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LOCALIZATION OF INJECTIVE MODULES OVER ARITHMETICAL RINGS

FRANÇOIS COUCHOT

Abstract. It is proved that localizations of injective $R$-modules of finite Goldie dimension are injective if $R$ is an arithmetical ring satisfying the following condition: for every maximal ideal $P$, $R_P$ is either coherent or not semi-coherent. If, in addition, each finitely generated $R$-module has finite Goldie dimension, then localizations of finitely injective $R$-modules are finitely injective too. Moreover, if $R$ is a Prüfer domain of finite character, localizations of injective $R$-modules are injective.

This is a sequel and a complement of [Couchot, 2006]. If $R$ is a noetherian or hereditary ring, it is well known that localizations of injective $R$-modules are injective. By [Couchot, 2006, Corollary 8] this property holds if $R$ is a h-local Prüfer domain. However [Couchot, 2006, Example 1] shows that this result is not generally true. E. C. Dade was probably the first to study localizations of injective modules. By [Dade, 1981, Theorem 25], there exist a ring $R$, a multiplicative subset $S$ and an injective module $G$ such that $S^{-1}G$ is not injective. In this example we can choose $R$ to be a coherent domain.

The aim of this paper is to study localizations of injective modules over arithmetical rings. We deduce from [Couchot, 2006, Theorem 3] the two following results: any localization of an injective $R$-module of finite Goldie dimension is injective if and only if any localization at a maximal ideal of $R$ is either coherent or non-semicoherent (Theorem 5) and each localization of any injective module over a Prüfer domain of finite character is injective (Theorem 10). Moreover, if any localization at a maximal ideal of $R$ is either coherent or non-semicoherent, and if each finitely generated $R$-module has a finite Goldie dimension, then each localization of any finitely injective $R$-module is finitely injective.

In this paper all rings are associative and commutative with unity and all modules are unital. A module is said to be uniserial if its submodules are linearly ordered by inclusion. A ring $R$ is a valuation ring if it is uniserial as $R$-module and $R$ is arithmetical if $R_P$ is a valuation ring for every maximal ideal $P$. An arithmetical domain $R$ is said to be Prüfer. We say that a module $M$ is of Goldie dimension $n$ if and only if its injective hull $E(M)$ is a direct sum of $n$ indecomposable injective modules. We say that a domain $R$ is of finite character if every non-zero element is contained in finitely many maximal ideals.

As in [Ramamurthi and Rangaswamy, 1973], a module $M$ over a ring $R$ is said to be finitely injective if every homomorphism $f : A \to M$ extends to $B$ whenever $A$ is a finitely generated submodule of an arbitrary $R$-module $B$.

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As in (Matlis, 1983) a ring $R$ is said to be semicoherent if $\text{Hom}_R(E, F)$ is a submodule of a flat $R$-module for any pair of injective $R$-modules $E$, $F$. An $R$-module $E$ is FP-injective if $\text{Ext}^1_R(F, E) = 0$ for any finitely presented $R$-module $F$, and $R$ is self FP-injective as $R$-module. We recall that a module $E$ is FP-injective if and only if it is a pure submodule of every overmodule. If each injective $R$-module is flat we say that $R$ is an IF-ring. By (Colby, 1974, Theorem 2), $R$ is an IF-ring if and only if it is coherent and self FP-injective.

We begin by some results on semicoherent rings.

**Proposition 1.** Let $R$ be a self FP-injective ring. Then $R$ is coherent if and only if it is semicoherent.

**Proof.** If $R$ is coherent then, by (Fuchs and Salce, 2001, Theorem XIII.6.4(b)), $\text{Hom}_R(E, F)$ is flat for any pair of injective modules $E$, $F$; so, $R$ is semicoherent. Conversely, let $E$ be the injective hull of $R$. Since $R$ is a pure submodule of $E$, then, for each injective $R$-module $F$, the following sequence is exact:

$$0 \to \text{Hom}_R(F \otimes_R E/R, F) \to \text{Hom}_R(F \otimes_R F) \to 0.$$

