F_1, F_2 \in \mathbf{F} \\ F_1 F_2 = F}} \langle F_1 \otimes F_2, f_G \otimes f_H \rangle. \end{aligned}$$

As $(f_F)_{F \in \mathbf{F}}$ is a basis of $\mathcal{H}$ and $\langle -, - \rangle$ is non degenerate, this proves the result. $\square$

Remark. As a consequence, the elements of $\mathbf{T}$ are primitive for this coproduct.

We now have defined three products, namely m , $\nearrow$, and $\searrow$, and two coproducts, namely $\tilde{\Delta}$ and $\tilde{\Delta}_{\nearrow}$, on $\mathcal{M}$, obtained from Δ and $\Delta_{\nearrow}$ by subtracting their primitive parts. The following properties sum up the different compatibilities.

Proposition 24 For all $x, y \in \mathcal{M}$:

$$\tilde{\Delta}(xy) = (x \otimes 1)\tilde{\Delta}(y) + \tilde{\Delta}(x)(1 \otimes y) + x \otimes y, \quad (4)$$

$$\tilde{\Delta}(x \nearrow y) = (x \otimes 1) \nearrow \tilde{\Delta}(y) + \tilde{\Delta}(x) \nearrow (1 \otimes y) + x \otimes y, \quad (5)$$

$$\tilde{\Delta}_{\nearrow}(xy) = (x \otimes 1)\tilde{\Delta}_{\nearrow}(y) + \tilde{\Delta}_{\nearrow}(x)(1 \otimes y) + x \otimes y, \quad (6)$$

$$\tilde{\Delta}_{\nearrow}(x \nearrow y) = (x \otimes 1) \nearrow \tilde{\Delta}_{\nearrow}(y), \quad (7)$$

$$\tilde{\Delta}_{\nearrow}(x \searrow y) = (x \otimes 1) \searrow \tilde{\Delta}_{\nearrow}(y). \quad (8)$$

Proof. It remains to consider the compatibility between $\nearrow$ or $\searrow$ and $\tilde{\Delta}_{\nearrow}$. Let $F, G \in \mathbf{F} - \{1\}$. We put $G = t_1 \dots t_n$, where the t_i 's are trees. Then $F \nearrow G = (F \nearrow t_1)t_2 \dots t_n$, and $F \nearrow t_1$ is a tree. Hence:

$$\begin{aligned} \tilde{\Delta}_{\nearrow}(F \nearrow G) &= \sum_{i=1}^{n-1} (F \nearrow t_1)t_2 \dots t_i \otimes t_{i+1} \dots t_n \\ &= \sum_{i=1}^{n-1} F \nearrow (t_1 t_2 \dots t_i) \otimes t_{i+1} \dots t_n \\ &= (F \otimes 1) \nearrow \tilde{\Delta}_{\nearrow}(G). \end{aligned}$$

The proof is similar for $F \searrow G$. So all these compatibilities are satisfied. $\square$

Remark. There is no similar compatibility between $\tilde{\Delta}$ and $\searrow$. In particular, lemma 19 is not available for $t \notin \mathbf{T}$.

This justifies the following definitions:

Definition 25

1. A $\mathbb{P}_{\nearrow}$ -bialgebra of type 1 is a family $(A, m, \nearrow, \tilde{\Delta})$, such that:
 - (a) $(A, m, \nearrow)$ is a $\mathbb{P}_{\nearrow}$ -algebra.
 - (b) $(A, \tilde{\Delta})$ is a coassociative, non counitary coalgebra.
 - (c) Compatibilities (4) and (5) are satisfied.
2. A $\mathbb{P}_{\nearrow}$ -bialgebra of type 2 is a family $(A, m, \nearrow, \tilde{\Delta}_{\nearrow})$, such that:
 - (a) $(A, m, \nearrow)$ is a $\mathbb{P}_{\nearrow}$ -algebra.
 - (b) $(A, \tilde{\Delta}_{\nearrow})$ is a coassociative, non counitary coalgebra.
 - (c) Compatibilities (6) and (7) are satisfied.
3. A $\mathbb{P}_{\searrow}$ -bialgebra is a family $(A, m, \searrow, \tilde{\Delta}_{\nearrow})$, such that:
 - (a) $(A, m, \searrow)$ is a $\mathbb{P}_{\searrow}$ -algebra.
 - (b) $(A, \tilde{\Delta}_{\nearrow})$ is a coassociative, non counitary coalgebra.
 - (c) Compatibilities (6) and (8) are satisfied.

Example. The augmentation ideal $\mathcal{M}$ of the infinitesimal Hopf algebra of trees $\mathcal{H}$ is both a $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra of type 1 and 2, and also a $\mathbb{P}_{\searrow}$ -infinitesimal bialgebra.

If A is a bialgebra of such a type, we denote by $\text{Prim}(A)$ the kernel of the coproduct. We deduce the definition of the following suboperads:

Definition 26 Let $n \in \mathbb{N}$. We put:

$$\left\{ \begin{array}{l} \text{PRIM}_{\nearrow}^{(1)}(n) = \left\{ p \in \mathbb{P}_{\nearrow}(n) / \begin{array}{l} \text{For all } A, \mathbb{P}_{\nearrow}\text{-infinitesimal bialgebra of type 1,} \\ \text{and for } a_1, \dots, a_n \in \text{Prim}(A), \\ p.(a_1, \dots, a_n) \in \text{Prim}(A). \end{array} \right\}, \\ \text{PRIM}_{\nearrow}^{(2)}(n) = \left\{ p \in \mathbb{P}_{\nearrow}(n) / \begin{array}{l} \text{For all } A, \mathbb{P}_{\nearrow}\text{-infinitesimal bialgebra of type 2,} \\ \text{and for } a_1, \dots, a_n \in \text{Prim}_{\nearrow}(A), \\ p.(a_1, \dots, a_n) \in \text{Prim}_{\nearrow}(A). \end{array} \right\}, \\ \text{PRIM}_{\searrow}(n) = \left\{ p \in \mathbb{P}_{\searrow}(n) / \begin{array}{l} \text{For all } A, \mathbb{P}_{\searrow}\text{-infinitesimal bialgebra,} \\ \text{and for } a_1, \dots, a_n \in \text{Prim}_{\nearrow}(A), \\ p.(a_1, \dots, a_n) \in \text{Prim}_{\nearrow}(A). \end{array} \right\}. \end{array} \right.$$

