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Listing all the minimal separators of a 3-connected planar graph

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Abstract

We present an efficient algorithm that lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Key words: minimal separator; planar graphs; enumeration

1 Introduction

In the last ten years, minimal separators have been increasingly studied in graph theory leading to many algorithmic applications \[5,9,10,12\].

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In \[5\], Bodlaender \textit{et al.} conjecture that for a class of graphs with a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Tödina introduced the concept of potential maximal clique \[2\] and showed that, if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can be solved in polynomial time. They later showed \[3\] that if a graph has a polynomial number of minimal separators, it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Extensive research has been performed to compute the set of the minimal separators of a graph \[1,6,7,11\]. Berry \textit{et al.} \[1\] proposed an algorithm of running time $O(nm)$ per separator\(^1\) that uses the concept of generating new minimal

\(^1\) The authors only proved a running time of $O(n^3)$ but the actual bound is $O(nm)$ \[8\].
separators from a previous minimal separator $S$ by finding the minimal separators contained in $S \cup N(x)$ for $x \in S$. This simple process can generate all the minimal separators of a graph. However, by using this algorithm a minimal separator can be generated many times.

The aim of this article is to address the problem of finding the minimal separators of a 3-connected planar graph $G$. In order to avoid the problem of recalculation, we define the set $S_a(S, O)$ of the $a, b$-minimal separators $S'$ for some $b$ such that the connected component of $a$ in $G \setminus S'$ contains the connected component of $a$ in $G \setminus S$ but avoids the set $O$. Therefore, it is possible to ensure that a given minimal separator is never computed more than five times.

2 Definitions

Throughout this paper, $G = (V, E)$ is a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x, y) \in E\}$. When the sets $A$ and $B$ are disjoint, their union is denoted by $A \sqcup B$.

A set $S \subseteq V$ is a separator if $G \setminus S$ has at least two connected components, an $a, b$-separator if $a$ and $b$ are in different connected components of $G \setminus S$, an $a, b$-minimal separator if no proper subset of $S$ is an $a, b$-separator. The connected component of $a$ in $G \setminus S$ is $C_a(S)$. The component $C_a(S)$ is a full connected component if $N(C_a(S)) = S$. For an $a, b$-minimal separator $S$, both $C_a(S)$ and $C_b(S)$ are full. A set $S$ is a minimal separator if there exist $a$ and $b$ such that $S$ is an $a, b$-minimal separator or, which is equivalent, if it has at least two full connected components. An $a, \ast$-minimal separator of a graph $G = (V, E)$ is an $a, b$-minimal separator of $G$ for some $b \in V$. The set of the $a, \ast$-minimal separators is denoted by $S_a$ and the set of the minimal separators of $G$ is denoted by $S(G)$.

It is possible to order the $a, \ast$-minimal separators in the following way:

$$S_1 \preceq_a S_2 \text{ if } C_a(S_1) \subseteq C_a(S_2).$$

The minimal separator $S_1$ is closer to $a$ than $S_2$. The set of $a, b$-minimal separators is a lattice for the relation $\preceq_a$[4] but we only need the following weaker lemma:

**Lemma 1** Let $C$ be a set of vertices of a graph $G$ inducing a connected subgraph of $G$, $a$ be a vertex of $C$ and $b$ be a vertex of $G \setminus (C \cup N(C))$. 


The neighbour \( S \) of \( C_b\left( C \cup N(C) \right) \) is an \( a, b \)-minimal separator such that \( C \) is a subset of \( C_a(S) \) that is closer to \( a \) than any \( a, b \)-minimal separator \( S' \) such that \( C \) is a subset of \( C_a(S') \).

**Proof.** By construction, \( C \) is a subset of \( C_a(S) \). By definition, the component \( C_b(S) \) is full and since \( S \) is a subset of \( N(C) \), the component \( C_a(S) \) is also a full component which implies that \( S \) is an \( a, b \)-minimal separator.