By using the natural isomorphisms $\text{Hom}_R(F \otimes_R B, F) \cong \text{Hom}_R(F, \text{Hom}_R(B, F))$ and $F \cong \text{Hom}_R(R, F)$ we get the following exact sequence:

$$0 \to \text{Hom}_R(F, \text{Hom}_R(E/R, F)) \to \text{Hom}_R(F, \text{Hom}_R(E, F)) \to \text{Hom}_R(F, F) \to 0.$$

So, the identity map on $F$ is the image of an element of $\text{Hom}_R(F, \text{Hom}_R(E, F))$. Consequently the following sequence splits:

$$0 \to \text{Hom}_R(E/R, F) \to \text{Hom}_R(E, F) \to F \to 0.$$

It follows that $F$ is a summand of a flat module. So, $R$ is an IF-ring. □

**Corollary 2.** Let $R$ be a ring such that its ring of quotients $Q$ is self FP-injective. Then $R$ is semicoherent if and only if $Q$ is coherent.

**Proof.** If $R$ is semicoherent, then so is $Q$ by (Matlis, 1983, Proposition 1.2). From Proposition 1 we deduce that $Q$ is coherent. Conversely, let $E$ and $F$ be injective $R$-modules. It is easy to check that the multiplication by a regular element of $R$ in $\text{Hom}_R(E, F)$ is injective. So, $\text{Hom}_R(E, F)$ is a submodule of the injective hull of $Q \otimes_R \text{Hom}_R(E, F)$ which is flat over $Q$ and $R$ because $Q$ is an IF-ring. □

From Corollary 2 and (Couchot, 2003, Theorem II.11) we deduce the following:

**Corollary 3.** Let $R$ be a valuation ring. Denote by $Z$ its subset of zerodivisors which is a prime ideal. Assume that $Z \neq 0$. Then the following conditions are equivalent:

1. $R$ is semicoherent;
2. $R_Z$ is an IF-ring;
3. $Z$ is not a flat $R$-module.

From Corollary 3 and (Couchot, 2000, Theorem 3) we deduce the following:

**Corollary 4.** Let $R$ be a valuation ring of maximal ideal $P$. Then the following conditions are equivalent:

1. $R$ is either coherent or non-semicoherent;
(2) for each multiplicative subset $S$ of $R$ and for each injective $R$-module $E$, $S^{-1}E$ is injective;
(3) for each multiplicative subset $S$ of $R$ and for each FP-injective $R$-module $E$, $S^{-1}E$ is FP-injective;
(4) $(E_R(R/P))_Z$ is FP-injective.

Proof. (1) ⇒ (2). Since $R$ is a valuation ring, $R \setminus S$ is a prime ideal. If $R$ is coherent then either $Z = 0$ or $Z = P$. In the first case $Z$ is flat and in the second $E$ is flat. So we conclude by (Couchot, 2006, Theorem 3). If $R$ is not semicoherent then $Z$ is flat. We conclude in the same way.

(2) ⇒ (3). $E$ is a pure submodule of its injective hull $H$. Then $S^{-1}E$ is a pure submodule of $S^{-1}H$ which is injective by (2). So, $S^{-1}E$ is FP-injective.

(3) ⇒ (4) is obvious.

(4) ⇒ (1). Suppose that $Z$ is not flat. If $R$ is not coherent, then, by (Couchot, 2003, Theorem II.11), $Z \neq 0$ and $Z \neq P$, and $R$ is not self FP-injective. Let $E = E_R(R/P)$. By (Couchot, 1982, Proposition 2.4) $E$ is not flat. Now, we do as in the last part of the proof of (Couchot, 2003, Theorem 3) to show that $E_Z$ is not FP-injective. This contradicts (4). The proof is now complete. □

Theorem 5. For any arithmetical ring $R$ the following conditions are equivalent:

1. for each maximal ideal $P$, $R_P$ is either coherent or non-semicoherent;
2. for each multiplicative subset $S$ and for each injective $R$-module $G$ of finite Goldie dimension, $S^{-1}G$ is injective;
3. for each multiplicative subset $S$ and for each FP-injective $R$-module $G$ of finite Goldie dimension, $S^{-1}G$ is FP-injective;
4. for each maximal ideal $P$, $Q(R_P) \otimes_R E_R(R/P)$ is FP-injective, where $Q(R_P)$ is the ring of fractions of $R_P$.