We identify $\mathbb{P}_{\nearrow}(n)$ and $\mathbb{P}_{\searrow}(n)$ with the homogeneous component of weight n of $\mathcal{M}$. We put $\text{Prim}(\mathcal{M}) = \text{Ker}(\tilde{\Delta})$ and $\text{Prim}_{\nearrow}(\mathcal{M}) = \text{Ker}(\tilde{\Delta}_{\nearrow})$. We obtain:

Proposition 27 1. For all $n \in \mathbb{N}$:

$$\text{PRIM}_{\nearrow}^{(1)}(n) = \{p \in \mathbb{P}_{\nearrow}(n) / p_{\bullet^{\nearrow}}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap \text{Prim}(\mathcal{M}).$$

2. For all $n \in \mathbb{N}$:

$$\text{PRIM}_{\nearrow}^{(2)}(n) = \{p \in \mathbb{P}_{\nearrow}(n) / p_{\bullet^{\nearrow}}(\cdot, \dots, \cdot) \in \text{Prim}_{\nearrow}(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap \text{Prim}_{\nearrow}(\mathcal{M}).$$

3. For all $n \in \mathbb{N}$:

$$\text{PRIM}_{\searrow}(n) = \{p \in \mathbb{P}_{\searrow}(n) / p_{\bullet^{\searrow}}(\cdot, \dots, \cdot) \in \text{Prim}_{\nearrow}(\mathcal{M})\} = \mathbb{P}_{\searrow}(n) \cap \text{Prim}_{\nearrow}(\mathcal{M}).$$

Proof. As $\mathcal{M}$ is a $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra, by definition:

$$\text{PRIM}_{\nearrow}^{(1)}(n) \subseteq \{p \in \mathbb{P}_{\nearrow}(n) / p_{\bullet^{\nearrow}}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})\}.$$

Moreover, $\{p \in \mathbb{P}_{\nearrow}(n) / p_{\bullet^{\nearrow}}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})\} = \mathbb{P}_{\nearrow}(n) \cap \text{Prim}(\mathcal{M})$, as, for all $p \in \mathbb{P}_{\nearrow}(n)$, $p_{\bullet^{\nearrow}}(\cdot, \dots, \cdot) = p \in \mathcal{M}$.

We now show that $\{p \in \mathbb{P}_{\nearrow}(n) / p_{\bullet^{\nearrow}}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})\} \subseteq \text{PRIM}_{\nearrow}^{(1)}(n)$. We take $p \in \mathbb{P}_{\nearrow}(n)$, such that $p_{\bullet^{\nearrow}}(\cdot, \dots, \cdot) \in \text{Prim}(\mathcal{M})$. Let $\mathcal{D} = \{1, \dots, n\}$ and let A be the free $\mathbb{P}_{\nearrow}$ -algebra generated by $\mathcal{D}$ (with a unit). It can be described as the associative algebra $\mathcal{H}^{\mathcal{D}}$ generated by the set of planar rooted trees decorated by $\mathcal{D}$, and can be given a structure of $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra. As $\mathcal{M}$ is freely generated by $\cdot$ as a $\mathbb{P}_{\nearrow}$ -algebra, there exists a unique morphism of $\mathbb{P}_{\nearrow}$ -algebras from $\mathcal{M}$ to $\mathcal{M}^{\mathcal{D}}$, augmentation ideal of $\mathcal{H}^{\mathcal{D}}$:

$$\xi : \begin{cases} \mathcal{M} & \longrightarrow \mathcal{M}^{\mathcal{D}} \\ \cdot & \longrightarrow \cdot_1 + \dots + \cdot_n. \end{cases}$$

As $\cdot \in \text{Prim}(\mathcal{M})$ and $\cdot_1 + \dots + \cdot_n \in \text{Prim}(A)$, ξ is a $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra morphism from $\mathcal{M}$ to $\mathcal{M}^{\mathcal{D}}$. So, $\xi(p_{\bullet^{\nearrow}}(\cdot, \dots, \cdot)) \in \text{Prim}(A)$.

Let $F \in A$ be a forest, and $s_1 \geq_{h,l} \dots \geq_{h,l} s_k$ its vertices. For all $i \in \{1, \dots, k\}$, we put d_i the decoration of s_i . The *decoration word* associated to F is the word $d_1 \dots d_n$. It belongs to $M(\mathcal{D})$, the free monoid generated by the elements of $\mathcal{D}$. For all $w \in M(\mathcal{D})$, Let A_w be the subspace of A generated by forests whose decoration word is w . This defines a $M(\mathcal{D})$ -gradation of A , as a $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra of type 1.

Consider the projection $\pi_{1,\dots,n}$ onto $A_{1,\dots,n}$. We get:

$$\begin{aligned}\pi_{1,\dots,n} \circ \xi(p \blacktriangleright (\cdot, \dots, \cdot)) &\in \text{Prim}(A), \\ &= \pi_{1,\dots,n}(p \blacktriangleright (\xi(\cdot), \dots, \xi(\cdot))) \\ &= \pi_{1,\dots,n}(p \blacktriangleright (\cdot_1 + \dots + \cdot_n, \dots, \cdot_1 + \dots + \cdot_n)) \\ &= p \blacktriangleright (\cdot_1, \dots, \cdot_n).\end{aligned}$$

So $p \blacktriangleright (\cdot_1, \dots, \cdot_n) \in \text{Prim}(A)$.

Let B be a $\mathbb{P}_{\blacktriangleright}$ -infinitesimal bialgebra and let $a_1, \dots, a_n \in \text{Prim}(B)$. As $\mathcal{M}^{\mathcal{D}}$ is freely generated by the $\cdot_i$'s, there exists a unique morphism of $\mathbb{P}_{\blacktriangleright}$ -algebras:

$$\chi : \begin{cases} A & \longrightarrow B \\ \cdot_i & \longrightarrow a_i. \end{cases}$$

As the $\cdot_i$ and the a_i 's are primitive, χ is a $\mathbb{P}$ -infinitesimal bialgebra morphism. So:

$$\xi(p \blacktriangleright (\cdot_1, \dots, \cdot_n)) = p.(\xi(\cdot_1), \dots, \xi(\cdot_n)) = p.(a_1, \dots, a_n) \in \chi(\text{prim}(\mathcal{M}^{\mathcal{D}})) \subseteq \text{Prim}(A).$$