Let \( S' \) be an \( a, b \)-minimal separator such that \( C \) is a subset of \( C_a(S') \). Let \( p \) be a path in \( C_b(S') \) with \( b \) as one of its ends. The vertices of \( S' \) are at least at distance 1 of \( C \) so the vertices of \( p \) are at least at distance 2 of \( C \). Since \( S \) is a subset of \( N(C) \), \( p \cap S = \emptyset \). In other words \( p \) is a subset of \( C_b(S) \) and \( C_b(S') \subseteq C_b(S) \). This last inclusion implies that \( C_a(S) \subseteq C_a(S') \) i.e. \( S \) is closer to \( a \) than \( S' \). □

For \( S \) an \( a, * \)-minimal separator and \( O \subseteq V \), the set \( S_a(S, O) \) is the set of the \( a, * \)-minimal separators \( S' \) further from \( a \) than \( S \) and such that \( O \cap C_a(S') = \emptyset \). If \( x \in V \), the set \( S_a^x(S, O) \) is the set of \( S' \in S_a(S, O) \) such that \( x \in C_a(S') \).

**Remark 2** If \( x \in S \), then \( S_a(S, O) \) is the disjoint union
\[
S_a(S, O) \cup \bigcup_{i \in I} S_a(S_i, O).
\]

More precisely, if \( (S_i)_{i \in I} \) are the elements of \( S_a^x(S, O) \) closest to \( a \), then
\[
S_a(S, O) = S_a\left(S, O \cup \{x\}\right) \bigcup \left( \bigcup_{i \in I} S_a(S_i, O) \right).
\]

This gives the skeleton of an algorithm to compute the set \( S_a(S, O) \).

**Remark 3** If \( S \) belongs to \( S_a^x(S, O) \), then \( S_a^x(S, O) = S_a(S, O) \).

The algorithm is based on remarks 2 and 3. To list \( S_a \), the algorithm computes the sets \( S_a(S, O) \) for every \( S \) closest to \( a \) in \( S_a \). During this calculation, it computes \( S_a(S, O) \) with \( O \subseteq S \). To do so, it chooses \( x \) in \( S \setminus O \) and calculates \( S_a^x(S, O) \) and \( S_a(S, O \cup \{x\}) \). The set \( S_a^x(S, O) \) is itself a union of \( S_a(S_i, O) \). But to obtain such a decomposition, one needs to find the elements of \( S_a^x(S, O) \) closest to \( a \), which the following proposition does.

**Proposition 4** Let \( G = (V, E) \) be a graph, \( S \) an \( a, * \)-minimal separator, \( O \subseteq S \), \( x \in S \setminus O \) and \( C = C_a(S) \cup \{x\} \)

The elements of \( S_a^x(S, O) \) closest to \( a \) are exactly the neighbourhoods of the connected components of \( G \setminus \{N(C) \cup C\} \) that contain \( O \) and that are maximal for inclusion.
PROOF. Let $S_1$ be an $a, b$-minimal separator of $S^x_a(S, O)$ closest to $a$. Let $S'$ be the neighbourhood of $C_b \left( N(C) \cup C \right)$. By lemma 1, $S'$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S')$ and $S'$ is closer to $a$ that $S_1$. Moreover, since $C_a(S_1) \cap O = \emptyset$ and $S'$ is closer to $a$ than $S_1$, $C_a(S') \cap O \subseteq C_a(S_1) \cap O = \emptyset$. Thus $S'$ belongs to $S^x_a(S, O)$ and is closer to $a$ than $S_1$. This proves that $S_1 = S'$. Since $S_1$ cannot be a subset of another element of $S^x_a(S, O)$, $S_1$ is the neighbourhood of a connected component of $G \setminus \left\{ N(C) \cup C \right\}$ which is maximal for inclusion.

Conversely, let $S_1$ be a neighbourhood of a connected component $D$ of $G \setminus \left\{ N(C) \cup C \right\}$ that contains $O$ and that is maximal for inclusion. By lemma 1, $S_1$ is an element of $S^x_a(S, O)$ that is closer to $a$ than any $a, b$-minimal separator of $S^x_a(S, O)$ with $b$ in $D$. So if $S_2$ is an $a, b$-minimal separator of $S^x_a(S, O)$ strictly closer to $a$ than $S_1$, $S_1$ is not an $a, b$-minimal separator. Suppose for a contradiction that such an $a, b$-minimal separator exists. It follows from the first part of the proof that such an $a, b$-minimal separator $S_2$ closest to $a$ is the neighbourhood of $C_b \left( N(C) \cup C \right)$. The set $S_2$ is an element of $S^x_a(S, O)$ that is closer to $a$ than $S_1$ and $S_1$ is a subset of $S_2$ (because $S_1 \setminus S_2 \subseteq C_a(S_2) \setminus C_a(S_1)$) and $S_2$ is closer to $a$ then $S_1$ and therefore $S_1$ is a strict subset of $S_2$ contradicting the fact that $S_1$ is maximal for inclusion.