Proof. (1) ⇒ (2). $G$ is a finite direct sum of indecomposable injective modules. We may assume that $G$ is indecomposable. Since $\text{End}_R G$ is local, there exists a maximal ideal $P$ such that $G$ is a module over $R_P$. If $S'$ is the multiplicative subset of $R_P$ generated by $S$, then $S^{-1}G = S'^{-1}G$. We conclude that $S^{-1}G$ is injective by Corollary 3.

We show (2) ⇒ (3) as in the proof of Corollary 3 and (3) ⇒ (4) is obvious.

(4) ⇒ (1) is an immediate consequence of Corollary 3. □

Remark 6. If $R$ is an arithmetical ring which is coherent or reduced, then $R$ satisfies the conditions of Theorem 5.

Corollary 7. Let $R$ be an arithmetical ring satisfying the following two conditions:

(a) for each maximal ideal $P$, $R_P$ is either coherent or non-semicoherent;
(b) every finitely generated $R$-module has a finite Goldie dimension.

Then, for each multiplicative subset $S$ and for each finitely injective (respectively FP-injective) $R$-module $G$, $S^{-1}G$ is finitely injective (respectively FP-injective).

Moreover, if $R_P$ is coherent for each maximal ideal $P$ then $R$ is coherent too.

Proof. Let $M$ be a finitely generated $S^{-1}R$-submodule of $S^{-1}G$. There exists a finitely generated submodule $M'$ of $G$ such that $M = S^{-1}M'$. If $G$ is finitely injective, by (Ramamurthi and Rangaswamy, 1973, Proposition 3.3) it contains an injective hull $E$ of $M'$. Then $E$ has finite Goldie dimension. By Theorem 5, $S^{-1}E$ is injective. It contains $M$ and it is contained in $S^{-1}G$. By using again
(Ramamurthi and Rangaswamy, 1973) Proposition 3.3) we conclude that $S^{-1}G$ is finitely injective.

If $G$ is FP-injective, it is a pure submodule of its injective hull $H$. Then $S^{-1}G$ is a pure submodule of $S^{-1}H$ which is finitely injective. So, $S^{-1}G$ is FP-injective.

The last assertion is an immediate consequence of (Couchot, 1982, Théorème 1.4). □

**Remark 8.** If $R$ is an arithmetical ring satisfying the condition (b) of Corollary 11 then $\text{Min } R/A$ is finite for each ideal $A$: we may assume that $A = 0$ and $R$ is reduced; its total ring of quotient is Von Neumann regular by (Lambek, 1966, Proposition 2 p. 106) and semisimple by (Lambek, 1966, Proposition 2 p. 103); it follows that $\text{Min } R$ is finite. However, the converse doesn’t hold. For instance, let $R = \{ (d, e) \mid d \in \mathbb{Z}, e \in \mathbb{Q}/\mathbb{Z} \}$ be the trivial extension of $\mathbb{Z}$ by $\mathbb{Q}/\mathbb{Z}$. Then $N = \{ (0, e) \mid e \in \mathbb{Q}/\mathbb{Z} \}$ is the only minimal prime. For each prime integer $p$, the localization $R_{(p)}$ is the trivial extension of $\mathbb{Z}_{(p)}$ by $\mathbb{Q}/\mathbb{Z}_{(p)}$. So it is a valuation ring. Consequently $R$ is arithmetical. But $N \cong \mathbb{Q}/\mathbb{Z}$ is an infinite direct sum, whence $R$ is not a module of finite Goldie dimension.