Hence, $p \in \text{PRIM}_{\blacktriangleright}^{(1)}(n)$. The proof is similar for $\text{PRIM}_{\blacktriangleright}^{(2)}$ and $\text{PRIM}_{\searrow}$. $\square$

4.2 Suboperad $\text{PRIM}_{\blacktriangleright}^{(1)}$

Lemma 28 *We define inductively the following elements of $\mathbb{P}_{\blacktriangleright}$:*

$$\begin{cases} q_1 &= \cdot, \\ q_{n+1} &= (\cdot - \mathbf{1}) \blacktriangleright (q_n, \cdot) = q_n \cdot - B^+(q_n), \text{ for } n \geq 1. \end{cases}$$

Then, for all $n \geq 1$, q_n belongs to $\text{PRIM}_{\blacktriangleright}^{(1)}$. Moreover, for all $x_1, \dots, x_n \in \text{Prim}(\mathcal{M})$:

$$\gamma(q_n \blacktriangleright (x_1, \dots, x_n)) = \gamma(x_1)x_2 \dots x_n.$$

Remark. These q_n 's are the same as the q_n 's defined in section 3.2.

Proof. Let us remark that $f_{\mathbf{1}} = \cdot - \mathbf{1} \in \text{Prim}(\mathcal{M})$. By proposition 27, $\cdot - \mathbf{1} \in \text{PRIM}_{\blacktriangleright}^{(1)}(2)$. As $\text{PRIM}_{\blacktriangleright}^{(1)}$ is a suboperad of $\mathbb{P}_{\blacktriangleright}$, it follows that all the q_n 's belongs to $\text{PRIM}_{\blacktriangleright}^{(1)}(n)$.

Let $x_1, \dots, x_n \in \text{Prim}(\mathcal{M})$. Let us show that $\gamma(q_n \blacktriangleright (x_1, \dots, x_n)) = \gamma(x_1)x_2 \dots x_n$ by induction on n . If $n = 1$, this is immediate. For $n = 2$, $q_2 \blacktriangleright (x_1x_2) = x_1x_2 - x_1 \blacktriangleright x_2$. Moreover, $x_1 \blacktriangleright x_2$ is a linear span of forests whose first tree is not $\cdot$. So $\gamma(q_2 \blacktriangleright (x_1, x_2)) = \gamma(x_1x_2) - 0 = \gamma(x_1)x_2$.

Suppose now the result true at rank $n - 1$. Then:

$$\begin{aligned}q_n \blacktriangleright (x_1, \dots, x_n) &= q_2 \blacktriangleright (\underbrace{q_{n-1} \blacktriangleright (x_1, \dots, x_{n-1})}_{\in \text{Prim}(\mathcal{M})}, x_n), \\ \gamma(q_n \blacktriangleright (x_1, \dots, x_n)) &= \gamma(q_2 \blacktriangleright (q_{n-1} \blacktriangleright (x_1, \dots, x_{n-1}), x_n)) \\ &= \gamma(q_{n-1} \blacktriangleright (x_1, \dots, x_{n-1}))x_n \\ &= \gamma(x_1)x_2 \dots x_n.\end{aligned}$$

$\square$

Theorem 29 *The non- Σ -operad $\text{PRIM}_{\blacktriangleright}^{(1)}$ is freely generated by $\mathbf{1} - \dots$.*

Proof. Let us first show that the family $(q_n)_{n \geq 1}$ generates $\text{PRIM}_{\nearrow}^{(1)}$. Let $\mathbb{P}$ be the suboperad of $\text{PRIM}_{\nearrow}^{(1)}$ generated by the q_n 's. Let us prove by induction on k that $\text{PRIM}_{\nearrow}^{(1)}(k) = \mathbb{P}(k)$. If $k = 1$, $\mathbb{P}(1) = \text{PRIM}_{\nearrow}^{(1)}(1) = K\bullet$. Suppose the result at all ranks $\leq k-1$. By the rigidity theorem for infinitesimal bialgebra of [10], a basis of $\mathcal{H}$ is $(f_{t_1} \cdots f_{t_n})_{t_1 \dots t_n \in \mathbf{F}}$, so a basis of $\text{Prim}(\mathcal{M})$ is:

$$\left(\gamma_{\text{Prim}(\mathcal{H})}^{-1}(f_{t_1} \cdots f_{t_n}) \right)_{t_1 \dots t_n \in \mathbf{F}}.$$

So, a basis of $\text{PRIM}_{\nearrow}^{(1)}(k)$ is $\left(\gamma_{\text{Prim}(\mathcal{H})}^{-1}(f_{t_1} \cdots f_{t_n}) \right)_{\substack{t_1 \dots t_n \in \mathbf{F} \\ \text{weight}(t_1 \dots t_n) = k-1}}$. By lemma 28:

$$\gamma_{\text{Prim}(\mathcal{H})}^{-1}(f_{t_1} \cdots f_{t_n}) = q_{n+1} \nearrow (\bullet, f_{t_1}, \dots, f_{t_n}).$$

By the induction hypothesis, the f_{t_i} 's belongs to $\mathbb{P}$. So:

$$\gamma_{\text{Prim}(\mathcal{H})}^{-1}(f_{t_1} \cdots f_{t_n}) = q_{n+1} \nearrow (\bullet, f_{t_1}, \dots, f_{t_n}) \in \mathbb{P}(n).$$

So $\text{PRIM}_{\nearrow}^{(1)} = \mathbb{P}$.

Moreover, if we denote by $\mathbb{P}'$ the suboperad of $\text{PRIM}_{\nearrow}^{(1)}$ generated by q_2 , then, immediately, $\mathbb{P}' \subseteq \mathbb{P}$. Finally, by induction on n , $q_n \in \mathbb{P}'(n)$ for all $n \geq 1$ and $\mathbb{P} \subseteq \mathbb{P}'$. So $\mathbb{P}' = \mathbb{P} = \text{PRIM}_{\nearrow}^{(1)}$ is generated by q_2 .

