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Treewidth of planar graphs: connection with duality

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1 Preliminaries

A graph is said to be chordal if each cycle with at least four vertices has a chord, that is an edge between two non-consecutive vertices of the cycle. Given an arbitrary graph $G = (V,E)$, a triangulation of $G$ is a chordal graph $H(= V,F)$ such that $E \subseteq F$. We say that $H$ is a minimal triangulation of $G$ if no proper subgraph of $H$ is a triangulation of $G$. The treewidth $tw(H)$ of a chordal graph is its maximum cliquesize minus one. The treewidth of an arbitrary graph $G$ is the minimum, over all triangulations $H$ of $G$, of $tw(H)$. When computing the treewidth of $G$, we can clearly restrict to minimal triangulations. Treewidth was introduced by Robertson and Seymour in connection with graph minors [5], but it has wide algorithmic applications since many NP-hard problems become polynomial when restricted to graphs of bounded treewidth.

Robertson and Seymour conjectures in [5] that the treewidth of a planar graph $G$ and its dual $G^*$ differ by at most one. This conjecture was recently proved by Lapoire [3], who gives a more general result, on hypergraphs of bounded genus. Nevertheless, the proof of Lapoire is rather long and technical. Here, we show that any minimal triangulation $H$ of a planar graph $G$ can be easily transformed into a triangulation $H^*$ of $G^*$ such that $tw(H^*) \leq tw(H) + 1$.

The minimal separators play a crucial role in the characterisation of the minimal triangulations of a graph. A subset $S \subseteq V$ separates two non-adjacent vertices $a, b \in V$ if $a$ and $b$ are in different connected components of $G \setminus S$. $S$ is a minimal $a,b$-separator if it separates $a$ and $b$ and no proper subset of $S$ separates $a$ and $b$. We say that $S$ is a minimal separator of $G$ if there are two vertices $a$ and $b$ such that $S$ is a minimal $a,b$-separator. Notice that a minimal separator can be strictly included into another. We denote by $\Delta_G$ the set of all minimal separators of $G$. Two minimal separators $S$ and $T$ cross if $T$ intersects at least two components of $G \setminus S$. Otherwise, $S$ and $T$ are parallel. Both relations are symmetric.

Let $S \in \Delta_G$ be a minimal separator. We denote by $G_S$ the graph obtained from $G$ by completing $S$, i.e. by adding an edge between every pair of non-adjacent vertices of $S$. If $\Gamma \subseteq \Delta_G$ is a set of separators of $G$, $G_\Gamma$ is the graph obtained by completing all the separators of $\Gamma$. The result of [2], concluded in [4], establish a
strong relation between the minimal triangulations of a graph and its minimal separators.

**Theorem 1.** $H$ is a minimal triangulation of $G$ if and only if there is a maximal set of pairwise parallel separators $\Gamma \subseteq \Delta_G$ such that $H = G_\Gamma$.

Since it is easy to extend our results to simply connected or disconnected graphs, we will restrict to 2-connected graphs.

## 2 Minimal separators in planar graphs

Consider a 2-connected planar graph $G = (V, E)$. We fix an embedding of $G$ in the plane $\mathbb{R}^2$. Let $F$ be the set of faces of this embedding. Let $F$ be the set of faces of this embedding. The intermediate graph $G_I$ has vertex set $V \cup F$. We place an edge in $G_I$ between an original vertex $v \in V$ and a face $f \in F$ whenever the corresponding vertex and face are incident in $G$. Notice that $(G^*_I)_I = G_I$.

Let $\nu$ be a cycle of $G_I$ (by “cycle” we will always mean a cycle which does not get through a same vertex twice). The drawing of $\nu$ forms a Jordan curve in the plane $\mathbb{R}^2$, denoted $\tilde{\nu}$. It is easy to see that if $\tilde{\nu}$ separates two original vertices $x$ and $y$ in the plane (i.e. $x$ and $y$ are in different regions of $\mathbb{R}^2 \setminus \nu$), then $\nu \cap V$ separates $x$ and $y$ in $G$. Therefore, the original vertices of $\nu$ form a separator in $G$. Conversely, to each minimal separator $S$ of $G$, we can associate a cycle $\nu$ of $G_I$ (see [1]).

**Proposition 1.** Let $S$ be a minimal separator of the planar graph $G$. Consider two connected components $C$ and $D$ of $G \setminus S$. There is a cycle $\nu_S$ of $G_I$ such that $\tilde{\nu}$ separates $C$ and $D$ in the plane.

This cycle is usually not unique. In the case of 3-connected planar graphs, notice that if $S$ is a minimal separator, then $G \setminus S$ has exactly two connected components $C$ and $D$. For each couple of original vertices $x$ and $y$ incident to a same face, fix a unique face $f(x, y)$ containing both $x$ and $y$. We say that a cycle $\nu$ of $G_I$ is well-formed if, for any two consecutive original vertices $x, y \in \nu$, the face-vertex between them if $f(x, y)$. If $G$ is a 3-connected planar graph, for any minimal separator $S$, there is a unique well-formed cycle of $G_I$ separating $C$ and $D$ in the plane.

In what follows, $G$ denotes a 3-connected planar graph. However, our main results can be easily extended to arbitrary planar graphs.