Proposition 4 gives us a way to find the minimal elements of $S^x_a(S, O)$, for example by using a graph search to compute the neighbourhoods of the connected components of $G \setminus \left\{ N(C) \cup C \right\}$ and then choosing among the minimal separators found the ones that contain $O$ and that are maximal by inclusion. Using the skeleton of remark 2, we can construct an algorithm to compute the set $S_a(S, O)$ that may look like:

Algorithm 1 _calc3_

begin
  if $S \setminus O = \emptyset$ then
    return ($\{S\}$)
  else
    let $x \in S \setminus O$
    $S \leftarrow _\text{calc3_}(G, a, S, O \cup \{x\})$

    for each $S_i$ in find_closest_elements($G, a, x, S, O$)
      $S \leftarrow S \cup _\text{calc3_}(G, a, S_i, O)$
    return ($S$)
end

However several problems need to be solved.
i. We do not know whether the sets $S_a(S, O)$ are disjoint or not. If not, a minimal separator could be computed many times, which would lead to a bad complexity.

ii. To implement the function find closest elements, proposition 4 states that we can start with a graph search of $G$.

But if $S_a(S, O) = \{S\}$, the recursive calls to the algorithm will try to find an element of $S_a^x(S, O)$ closest to $a$ for every $x \in S \setminus O$. Each call to find min elements costs at least $O(m)$ and finally, we would have spent at least $O(nm)$ to realise that $S_a(S, O) = \{S\}$.

Proposition 6 in section 3.1 ensures that for 3-connected planar graphs, problem (i) is true, i.e. if $S_1$ and $S_2$ are two minimal elements of $S_a^x(S, O)$, the sets $S_a(S_1, O)$ and $S_a(S_2, O)$ are disjoint. Section 3.3 then shows how to determine whether $S_a^x(S, O)$ is empty or not in an overall $O(n)$.

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let $\Sigma$ be the plane. A plane graph $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of $\Sigma$ between two vertices of $V_\Sigma$ in such a way that the interiors of two distinct edges do not meet. We will denote by $\tilde{G}_\Sigma$ the drawing of $G_\Sigma$. A planar graph is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A face of $G_\Sigma$ is a connected component of $\Sigma \setminus \tilde{G}_\Sigma$.

3.1 Minimal separators of 3-connected planar graphs

Proposition 5 In a 3-connected planar graph, minimal separators are minimal for inclusion.

PROOF. Suppose that $S \subset S'$ are two minimal separators of a 3-connected planar graph.

Let $a, b, c$ and $d$ be vertices such that $S'$ is an $a, b$-minimal separator and $S$ is a $c,d$-minimal separator. Since $S$ is not an $a, b$-minimal separator, either $C_c(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_c(S')$ is such a component. In this case, $C_c(S)$ and $N(C_c(S))$ are respectively equal to $C_c(S')$ and $S$.
But then $G$ admits $K_{3,3}$ as a minor if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices $a'$, $b'$ and $c'$, all these vertices have $S$ in their neighbourhood and since $G$ is 3-connected, $|S| \geq 3$. This contradicts the fact that $G$ is planar. \qed

**Proposition 6** Let $G = (V, E)$ be a 3-connected planar graph, $a$ a vertex of $G$, $S$ an $a, \ast$-minimal separator, $O$ a subset of $S$ and $x$ a vertex of $S \setminus O$.

If $S_1$ and $S_2$ are two distinct elements of $\mathcal{S}_a^x(S, O)$ that are closest to $a$, then

$$\mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O) = \emptyset.$$ 

**PROOF.** Let $C = C_a(S) \cup \{x\}$ and suppose for a contradiction that $S_3$ is a minimal separator of $\mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O)$ with $S_1$ and $S_2$ two distinct elements of $\mathcal{S}_a^x(S, O)$ closest to $a$. Let $b$ be a vertex such that $S_3$ is an $a, b$-minimal separator.