Let $\mathbb{P}_{q_2}$ be the non- Σ -operad freely generated by q_2 . There is a non- Σ -operad epimorphism:

$$\Psi : \begin{cases} \mathbb{P}_{q_2} & \longrightarrow \text{PRIM}_{\nearrow}^{(1)} \\ q_2 & \longrightarrow q_2. \end{cases}$$

The dimension of $\mathbb{P}_{q_2}(n)$ is the number of planar binary rooted trees with n leaves, that is to say the Catalan number $c_n = \frac{(2n-2)!}{(n-1)!n!}$. On the other side, the dimension of $\text{PRIM}_{\nearrow}^{(1)}(n)$ is the number of planar rooted trees with n vertices, that is to say c_n . So Ψ is an isomorphism. $\square$

In other terms, in the language of [9]:

Theorem 30 *The triple of operads $(\text{Ass}, \mathbb{P}_{\nearrow}^{\Sigma}, \text{FREE}_2)$, where $\mathbb{P}_{\nearrow}^{\Sigma}$ is the symmetrisation of $\mathbb{P}_{\nearrow}$ and FREE_2 is the free operad generated by an element in $\text{FREE}_2(2)$, is a good triple of operads.*

Remark. Note that if A is a $\mathbb{P}_{\nearrow}$ -bialgebra of type 1, then $(A, m, \tilde{\Delta})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta)$ has an antipode S , then $-S$ is an eulerian idempotent for A .

4.3 Another basis of $\text{Prim}(\mathcal{H})$

Recall that $\mathbf{T}_b$ is freely generated (as a non- Σ -operad) by Υ . In particular, if $t_1, t_2 \in \mathbf{T}_b$, we denote:

$$t_1 \vee t_2 = \Upsilon \circ (t_1, t_2).$$

Every element $t \in \mathbf{T}_b - \{\mathbf{1}\}$ can be uniquely written as $t = t^l \vee t^r$.

There exists a morphism of operads:

$$\Theta : \begin{cases} \mathbf{T}_b & \longrightarrow \mathbb{P}_{\nearrow} \\ \Upsilon & \longrightarrow \bullet - \mathbf{1}. \end{cases}$$

By theorem 29, Θ is injective and its image is $\text{PRIM}^{(1)}_{\nearrow}$. So, we obtain a basis of $\text{PRIM}^{(1)}_{\nearrow}$ indexed by $\mathbf{T}_b$, given by $p_t = \Theta(t)$. It is also a basis of $\text{Prim}(\mathcal{M})$, which can be inductively computed by:

$$\begin{cases} p_{\mathbf{1}} &= \bullet, \\ p_{t_1 \vee t_2} &= (\bullet - \mathbf{1}) \nearrow (p_{t_1}, p_{t_2}) = p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}. \end{cases}$$

4.4 From the basis $(f_t)_{t \in \mathbf{T}}$ to the basis $(p_t)_{t \in \mathbf{T}_b}$

We define inductively the application $\kappa : \mathbf{T}_b \longrightarrow \mathbf{T}$ in the following way:

$$\kappa : \begin{cases} \mathbf{T}_b &\longrightarrow \mathbf{T} \\ \mathbf{1} &\longrightarrow \bullet, \\ t_1 \vee t_2 &\longrightarrow \kappa(t_2) \searrow \kappa(t_1). \end{cases}$$

Examples.

$$\begin{array}{c|c|c|c} \begin{array}{c} \diagup \\ \diagdown \end{array} \longrightarrow \mathbf{1} & \begin{array}{c} \diagup \\ \diagdown \end{array} \longrightarrow \mathbf{v} & \begin{array}{c} \diagup \\ \diagdown \end{array} \longrightarrow \mathbf{1} & \begin{array}{c} \diagup \\ \diagdown \end{array} \longrightarrow \mathbf{1} \\ \begin{array}{c} \diagup \\ \diagdown \end{array} \longrightarrow \mathbf{v} & \begin{array}{c} \diagup \\ \diagdown \end{array} \longrightarrow \mathbf{v} & \begin{array}{c} \diagup \\ \diagdown \end{array} \longrightarrow \mathbf{v} & \begin{array}{c} \diagup \\ \diagdown \end{array} \longrightarrow \mathbf{1} \end{array}$$

It is easy to show that κ is bijective, with inverse given by:

$$\kappa^{-1} : \begin{cases} \mathbf{T} &\longrightarrow \mathbf{T}_b \\ \bullet &\longrightarrow \mathbf{1}, \\ B^+(s_1 \dots s_m) &\longrightarrow \kappa^{-1}(B^+(s_2 \dots s_m)) \vee \kappa^{-1}(s_1). \end{cases}$$

Let us recall the partial order $\leq$, defined in [4], on the set $\mathbf{F}$ of planar forests, making it isomorphic to the Tamari poset.

Definition 31 Let $F \in \mathbf{F}$.

1. An *admissible transformation* on F is a local transformation of F of one of the following types (the part of F which is not in the frame remains unchanged):

2. Let F and $G \in \mathbf{F}$. We shall say that $F \leq G$ if there exists a finite sequence $F_0, \dots, F_k$ of elements of $\mathbf{F}$ such that:

- (a) For all $i \in \{0, \dots, k-1\}$, F_{i+1} is obtained from F_i by an admissible transformation.
- (b) $F_0 = F$.
- (c) $F_k = G$.

The aim of this section is to prove the following result:

Theorem 32 Let $t \in \mathbf{T}_b$. Then $p_t = \sum_{\substack{s \in \mathbf{T} \\ s \leq \kappa(t)}} f_s$.

Proof. By induction on the number n of leaves of t . If $n = 1$, then $t = \mathbf{1}$ and $p_{\mathbf{1}} = \bullet = f_{\bullet}$. Suppose the result at all ranks $\leq n - 1$. As p_t is primitive, we can put:

$$p_t = \sum_{s \in \mathbf{T}} a_s f_s.$$

Write $t = t_1 \vee t_2$. By the induction hypothesis:

$$p_{t_1} = \sum_{\substack{s_1 \in \mathbf{T} \\ s_1 \leq \kappa(t_1)}} f_{s_1} \text{ and } p_{t_2} = \sum_{\substack{s_2 \in \mathbf{T} \\ s_2 \leq \kappa(t_2)}} f_{s_2}.$$

As $t = t_1 \vee t_2$, $p_t = (\bullet - \mathbf{1}) \nearrow (p_{t_1}, p_{t_2}) = p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}$. So, for all $s \in \mathbf{T}$, as s is primitive for $\Delta_{\nearrow}$:

$$\begin{aligned} a_s &= \langle p_t, s \rangle \\ &= \langle p_{t_1} p_{t_2} - p_{t_1} \nearrow p_{t_2}, s \rangle \\ &= \langle p_{t_2} \otimes p_{t_1}, \Delta(s) - \Delta_{\nearrow}(s) \rangle \\ &= \langle p_{t_2} \otimes p_{t_1}, \Delta(s) \rangle \\ &= \sum_{\substack{s_1 \in \mathbf{T} \\ s_1 \leq \kappa(t_1)}} \sum_{\substack{s_2 \in \mathbf{T} \\ s_2 \leq \kappa(t_2)}} \langle f_{s_2} \otimes f_{s_1}, \Delta(s) \rangle. \end{aligned}$$

So a_s is the number of left-admissible cuts c of s , such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$.

