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Treewidth of planar graphs: connection with duality

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1 Preliminaries

A graph is said to be chordal if each cycle with at least four vertices has a chord, that is an edge between two non-consecutive vertices of the cycle. Given an arbitrary graph $G = (V, E)$, a triangulation of $G$ is a chordal graph $H(= V, F)$ such that $E \subseteq F$. We say that $H$ is a minimal triangulation of $G$ if no proper subgraph of $H$ is a triangulation of $G$. The treewidth $\text{tw}(H)$ of a chordal graph is its maximum cliquesize minus one. The treewidth of an arbitrary graph $G$ is the minimum, over all triangulations $H$ of $G$, of $\text{tw}(H)$. When computing the treewidth of $G$, we can clearly restrict to minimal triangulations. Treewidth was introduced by Robertson and Seymour in connection with graph minors [5], but it has wide algorithmic applications since many NP-hard problems become polynomial when restricted to graphs of bounded treewidth.

Robertson and Seymour conjectures in [5] that the treewidth of a planar graph $G$ and its dual $G^*$ differ by at most one. This conjecture was recently proved by Lapoire [3], who gives a more general result, on hypergraphs of bounded genus. Nevertheless, the proof of Lapoire is rather long and technical. Here, we show that any minimal triangulation $H$ of a planar graph $G$ can be easily transformed into a triangulation $H^*$ of $G^*$, such that $\text{tw}(H^*) \leq \text{tw}(H) + 1$.

The minimal separators play a crucial role in the characterisation of the minimal triangulations of a graph. A subset $S \subseteq V$ separates two non-adjacent vertices $a, b \in V$ is $a$ and $b$ are in different connected components of $G \setminus S$. $S$ is a minimal $a,b$-separator if it separates $a$ and $b$ and no proper subset of $S$ separates $a$ and $b$. We say that $S$ is a minimal separator of $G$ if there are two vertices $a$ and $b$ such that $S$ is a minimal $a,b$-separator. Notice that a minimal separator can be strictly included into another. We denote by $\Delta_G$ the set of all minimal separators of $G$. Two minimal separators $S$ and $T$ cross if $T$ intersects at least two components of $G \setminus S$. Otherwise, $S$ and $T$ are parallel. Both relations are symmetric.

Let $S \in \Delta_G$ be a minimal separator. We denote by $G_S$ the graph obtained from $G$ by completing $S$, i.e. by adding an edge between every pair of non-adjacent vertices of $S$. If $\Gamma \subseteq \Delta_G$ is a set of separators of $G$, $G_\Gamma$ is the graph obtained by completing all the separators of $\Gamma$. The result of [2], concluded in [4], establish a
strong relation between the minimal triangulations of a graph and its minimal separators.

**Theorem 1.** Let \( H \) be a minimal triangulation of \( G \) if and only if there is a maximal set of pairwise parallel separators \( \Gamma \subseteq \Delta_G \) such that \( H = G_\Gamma \).

Since it is easy to extend our results to simply connected or disconnected graphs, we will restrict to 2-connected graphs.

## 2 Minimal separators in planar graphs

Consider a 2-connected planar graph \( G = (V, E) \). We fix an embedding of \( G \) in the plane \( \mathbb{R}^2 \). Let \( F \) be the set of faces of this embedding. Let \( F \) be the set of faces of this embedding. The intermediate graph \( G_I \) has vertex set \( V \cup F \). We place an edge in \( G_I \) between an original vertex \( v \in V \) and a face \( f \in F \) whenever the corresponding vertex and face are incident in \( G \). Notice that \( (G^*)_I = G_I \).