Since $S_3$ is further from $a$ than $S_1$ and $S_2$, both $S_1$ and $S_2$ are $a, b$-separators. There exists an $a, b$-minimal separator $S'$ included in $S_1$. By proposition 5, a minimal separator of $G$ is minimal for inclusion which proves that $S_1 = S'$ and $S_1$ is an $a, b$-minimal separator. By lemma 1, the neighbourhood $S_4$ of $C_b(N(C) \cup C)$ is an $a, b$-minimal separator such that $C$ is a subset of $C_a(S_4)$ that is closer to $a$ than $S_1$. So $C_a(S_4) \cap O \subseteq C_a(S_1) \cap O = \emptyset$, and $S_4$ is an element of $\mathcal{S}_a^x(S, O)$ that is closer to $a$ than $S_1$. Similarly, $S_2$ is an $a, b$-minimal separator and $S_4$ is closer to $a$ than $S_2$ which contradicts the fact that $S_1$ and $S_2$ are two distinct elements of $\mathcal{S}_a^x(S, O)$ closest to $a$. \qed

### 3.2 The intermediate graph

**Definition 7** Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a 3-connected plane graph. Let $F$ be the set of its faces. In each face $f \in F$ pick up one point $v_f$. Let $R_F$ be the set $\{v_f \mid f \in F\}$. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_\Sigma \cup R_F$. We place an edge between a vertex $v \in V$ and $v_f \in R_F$ if and only if the vertex $v$ is incident to the face $f$.

For $G'$ a subgraph of $G_I$, the set $G' \cap V_\Sigma$ will be denoted by $V(G')$.

**Proposition 8** Let $\mu$ be a cycle of $G_I$ such that the curve $\tilde{\mu}$ separates at least two vertices $a$ and $b$ of $V_\Sigma$.

The set $V(\mu)$ is an $a, b$-separator of $G_\Sigma$.

**PROOF.** Let $p$ be a path in $G_\Sigma$ from $a$ to $b$. Since $a$ and $b$ are not in the
same connected component of $\Sigma \setminus \tilde{\mu}$, $\tilde{p}$ intersect $\tilde{\mu}$. By construction, $p \cap \mu \subseteq V_\Sigma$. This implies that every path from $a$ to $b$ meets $V(\mu)$ and so $V(\mu)$ is an $a, b$-separator. □

**Proposition 9** Let $S$ be an $a, b$-minimal separator of $G$. There exists a simple cycle $\mu$ of $G_I$ such that the Jordan curve defined by $\mu$ separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.

**PROOF.** Let $C$ be the connected component of $a$ in $G \setminus S$. Let us contract $C$ into a supervertex $v_C$ to build the graph $G/C$. There is a cycle $\mu/C$ in $(G/C)_I$ such that $V(\mu/C)$ is the neighbourhood of $v_C$ in $G/C$. Therefore, the neighbourhood of $C$ in $G_I$ has the structure of a cycle $\mu$.

Suppose $\tilde{\mu}$ is not a Jordan curve, the border $\mu'$ of the connected component of $b$ in $\Sigma \setminus \tilde{\mu}$ is a strict sub-lace of $\tilde{\mu}$ which separates $a$ and $b$. However, proposition 8 shows that $V(\mu')$ which is a strict subset of $S$ is an $a, b$-separator. This contradicts the fact that $S$ is a $a, b$-minimal separator. □

Proposition 9 shows that the minimal separators of a 3-connected planar graph correspond to cycles of the intermediate graph. Thus, when a set corresponds to no cycle of the $G_I$, it is not a minimal separator. However, this is not a characterisation of the minimal separators of a 3-connected planar graph for some cycles of $G_I$ correspond to no minimal separator of $G$.

There are several ways to find an exact criterion for minimal separators. The following section presents a criterion that is well suited to our purpose.

### 3.3 Ordered separators

**Definition 10** An ordered separator of $G$ is a sequence of distinct vertices

$$(v_0, \ldots, v_{p-1}), \ (p > 2)$$

such that

i. there exists a face to which $v_i$ and $v_{i+1}\left[ p \right]$ are incident;

ii. $v_i$ and $v_j$ are incident to a common face only if $i = j+1\left[ p \right]$ or $j = i+1\left[ p \right]$;

iii. if $p = 3$, no face is incident to $v_i$, $v_{i+1}\left[ p \right]$ and $v_{i+2}\left[ p \right]$.

The notation $i\left[ p \right]$ means $i$ modulo $p$.

A set $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator if there exists a permutation $\sigma$ such that $(v_{\sigma(0)}, \ldots, v_{\sigma(p-1)})$ is an ordered separator.
If \( S = (v_0, \ldots, v_{p-1}) \) is an ordered separator of \( G \), then \( S \) is naturally associated to the set \( \{v_0, \ldots, v_{p-1}\} \). We will use an ordered separator either as a sequence or as the corresponding set.