Suppose that $a_s \neq 0$. Then, there exists a left-admissible cut c of s , such that $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. As s is a tree, $s \leq \kappa(t_2) \searrow \kappa(t_1) = \kappa(t)$. Moreover, by considering the degree of $P^c(s)$, this cut c is unique, so $a_s = 1$. Reciproquely, if $s \leq \kappa(t)$, if c is the unique left admissible cut such that $\text{weight}(P^c(s)) = \text{weight}(t_2)$, then $P^c(s) \leq \kappa(t_2)$ and $R^c(s) \leq \kappa(t_1)$. So $a_s \neq 0$. Hence, $(s \leq \kappa(t)) \implies (a_s \neq 0) \implies (a_s = 1) \implies (s \leq \kappa(t))$. This proves theorem 32. $\square$

Let μ be the Möbius function of the poset $\mathbf{F}$ ([12, 13]). By the Möbius inversion formula:

Corollary 33 Let $s \in \mathbf{T}$. Then $f_s = \sum_{t \in T_b, \kappa(t) \leq s} \mu(\kappa(t), s) p_t$.

4.5 Suboperad $\text{PRIM}_{\nearrow}^{(2)}$

For all $n \in \mathbb{N}$, we put $c_{n+1} = B^+(\bullet^n)$. In other terms, c_{n+1} is the corolla tree with $n + 1$ vertices, or equivalently with n leaves.

Examples. $c_1 = \bullet$, $c_2 = \mathbf{1}$, $c_3 = \mathbf{V}$, $c_4 = \mathbf{\Psi}$, $c_5 = \mathbf{\Psi} \nearrow \bullet \dots$

Lemma 34 The set $\mathbf{T}$ is a basis of the operad $\text{PRIM}_{\nearrow}^{(2)}$. As an operad, $\text{PRIM}_{\nearrow}^{(2)}$ is generated by the c_n 's, $n \geq 2$. Moreover, for all $k, l \geq 2$,

$$c_k \nearrow (c_l, \underbrace{\bullet, \dots, \bullet}_{k-1 \text{ times}}) = c_l \nearrow (\underbrace{\bullet, \dots, \bullet}_{l-1 \text{ times}}, c_k).$$

Proof. The operad $\text{PRIM}_{\nearrow}^{(2)}$ is identified with $\text{Prim}_{\nearrow}(\mathcal{M})$ by proposition 27. So $\text{Prim}_{\nearrow}(\mathcal{M})$ is equal to $\text{Vect}(\mathbf{T})$. Let $\mathbb{P}$ be the suboperad of $\text{PRIM}_{\nearrow}^{(2)}$ generated by the corollas. Let $t \in \mathbf{T}$, of weight n . Let us prove that $t \in \mathbb{P}$ by induction on n . If $n = 1$, then $t = \bullet \in \mathbb{P}$. If $n \geq 2$, we can suppose that $t = B^+(t_1 \dots t_k)$, with $t_1, \dots, t_k \in \mathbb{P}$. Then, by theorem 11:

$$c_{k+1} \nearrow (t_1, \dots, t_k, \bullet) = (\bullet^k \nearrow (t_1, \dots, t_k)) \nearrow \bullet = (t_1 \dots t_k) \nearrow \bullet = B^+(t_1 \dots t_k) = t.$$

So $t \in \mathbb{P}$. hence, $\mathbb{P} = \text{PRIM}_{\nearrow}^{(2)}$.

Let $k, l \geq 2$. Then, by theorem 11:

$$\begin{aligned} c_k \nearrow (c_l, \bullet, \dots, \bullet) &= (\bullet^{k-1} \nearrow (c_l, \bullet, \dots, \bullet)) \nearrow \bullet \\ &= (c_l \bullet^{k-2}) \nearrow \bullet \\ &= B^+(c_l \bullet^{k-2}) \\ &= B^+(B^+(\bullet^{l-1}) \bullet^{k-2}). \end{aligned}$$

On the other hand:

$$\begin{aligned} c_l \nearrow (\bullet, \dots, \bullet, c_k) &= (\bullet^{l-1} \nearrow (\bullet, \dots, \bullet)) \nearrow c_k \\ &= (\bullet^{l-1}) \nearrow c_k \\ &= (\bullet^{l-1}) \nearrow B^+(\bullet^{k-1}) \\ &= B^+(((\bullet^{l-1}) \nearrow \bullet) \bullet^{k-2}) \\ &= B^+(B^+(\bullet^{l-1}) \bullet^{k-2}). \end{aligned}$$

So $c_k \nearrow (c_l, \bullet, \dots, \bullet) = c_l \nearrow (\bullet, \dots, \bullet, c_k)$. □

Definition 35 The operad $\mathbb{T}$ is the non- Σ -operad generated by elements $c_n \in \mathbb{T}(n)$, for $n \geq 2$, and the following relations: for all $k, l \geq 2$,

$$c_k \circ (c_l, \underbrace{I, \dots, I}_{k-1 \text{ times}}) = c_l \circ (\underbrace{I, \dots, I}_{l-1 \text{ times}}, c_k).$$

In other terms, a $\mathbb{T}$ -algebra A has a family of n -multilinear products $[\cdot, \dots, \cdot] : A^{\otimes n} \longrightarrow A$ for all $n \geq 2$, with the associativity condition:

$$[[a_1, \dots, a_l], a_{l+1}, \dots, a_{l+k}] = [a_1, \dots, a_{l-1}, [a_l, \dots, a_{l+k}]].$$

In particular, $[\cdot, \cdot]$ is associative.

Theorem 36 The operads $\mathbb{T}$ and $\text{PRIM}_{\nearrow}^{(2)}$ are isomorphic.