We say that two Jordan curves $\tilde{\nu}_1$ and $\tilde{\nu}_2$ cross if $\tilde{\nu}_1$ intersects the two regions defined by $\tilde{\nu}_2$. Otherwise, they are parallel. Two cycles $\nu_1$ and $\nu_2$ of $G_I$ cross if and only if $\tilde{\nu}_1$ and $\tilde{\nu}_2$ cross. Notice that the parallel and crossing relations between curves and cycles are symmetric.

**Proposition 2.** Two minimal separators $S$ and $T$ of $G$ are parallel if and only if the corresponding cycles $\nu_S$ and $\nu_T$ of $G_I$ are parallel.
Let $\tilde{\nu}$ be a Jordan curve in the plane. Let $R$ be one of the regions of $\mathbb{R}^2 \setminus \tilde{\nu}$. We say that $(\tilde{\nu}, R) = \tilde{\nu} \cup R$ is a one-block region of the plane, bordered by $\tilde{\nu}$. Let $\tilde{C}$ be a set of curves such that for each $\tilde{\nu} \in \tilde{C}$, there is a one-block region $(\tilde{\nu}, R(\tilde{\nu}))$ containing all the curves of $\tilde{C}$. We define the region between the elements of $\tilde{C}$ as $RB(\tilde{C}) = \bigcap_{\nu \in \tilde{C}}(\tilde{\nu}, R(\tilde{\nu}))$. A subset $Br \subseteq \mathbb{R}^2$ of the plane is a block region if $BR$ is a one-block region $(\tilde{\nu}, R)$ or $BR$ is the region between some set of curves $\tilde{C}$.

3 Minimal triangulations of $G$ and $G^*$

Let $G$ be a 3-connected planar graph and let $H$ be a minimal triangulation of $G$. According to Theorem 1, there is a maximal set of pairwise parallel separators $I \subseteq \Delta_G$ such that $H = G_I$. Let $\hat{C}(I) = \{S \mid S \subseteq I\}$ be the cycles of $G_I$ associated to the minimal separators of $I$ and let $\check{C}(I) = \{\hat{\nu}_S \mid S \subseteq I\}$ be the curves associated to these cycles. According to Proposition 2, the cycles of $\check{C}(I)$ are pairwise parallel. Thus, the curves of $\check{C}(I)$ split the plane into block regions. Consider the set of all the block regions bordered by some elements of $\hat{C}$. We show that any maximal clique $\Omega$ of $H$ corresponds to the original vertices contained in a minimal block regions defined by $\check{C}(I)$.

**Theorem 2.** Let $G$ be a 3-connected planar graph and let $H = G_I$ be a minimal triangulation of $G$. $\Omega \subseteq V$ is a maximal clique of $H$ if and only if there is a minimal block region $BR$ defined by $\check{C}(I)$, such that $\Omega = BR \cap V$.

Let now $\tilde{C}$ be an arbitrary set of pairwise parallel cycles of $G_I$. This family $\tilde{C}$ of curves associated to these cycles splits the plane into block regions. Let $G^*$ be the dual of $G$. The graph $H^*(\tilde{C}) = (F, E_H)$ has vertex set $F$. We place an edge between two face-vertices $f$ and $f'$ of $H$ if and only if $f$ and $f'$ are in a same minimal block region defined by $\check{C}$. Equivalently, $f$ and $f'$ are non-adjacent in $H^*(\tilde{C})$ if and only if there is a $\tilde{\nu} \in \tilde{C}$ separating $f$ and $f'$ in the plane.

**Theorem 3.** $H^*(\tilde{C})$ is a triangulation of $G^*$. Moreover, any clique $\Omega^*$ of $H^*$ is contained in some minimal block region $BR$ defined by $\check{C}$.

Let $H = G_I$ be a minimal triangulation of $G$. Consider the cycles $G(I)$ associated to the minimal separators of $I$ and the corresponding curves $\check{C}(I)$. We could try to considerate the triangulation $H^*(\check{C}(I))$ of $G^*$, but unfortunately it does not satisfy $tw(H^*) \leq tw(H) + 1$.

Thus, we consider a maximal set of pairwise parallel cycles $C'$ of $G_I$ such that $\hat{C}(I) \subseteq C'$. Clearly, each minimal block region defined by $C'$ is contained in a minimal block region defined by $\check{C}(I)$.

**Theorem 4.** Let $C'$ be a maximal set of pairwise parallel cycles of $G_I$. Let $BR$ be a minimal block region of $C'$. Then $Br \cap G_I$ is either formed by a cycle $\tilde{\nu}$ and a path $\tilde{\mu}$ joining two vertices of $\tilde{\nu}$ or $BR$ is a one-block region $(\tilde{\nu}, R)$ and $BR \cap G_I = \nu$ where $\nu$ is the cycle of $G_I$ associated to $\tilde{\nu}$. In particular, $|BR \cap V^*| \leq |BR \cap V| + 1$. 
According to theorem 3, each maximal clique $\Omega^*$ of $H^*$ is contained in some minimal block region $BR$, and by the previous theorem it has at most one more vertex than $\Omega = BR \cap V$. By theorem 2, $\Omega$ is a clique of $H$. Hence, $|\Omega^*| \leq |\Omega| + 1$ and thus $tw(H^*) \leq tw(H) + 1$. By considering an optimal triangulation $H$ of $G$, we obtain a triangulation $H^*$ of $G^*$ of width at most $tw(G) + 1$. We conclude that $tw(G^*) \leq tw(G) + 1$.

So we can state:

**Theorem 5 (Main theorem).** Let $G = (V,E)$ be a planar graph.

$$|tw(G) - tw(G^*)| \leq 1.$$ 

**References**