**Lemma 11** Every minimal separator \( S \) of \( G \) is ordered.

**PROOF.** Let \( S \) be an \( a, b \)-minimal separator of \( G \).

Proposition 9 states that there exists a simple cycle of \( G_I \)

\[
\mu = (v_0, f_0, \ldots, v_{p-1}, f_{p-1})
\]

such that \( V(\mu) = S \).

Let us prove that \( T = (v_0, \ldots, v_{p-1}) \) is an ordered separator corresponding to \( S \).

i. The construction of \( T \) ensures that \( v_i \) and \( v_{i+1} \) are incident to a common face \((f_i)\).

ii. Suppose that \( v_i \) et \( v_j \) are incident to a common face \( f \) and that \( i + 1 \neq j \) [p] and \( j + 1 \neq i \) [p].

\[
\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_j, f) \quad \text{and} \quad \mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \ldots, v_i, f)
\]

are laces of \( G_I \). Moreover, since either \( \mu_1 \) or \( \mu_2 \) separates \( a \) and \( b \), there exists an \( a, b \)-separator strictly included in \( S \) which is absurd.

iii. Suppose that \( p = 3 \) and that \( v_0, v_1 \) et \( v_2 \) are all incident to a common face \( f \). If we add a vertex \( f \) to \( G \) connected to the vertices \( v_0, v_1 \) and \( v_2 \), the graph remains planar which is absurd because this graph has \( K_{3,3} \) as a minor. Indeed, the connected component of \( a \), the connected component of \( b \) and the vertex \( f \) are all incident to \( v_0, v_1 \) and \( v_2 \) which builds up a \( K_{3,3} \).

The sequence \( T \) is an ordered separator corresponding to \( S \). \( \square \)

Conversely,

**Lemma 12** Every ordered separator of \( G \) is a minimal separator of \( G \).

**PROOF.** Let \( S = (v_0, \ldots, v_{p-1}) \) be an ordered separator of \( G \).

First, \( S \) is a separator. Otherwise, \( G \setminus S \) would be connected or empty. In both cases, all the vertices of \( S \) would be incident to a common face.

Let \( S' \) be a minimal separator included in \( S \). By lemma 11, \( S' \) is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, \( S' = S \). The ordered separator \( S \) is a minimal separator. \( \square \)
From lemmata 11 and 12, we have the following proposition:

**Proposition 13** A set $S \subseteq V$ is a minimal separator of a 3-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of $G$.

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether $S^x_a(S, O)$ is empty or not ($O \subseteq S$ and $x \in S \setminus O$).

**Proposition 14** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered $a, *$-separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of $S$.

If there exists a face that is incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and $v_j$ with $0 < j < i$, then $S^v_{a+1}(S, O)$ is empty.

**PROOF.** Let $b$ be such that $S$ is an $a, b$-minimal separator and suppose that $y \in N(v_{i+1})$ and $v_j$ with $0 < j < i$ are both incident to a face $f$. Since $S$ is an ordered separator, there exists a cycle $(v_0, f_0, \ldots, v_k, f_k)$ of $G$ corresponding to a Jordan curve $\tilde{\mu}$. Let $\Sigma_b$ be the connected component of $\Sigma \setminus \tilde{\mu}$ that contains $b$. Since $y$ and $v_j$ are incident to $f$, there exists a path $(v_{i+1}, g, y, f, v_j)$ that corresponds to a curve $\tilde{\nu}$ that cuts $\Sigma_b$ in two parts $\Sigma_1^b$ and $\Sigma_2^b$ whose borders are $\tilde{\mu}_1$ and $\tilde{\mu}_2$ respectively. Since $0 < j < i$, neither $V(\tilde{\mu}_1)$ nor $V(\tilde{\mu}_2)$ contains $O$.

Suppose that $S'$ is an element of $S^v_{a+1}(S, O)$ closest to $a$. Let $c$ be such that $S'$ is an $a, c$-minimal separator. The vertex $c$ belongs to $\Sigma_b$. We may suppose that $c$ belongs to $\Sigma_1^b$. By proposition 4, $S'$ is the neighbourhood of $C_c(S \cup N(v_{i+1}))$ i.e. $S' = V(\tilde{\mu}_1)$, but $O$ is not a subset of $S'$ which is absurd. □

Conversely,

**Proposition 15** Let $S = (v_0, \ldots, v_{p-1})$ be an ordered $a, *$-separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of $S$.