Proof. By lemma 34, there is an epimorphism of operads:

$$\begin{cases} \mathbb{T} & \longrightarrow \text{PRIM}_{\nearrow}^{(2)} \\ c_n & \longrightarrow c_n. \end{cases}$$

In order to prove this is an isomorphism, it is enough to prove that $\dim(\mathbb{T}(n)) \leq \dim(\text{PRIM}_{\nearrow}^{(2)}(n))$ for all $n \geq 2$. By lemma 34, $\dim(\text{PRIM}_{\nearrow}^{(2)}(n))$ is the n -th Catalan number. Because of the defining relations, $\mathbb{T}(n)$ is generated as a vector space by elements of the form $c_l \circ (I, b_2, \dots, b_l)$, with $b_i \in \mathbb{T}(n_i)$, such that $n_1 + \dots + n_l = n - 1$. Hence, we define inductively the following subsets the free non- Σ -operad generated by the c_n 's, $n \geq 2$:

$$X(n) = \begin{cases} \{I\} & \text{if } n = 1, \\ \bigcup_{l=2}^n \bigcup_{i_2+\dots+i_l=n-1} c_l \circ (I, X(i_2), \dots, X(i_l)) & \text{if } n \geq 2. \end{cases}$$

Then the images of the elements of $X(n)$ linearly generate $\mathbb{T}(n)$, so $\dim(\mathbb{T}(n)) \leq \text{card}(X(n))$ for all n . We now put $a_n = \text{card}(X(n))$ and prove that a_n is the n -th Catalan number. We denote by $A(h)$ their generating formal series. Then:

$$\begin{cases} a_1 = 1, \\ a_n = \sum_{l=2}^n \sum_{i_2+\dots+i_l=n-1} a_{i_1} \dots a_{i_l} \text{ if } n \geq 2. \end{cases}$$

In terms of generating series:

$$A(h) - a_1 h = h \frac{A(x)}{1 - A(x)}.$$

So $A(h)^2 - A(h) + h = 0$. As $A(h) = 1$:

$$A(h) = \frac{1 - \sqrt{1 - 4h}}{2}.$$

So a_n is the n -th Catalan number for all $n \geq 1$. □

In other terms:

Theorem 37 *The triple of operads $(\text{Ass}, \mathbb{P}_{\nearrow}^{\Sigma}, \mathbb{T}^{\Sigma})$ is a good triple of operads.*

Remark. Note that if A is a $\mathbb{P}_{\nearrow}$ -bialgebra of type 2, then $(A, m, \tilde{\Delta}_{\nearrow})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta_{\nearrow})$ has an antipode $S_{\nearrow}$, then $-S_{\nearrow}$ is an eulerian idempotent for A .

4.6 Suboperad $\text{PRIM}_{\searrow}$

Lemma 38 *The set $\mathbf{T}$ is a basis of the operad $\text{PRIM}_{\searrow}$. As an operad, $\text{PRIM}_{\searrow}$ is generated by $\mathbf{1}$.*

Proof. Let $\mathbb{P}$ be the suboperad of $\text{PRIM}_{\searrow}$ generated by $\mathbf{1}$. Let $t \in \mathbf{T}$, of weight n . Let us prove that $t \in \mathbb{P}$ by induction on n . If $n = 1$ or 2 , this is obvious. If $n \geq 2$, suppose that $t = B^+(t_1 \dots t_k)$. By the induction hypothesis, t_1 and $B^+(t_2 \dots t_k)$ belong to $\mathbb{P}$. Then:

$$t = t_1 \searrow B^+(t_2 \dots t_k) = \mathbf{1} \searrow_{\searrow} (t_1, B^+(t_2 \dots t_k)).$$

So $t \in \mathbb{P}$. □

Theorem 39 *The non- Σ -operad $\text{PRIM}_{\searrow}$ is freely generated by $\mathbf{1}$.*

Proof. Similar as the proof of theorem 29. □

In other terms:

Theorem 40 *The triple of operads $(\text{Ass}, \mathbb{P}_{\searrow}^{\Sigma}, \mathbb{F}_2)$, where $\mathbb{F}_2$ is the free operad generated by an element in $\mathbb{F}_2(2)$, is a good triple of operads.*

Remark. Note that if A is a $\mathbb{P}_{\searrow}$ -bialgebra, then $(A, m, \tilde{\Delta})$ is a non unitary infinitesimal bialgebra. Hence, if $(K \oplus A, m, \Delta)$ has an antipode S , then $-S$ is an eulerian idempotent for A .

5 A rigidity theorem for $\mathbb{P}_{\nearrow}$ -algebras

5.1 Double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebras

Definition 41 A double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra is a family $(A, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$ where $m, \nearrow: A \otimes A \longrightarrow A$, $\tilde{\Delta}, \tilde{\Delta}_{\nearrow}: A \longrightarrow A \otimes A$, with the following compatibilities:

1. $(A, m, \nearrow)$ is a (non unitary) $\mathbb{P}_{\nearrow}$ -algebra.
2. For all $x \in A$:

$$\begin{cases} (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}(x) &= (Id \otimes \tilde{\Delta}) \circ \tilde{\Delta}(x), \\ (\tilde{\Delta}_{\nearrow} \otimes Id) \circ \tilde{\Delta}_{\nearrow}(x) &= (Id \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}_{\nearrow}(x), \\ (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}_{\nearrow}(x) &= (Id \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}(x). \end{cases}$$

In other terms, $(A, \tilde{\Delta}^{cop}, \tilde{\Delta}_{\nearrow}^{cop})$ is a $\mathbb{P}_{\nearrow}$ -coalgebra.

3. $(A, m, \nearrow, \tilde{\Delta})$ is a $\mathbb{P}_{\nearrow}$ -bialgebra of type 1.
4. $(A, m, \nearrow, \tilde{\Delta}_{\nearrow})$ is a $\mathbb{P}_{\nearrow}$ -bialgebra of type 2.

Remark. If $(A, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$ is a graded double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra, with finite-dimensional homogeneous components, then its graded dual $(A^*, \tilde{\Delta}^{*,op}, \tilde{\Delta}_{\nearrow}^{*,op}, m^{*,cop}, \nearrow^{*,cop})$ also is.

Theorem 42 $(\mathcal{M}, m, \nearrow, \tilde{\Delta}, \tilde{\Delta}_{\nearrow})$ is a double $\mathbb{P}$ -infinitesimal bialgebra.