Let \( \nu \) be a cycle of \( G_I \) (by “cycle” we will always mean a cycle which does not get through a same vertex twice). The drawing of \( \nu \) forms a Jordan curve in the plane \( \mathbb{R}^2 \), denoted \( \tilde{\nu} \). It is easy to see that if \( \tilde{\nu} \) separates two original vertices \( x \) and \( y \) in the plane (i.e. \( x \) and \( y \) are in different regions of \( \mathbb{R}^2 \setminus \nu \)), then \( v \cap V \) separates \( x \) and \( y \) in \( G \). Therefore, the original vertices of \( \nu \) form a separator in \( G \). Conversely, to each minimal separator \( S \) of \( G \), we can associate a cycle \( \nu \) of \( G_I \) (see [1]).

**Proposition 1.** Let \( S \) be a minimal separator of the planar graph \( G \). Consider two connected components \( C \) and \( D \) of \( G \setminus S \). There is a cycle \( \nu_S \) of \( G_I \) such that \( \tilde{\nu} \) separates \( C \) and \( D \) in the plane.

This cycle is usually not unique. In the case of 3-connected planar graphs, notice that if \( S \) is a minimal separator, then \( G \setminus S \) has exactly two connected components \( C \) and \( D \). For each couple of original vertices \( x \) and \( y \) incident to a same face, fix a unique face \( f(x, y) \) containing both \( x \) and \( y \). We say that a cycle \( \nu \) of \( G_I \) is well-formed if, for any two consecutive original vertices \( x, y \in \nu \), the face-vertex between them if \( f(x, y) \). If \( G \) is a 3-connected planar graph, for any minimal separator \( S \), there is a unique well-formed cycle of \( G_I \) separating \( C \) and \( D \) in the plane.

**In what follows, \( G \) denotes a 3-connected planar graph.** However, our main results can be easily extended to arbitrary planar graphs.

We say that two Jordan curves \( \tilde{\nu}_1 \) and \( \tilde{\nu}_2 \) cross if \( \tilde{\nu}_1 \) intersects the two regions defined by \( \tilde{\nu}_2 \). Otherwise, they are parallel. Two cycles \( \nu_1 \) and \( \nu_2 \) of \( G_I \) cross if and only if \( \tilde{\nu}_1 \) and \( \tilde{\nu}_2 \) cross. Notice that the parallel and crossing relations between curves and cycles are symmetric.

**Proposition 2.** Two minimal separators \( S \) and \( T \) of \( G \) are parallel if and only if the corresponding cycles \( \nu_S \) and \( \nu_T \) of \( G_I \) are parallel.
Let \( \tilde{v} \) be a Jordan curve in the plane. Let \( R \) be one of the regions of \( \mathbb{R}^2 \setminus \tilde{v} \). We say that \((\tilde{v}, R) = \tilde{v} \cup R\) is a one-block region of the plane, bordered by \( \tilde{v} \). Let \( \mathcal{C} \) be a set of curves such that for each \( \tilde{v} \in \mathcal{C} \), there is a one-block region \((\tilde{v}, R(\tilde{v}))\) containing all the curves of \( \mathcal{C} \). We define the region between the elements of \( \mathcal{C} \) as \( RB(\mathcal{C}) = \bigcap_{\tilde{v} \in \mathcal{C}}(\tilde{v}, R(\tilde{v})) \). A subset \( BR \subseteq \mathbb{R}^2 \) of the plane is a block region if \( BR \) is a one-block region \((\tilde{v}, R)\) or \( BR \) is the region between some set of curves \( \mathcal{C} \).

\[\text{3 Minimal triangulations of } G \text{ and } G^*\]

Let \( G \) be a 3-connected planar graph and let \( H \) be a minimal triangulation of \( G \). According to Theorem 1, there is a maximal set of pairwise parallel separators \( I \subseteq \Delta_G \) such that \( H = G_I \). Let \( \mathcal{C}(I) = \{v_S \mid S \in I\} \) be the cycles of \( G_I \) associated to the minimal separators of \( I \) and let \( \hat{\mathcal{C}}(I) = \{v_S \mid S \in I\} \) be the curves associated to these cycles. According to Proposition 2, the cycles of \( \mathcal{C}(I) \) are pairwise parallel. Thus, the curves of \( \hat{\mathcal{C}}(I) \) split the plane into block regions. Consider the set of all the block regions bordered by some elements of \( \mathcal{C} \). We show that any maximal clique \( \Omega \) of \( H \) corresponds to the original vertices contained in a minimal block regions defined by \( \hat{\mathcal{C}}(I) \).