By (Couchot, 2006, Corollary 8), if $R$ is a $h$-local Prüfer domain, all localizations of injective $R$-modules are injective. Now, we extend this result to each Prüfer domain of finite character. A such ring satisfies condition (b) of Corollary 8. But $\mathbb{Z} + X\mathbb{Q}[[X]]$ is an example showing that the converse doesn’t hold.

**Lemma 9.** Let $R$ be a Prüfer domain of finite character. For each maximal ideal $P$, let $F_{(P)}$ be an injective $R_P$-module and let $F = \prod_{P \in \text{Max } R} F_{(P)}$. Then $S^{-1}F$ is injective for every multiplicative subset $S$ of $R$.

**Proof.** Let $T_{(P)}$ be the torsion submodule of $F_{(P)}$, let $G_{(P)} = F_{(P)}/T_{(P)}$, let $T = \prod_{P \in \text{Max } R} T_{(P)}$ and let $G = \prod_{P \in \text{Max } R} G_{(P)}$. Then $G$ is torsion-free and $F \cong T \oplus G$. It is obvious that $S^{-1}G$ is injective. Let $T' = \oplus_{P \in \text{Max } R} rT_{(P)}$. Since $R$ has finite character, it is easy to check that $T'$ is the torsion submodule of $T$. So, $T'$ is injective and $S^{-1}(T/T')$ is injective. For each maximal ideal $P$, $S^{-1}T_{(P)}$ is injective by (Couchot, 2006, Théorème 3). Since $S^{-1}T'$ is the torsion submodule of $\prod_{P \in \text{Max } R} S^{-1}T_{(P)}$, we successively deduce the injectivity of $S^{-1}T'$ and $S^{-1}T$. □

**Theorem 10.** Let $R$ be a Prüfer domain of finite character. Then, for each injective module $G$, $S^{-1}G$ is injective for every multiplicative subset $S$ of $R$.

**Proof.** Let $E = \prod_{P \in \text{Max } R} E_R(R/P)$ and let $F = \text{Hom}_R(\text{Hom}_R(G, E), E)$. Then $E$ is an injective cogenerator and $G$ is isomorphic to a submodule of $F$. Since $R$ is coherent, $\text{Hom}_R(G, E)$ is flat by (Fuchs and Salce, 2001, Théorème XIII.6.4(b)). Thus $F$ is injective. We put $F_{(P)} = \text{Hom}_R(\text{Hom}_R(G, E), E_R(R/P))$. Then $F_{(P)}$ is an injective $R_P$-module and $F \cong \prod_{P \in \text{Max } R} F_{(P)}$. By Lemma 9, $S^{-1}F$ is injective. We conclude that $S^{-1}G$ is injective too. □

**Corollary 11.** Let $R$ be a semilocal Prüfer domain. Then, for each injective module $G$, $S^{-1}G$ is injective for every multiplicative subset $S$ of $R$.

The following example shows that the finite character is not a necessary condition in order that localizations of injective modules at multiplicative subsets are still injective.
Example 12. Let $R$ be the ring defined in (Hutchins, 1981, Example 39). Then $R$ is a Prüfer domain which is not of finite character. But, since $R$ is the union of a countable family of principal ideal subrings, it is easy to check that, for any multiplicative subset $S$, $R$ satisfies (Dade, 1981, Condition 14). So, for each injective module $G$, $S^{-1}G$ is injective by (Dade, 1981, Theorem 15).

Here another example communicated to me by L. Salce. Take $R$ constructed as in Chapter III, Example 5.5 of (Fuchs and Salce, 2001), which is a classical example by Heinzer-Ohm of almost Dedekind domain not of finite character. If you start with a countable field $K$, then $R$ is countable, hence conditions (14a) and (14c) of (Dade, 1981) are satisfied. Condition (14b) must be checked only for $I$ principal ideal, and it is easy to see that it holds true.

Consequently, the following question is unsolved:

**Open question:** characterize the Prüfer domains such that localizations of injective at multiplicative subsets are still injective.

References


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