Proof. We already now that $(\mathcal{M}, m, \nearrow)$ is a $\mathbb{P}_{\nearrow}$ -algebra. Moreover, $(\mathcal{M}, \tilde{\Delta}^{cop}, \tilde{\Delta}_{\nearrow}^{cop})$ is isomorphic to $(\mathcal{M}^*, m^*, \nearrow^*)$ via the pairing $\langle -, - \rangle$, so it is a $\mathbb{P}_{\nearrow}$ -coalgebra. It is already known that $(\mathcal{M}, m, \tilde{\Delta})$ and $(\mathcal{M}, \nearrow, \tilde{\Delta})$ are infinitesimal bialgebras. As $(\mathcal{M}, \nearrow, \tilde{\Delta})$ is isomorphic to $(\mathcal{M}^{op}, m^{op}, \tilde{\Delta}_{\nearrow}^{cop})$ via the pairing $\langle -, - \rangle$, it is also an infinitesimal bialgebra. So all the needed compatibilities are satisfied. $\square$

Remarks.

1. Via the pairing $\langle -, - \rangle$, $\mathcal{M}$ is isomorphic to its graded dual as an double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra. As a consequence, as $\mathcal{M}$ is the free $\mathbb{P}_{\nearrow}$ -algebra generated by $\bullet$, then $\mathcal{M}^{cop}$ is also the cofree $\mathbb{P}_{\nearrow}$ -coalgebra cogenerated by $\bullet$.
2. All these results can be easily extended to infinitesimal Hopf algebras of decorated planar rooted trees, in other terms to every free $\mathbb{P}_{\nearrow}$ -algebras.

Lemma 43 In the double infinitesimal $\mathbb{P}_{\nearrow}$ -algebra $\mathcal{M}$, $Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow}) = Vect(\bullet)$.

Proof. $\supseteq$. Obvious.

$\subseteq$. Let $x \in Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow})$. Then $\tilde{\Delta}_{\nearrow}(x) = 0$, so x is a linear span of trees. We can write:

$$x = \sum_{t \in \mathbf{T}} a_t t.$$

Consider the terms in $\mathcal{M} \otimes \bullet$ of $\tilde{\Delta}(x)$. We get $\sum_{t \in \mathbf{T} - \{\bullet\}} a_t B^-(t) \otimes \bullet = 0$, where $B^-(t)$ is the forest obtained by deleting the root of t . So, if $t \neq \bullet$, then $a_t = 0$. So $x \in vect(\bullet)$. $\square$

Remark. This lemma can be extended to any free $\mathbb{P}_{\nearrow}$ -algebra: if V is a vector space, then the free $\mathbb{P}_{\nearrow}$ -algebra $F_{\mathbb{P}_{\nearrow}}(V)$ generated by V is given a structure of double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra by $\tilde{\Delta}(v) = \tilde{\Delta}_{\nearrow}(v) = 0$ for all $v \in V$. In this case, $Ker(\tilde{\Delta}) \cap Ker(\tilde{\Delta}_{\nearrow}) = V$ for $F_{\mathbb{P}_{\nearrow}}(V)$.

5.2 Connected double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebras

Notations. Let A be a double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra. The iterated coproducts will be denoted in the following way: for all $n \in \mathbb{N}$,

$$\begin{aligned}\tilde{\Delta}^n &: \begin{cases} A & \longrightarrow A^{\otimes(n+1)} \\ a & \longrightarrow a^{(1)} \otimes \dots \otimes a^{(n+1)}, \end{cases} \\ \tilde{\Delta}_{\nearrow}^n &: \begin{cases} A & \longrightarrow A^{\otimes(n+1)} \\ a & \longrightarrow a_{\nearrow}^{(1)} \otimes \dots \otimes a_{\nearrow}^{(n+1)}. \end{cases}\end{aligned}$$

Definition 44 Let A be a double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra. It will be said *connected* if, for any $a \in A$, every iterated coproduct $A \longrightarrow A^{\otimes(n+1)}$ vanishes on a for a great enough n .

Theorem 45 Let A be a connected double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra. Then A is isomorphic to the free $\mathbb{P}_{\nearrow}$ -algebra generated by $\text{Prim}(A) = \text{Ker}(\tilde{\Delta}) \cap \text{Ker}(\tilde{\Delta}_{\nearrow})$ as a double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebra.

Proof. *First step.* We shall use the results on infinitesimal Hopf algebras of [4]. We show that $A = \text{Prim}(A) + A.A + A \nearrow A$. As $(A, \nearrow, \tilde{\Delta})$ is a connected non unitary infinitesimal bialgebra, it (or more precisely its unitarisation) has an antipode $S_{\nearrow}$, defined by:

$$S_{\nearrow} : \begin{cases} A & \longrightarrow A \\ a & \longrightarrow \sum_{i=0}^{\infty} (-1)^{i+1} a^{(1)} \nearrow \dots \nearrow a^{(i+1)}. \end{cases}$$

As $(A, \tilde{\Delta})$ is connected, this makes sense. Moreover, $-S_{\nearrow}$ is the projector on $\text{Ker}(\tilde{\Delta})$ in the direct sum $A = \text{Ker}(\tilde{\Delta}) \oplus A \nearrow A$.

In the same order of idea, as $(A, m, \tilde{\Delta}_{\nearrow})$ is a connected infinitesimal bialgebra, we can define its antipode $S^{\nearrow}$ by:

$$S^{\nearrow} : \begin{cases} A & \longrightarrow A \\ a & \longrightarrow \sum_{i=0}^{\infty} (-1)^{i+1} a_{\nearrow}^{(1)} \dots a_{\nearrow}^{(i+1)}, \end{cases}$$

and $-S^{\nearrow}$ is the projector on $\text{Ker}(\tilde{\Delta}_{\nearrow})$ in the direct sum $A = \text{Ker}(\tilde{\Delta}_{\nearrow}) \oplus A.A$.

Let $a \in A$, $b \in \text{Ker}(\tilde{\Delta}_{\nearrow})$. Then $\tilde{\Delta}_{\nearrow}(a \nearrow b) = (a \otimes 1) \tilde{\Delta}_{\nearrow}(b) = 0$. So $A \nearrow \text{Ker}(\tilde{\Delta}_{\nearrow})$ is a subset of $\text{Ker}(\tilde{\Delta}_{\nearrow})$. Moreover, if $\tilde{\Delta}_{\nearrow}(a) = 0$, then $(\text{Id} \otimes \tilde{\Delta}_{\nearrow}) \circ \tilde{\Delta}(a) = (\tilde{\Delta} \otimes \text{Id}) \circ \tilde{\Delta}_{\nearrow}(a) = 0$. So $\tilde{\Delta}(a) \in A \otimes \text{Ker}(\tilde{\Delta}_{\nearrow})$. As a consequence, if $n \geq 1$:

$$\tilde{\Delta}^n(a) = (\tilde{\Delta}^{n-1} \otimes \text{Id}) \circ \tilde{\Delta}(a) \in A^{\otimes n} \otimes \text{Ker}(\tilde{\Delta}_{\nearrow}).$$

Hence, for all $n \in \mathbb{N}$, $\tilde{\Delta}^n(\text{Ker}(\tilde{\Delta}_{\nearrow})) \in A^{\otimes n} \otimes \text{Ker}(\tilde{\Delta}_{\nearrow})$. Finally, we deduce that $S_{\nearrow}(\text{Ker}(\tilde{\Delta}_{\nearrow})) \subseteq \text{Ker}(\tilde{\Delta}_{\nearrow})$.