\[\text{Theorem 2. Let } G \text{ be a 3-connected planar graph and let } H = G_I \text{ be a minimal triangulation of } G. \text{ \( \Omega \subseteq V \) is a maximal clique of } H \text{ if and only if there is a minimal block region } BR \text{ defined by } \hat{\mathcal{C}}(I) \text{ such that } \Omega = BR \cap V.\]

Let now \( \mathcal{C} \) be an arbitrary set of pairwise parallel cycles of \( G_I \). This family \( \hat{\mathcal{C}} \) of curves associated to these cycles splits the plane into block regions. Let \( G^* \) be the dual of \( G \). The graph \( H^*(\mathcal{C}) = (F, E_H) \) has vertex set \( F \). We place an edge between two face-vertices \( f \) and \( f' \) of \( H \) if and only if \( f \) and \( f' \) are in a same minimal block region defined by \( \hat{\mathcal{C}} \). Equivalently, \( f \) and \( f' \) are non-adjacent in \( H^*(\mathcal{C}) \) if and only if there is a \( \tilde{v} \in \hat{\mathcal{C}} \) separating \( f \) and \( f' \) in the plane.

\[\text{Theorem 3. } H^*(\mathcal{C}) \text{ is a triangulation of } G^*. \text{ Moreover, any clique } \Omega^* \text{ of } H^* \text{ is contained in some minimal block region } BR \text{ defined by } \hat{\mathcal{C}}.\]

Let \( H = G_I \) be a minimal triangulation of \( G \). Consider the cycles \( \mathcal{C}(I) \) associated to the minimal separators of \( I \) and the corresponding curves \( \hat{\mathcal{C}}(I) \). We could try to consider the triangulation \( H^*(\mathcal{C}(I)) \) of \( G^* \), but unfortunately it does not satisfy \( tw(H^*) \leq tw(H) + 1 \).

Thus, we consider a maximal set of pairwise parallel cycles \( \mathcal{C} \) of \( G_I \) such that \( \mathcal{C}(I) \subseteq \mathcal{C} \). Clearly, each minimal block region defined by \( \mathcal{C} \) is contained in a minimal block region defined by \( \hat{\mathcal{C}}(I) \).

\[\text{Theorem 4. Let } \mathcal{C} \text{ be a maximal set of pairwise parallel cycles of } G_I. \text{ Let } BR \text{ be a minimal block region of } \mathcal{C}. \text{ Then } BR \cap G_I \text{ is either formed by a cycle } \tilde{v} \text{ and a path } \tilde{\mu} \text{ joining two vertices of } \tilde{v} \text{ or } BR \text{ is a one-block region } (\tilde{v}, R) \text{ and } BR \cap G_I = \nu \text{ where } \nu \text{ is the cycle of } G_I \text{ associated to } \tilde{v}. \text{ In particular, } |BR \cap V^*| \leq |BR \cap V| + 1.\]
According to theorem 3, each maximal clique $\Omega^*$ of $H^*$ is contained in some minimal block region $BR$, and by the previous theorem it has at most one more vertex than $\Omega = BR \cap V$. By theorem 2, $\Omega$ is a clique of $H$. Hence, $|\Omega^*| \leq |\Omega| + 1$ and thus $\tw(H^*) \leq \tw(H) + 1$. By considering an optimal triangulation $H$ of $G$, we obtain a triangulation $H^*$ of $G^*$ of width at most $\tw(G) + 1$. We conclude that $\tw(G^*) \leq \tw(G) + 1$.

So we can state:

**Theorem 5 (Main theorem).** Let $G = (V, E)$ be a planar graph.

$$|\tw(G) - \tw(G^*)| \leq 1.$$  

**References**