Let $a \in A$. Then $S^{\nearrow}(a) \in \text{Ker}(\tilde{\Delta}_{\nearrow})$ and $S_{\nearrow} \circ S^{\nearrow}(a) \in \text{Ker}(\tilde{\Delta}) \cap \text{Ker}(\tilde{\Delta}_{\nearrow}) = \text{Prim}(A)$ by the preceding point. Moreover:

$$\begin{aligned}S^{\nearrow}(a) &= -a + A.A, \\ S_{\nearrow} \circ S^{\nearrow}(a) &= -S^{\nearrow}(a) + A \nearrow A, \\ S_{\nearrow} \circ S^{\nearrow}(a) &= a + A.A + A \nearrow A.\end{aligned}$$

This proves the first step.

Second step. As A is connected, it classically inherits a filtration of $\mathbb{P}_{\nearrow}$ -algebra given by the kernels of the iterated coproducts. We denote by $\deg_p$ the associated degree. In particular, for

all $x \in A$, $\deg_p(x) \leq 1$ if, and only if, $x \in \text{Prim}(A)$. Let B be the $\mathbb{P}_{\nearrow}$ -subalgebra of A generated by $\text{Prim}(A)$. Let $a \in A$, let us show that $a \in B$ by induction on $n = \deg_p(a)$. If $n \leq 1$, then $a \in \text{Prim}(A) \subseteq B$. Suppose the result true at all ranks $\leq n-1$. Then, by the first step, we can write:

$$a = b + \sum_i a_i b_i + \sum_j c_j d_j,$$

with $b \in \text{Prim}(A)$, $a_i, b_i, c_j, d_j \in A$. Because of the filtration, we can suppose that $\deg_p(a_i), \deg_p(b_i), \deg_p(c_j), \deg_p(d_j) < n$. By the induction hypothesis, they belong to B , so $a \in B$.

Last step. So, there is an epimorphism of $\mathbb{P}_{\nearrow}$ -algebras:

$$\phi : \begin{cases} F_{\mathbb{P}_{\nearrow}}(\text{Prim}(A)) & \longrightarrow A \\ a \in \text{Prim}(A) & \longrightarrow a, \end{cases}$$

where $F_{\mathbb{P}_{\nearrow}}(\text{Prim}(A))$ is the free $\mathbb{P}_{\nearrow}$ -algebra generated by $\text{Prim}(A)$. As the elements of $\text{Prim}(A)$ are primitive both in A and $F_{\mathbb{P}_{\nearrow}}(\text{Prim}(A))$, this is a morphism of double $\mathbb{P}_{\nearrow}$ -infinitesimal bialgebras. Suppose that it is not monic. Take then $x \in \text{Ker}(\phi)$, non-zero, of minimal degree. Then it is primitive, so belongs to $\text{Prim}(A)$ (lemma 43). Hence, $\phi(a) = a = 0$: this is a contradiction. So ϕ is a bijection. $\square$

In other terms:

Corollary 46 *The triple of operads $\left((\mathbb{P}_{\nearrow}^{\Sigma})^{op}, \mathbb{P}_{\nearrow}^{\Sigma}, \text{VECT}\right)$ is a good triple. Here, VECT denotes the operad of vector spaces:*

$$\text{VECT}(k) = \begin{cases} KI & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases}$$

where I is the unit of VECT .

We also showed that $S_{\nearrow} \circ S_{\nearrow}$ is the projection on $\text{Prim}(A)$ in the direct sum $A = \text{Prim}(A) \oplus (A.A + A \nearrow A)$. So $S_{\nearrow} \circ S_{\nearrow}$ is the eulerian idempotent.

References

- [1] Alain Connes and Dirk Kreimer, *Hopf algebras, Renormalization and Noncommutative geometry*, Comm. Math. Phys. **199** (1998), no. 1, 203–242, arXiv:hep-th/98 08042.
- [2] Loïc Foissy, *Koszularity of the operads of forests*, in preparation.
- [3] ———, *Les algèbres de Hopf des arbres enracinés, I*, Bull. Sci. Math. **126** (2002), 193–239.
- [4] ———, *The infinitesimal Hopf algebra and the poset of planar rooted forests*, J. Algebraic Combin. (2009), arXiv: 08 02.0442.
- [5] Ralf Holtkamp, *Comparison of Hopf Algebras on Trees*, Arch. Math. (Basel) **80** (2003), no. 4, 368–383.
- [6] Dirk Kreimer, *On the Hopf algebra structure of perturbative quantum field theories*, Adv. Theor. Math. Phys. **2** (1998), no. 2, 303–334, arXiv:q-alg/97 07029.
- [7] ———, *On Overlapping Divergences*, Comm. Math. Phys. **204** (1999), no. 3, 669–689, arXiv:hep-th/98 10022.
- [8] ———, *Combinatorics of (perturbative) Quantum Field Theory*, Phys. Rep. **4–6** (2002), 387–424, arXiv:hep-th/00 10059.

- [9] Jean-Louis Loday, *Generalized bialgebras and triples of operads*, arXiv: math/0611885, 2006.
- [10] Jean-Louis Loday and Maria O. Ronco, *On the structure of cofree hopf algebras*, J. Reine Angew. Math. **592** (2006), 123–155, available at <http://www-irma.u-strasbg.fr/~loday/>.
- [11] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, 2002.
- [12] Richard P. Stanley, *Enumerative combinatorics. Vol. 1.*, Cambridge Studies in Advanced Mathematics, no. 49, Cambridge University Press, Cambridge, 1997.
- [13] ———, *Enumerative combinatorics. Vol. 2.*, Cambridge Studies in Advanced Mathematics, no. 62, Cambridge University Press, Cambridge, 1999.