If there is no face incident to both $y \in N(v_{i+1}) \setminus C_a(S)$ and $v_j$ $(0 < j < i)$, then there is an ordered separator in $S \cup N(v_{i+1}) \setminus C_a(S)$ that contains $O$.

**PROOF.** The neighbours $(y_1, \ldots, y_l)$ of $v_{i+1}$ taken in clockwise order are such that $y_i$ and $y_{i+1}$ are incident to a common face. Moreover, since $v_{i+1}$ and $v_i$ are both incident to a face $f_1$ and since $v_{i+1}$ and $v_{i+2}$ are both incident to a face $f_2$, there is a sequence $P = (v_i, x_1, \ldots, x_k, v_0)$ such that there exists a
face incident to any two consecutive vertices of $P$ and such that $P$ uses only vertices of $N(v_{i+1}) \setminus C_a(S)$ and $v_{i+2}, \ldots, v_{p-1}$. One such sequence is 

$$(v_i, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0).$$

Let $P$ be such a sequence between $v_i$ and $v_0$ of minimal length. Together with $(v_1, \ldots, v_{i-1})$, $P$ forms an ordered separator of $G$ as required. □

4 The algorithm

The properties of the previous section allow us to build up an algorithm to compute the set $S_a(S, O)$ with $O \subseteq S$.

**Algorithm 2 calc3 aux**

**input:**

G a 3-connected planar graph
a a vertex of $G$
$S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \notin S$
$O = (v_0, \ldots, v_i)$ with $i \leq p - 1$ a subset of $S$

The vertices that have an incident face in common with $v_l$ ($l \neq 0$) are tagged $l$

unless they can be tagged $j$ ($1 \leq j \leq l - 1$).

These vertices are the forbidden vertices.

The vertices of $C_a(S)$ are also tagged “$C_a(S)$”.

**output:**

$S_a(S, O)$ the set of a, b-minimal separators $S'$ further from $a$ than $S$
such that $C_a(S') \cap O = \emptyset$.

**begin**

if $i = p - 1$ then

return ($\{S\}$)

else

$x \leftarrow v_{i+1}$

$S \leftarrow$ calc3_aux($G, a, S, (v_0, \ldots, v_i, x)$)

for each $y \in N(x)$ not tagged “$C_a(S)$”

if $y$ is tagged $j < i$ then

return ($S$)

for each $S'$ in find_closest_elements($G, a, x, S, O$)

tag the vertices according to $S'$

$S \leftarrow S \cup$ calc3_aux($G, a, S', (v_0, \ldots, v_i)$)

end

**Proposition 16** The algorithm calc3_aux is correct. It computes the set
\( S_a(S, O) \) of a 3-connected planar graph.

**PROOF.** The algorithm is an application of remark 2 and proposition 13, 14 and 15. □

**Proposition 17** The algorithm can be implemented to compute the set \( S_a(S, O) \) in time \( O(n|S_a(S, O)|) \).

**PROOF.** The algorithm \_calc\_3\_aux is a recursive version of the for loop below:

```plaintext
for l from i + 1 to p - 1
    empty ← FALSE
    for each \( y \in N(v_l) \) not tagged “\( C_a(S) \)”
        if \( y \) is tagged \( j < l - 1 \) then
            empty ← TRUE
        if not empty then
            for each \( S' \) in find\_closest\_elements\((G, a, v_l, S, (v_0, \ldots, l - 1))\)
                tag the vertices according to \( S' \)
                \( S \leftarrow S \cup \text{calc3}\_aux\((G, a, S', (v_0, \ldots, l))\)\)

return(S)
```

For each minimal separator \( S \), the algorithm performs the following operations:

i. the function \text{find\_closest\_elements} produces \( S \);
ii. the vertices of \( G \) are tagged;
iii. the for loop is executed in the recursive call to \text{calc3}\_aux;
iv. \( S \) is returned.

The function \text{find\_closest\_elements} can be implemented in linear time. Computing the neighbourhoods of the connected component of \( G \setminus \{N(C) \cup C\} \) that contain \( O \) can clearly be done in linear time with a graph search, but not computing those that are maximal for inclusion. However, since the graph is 3-connected planar, anyone of these neighbourhoods is necessarily maximal for inclusion, because if some neighbourhood \( S \) was a strict subset of some other neighbourhood \( S' \) then \( S' \) would be a minimal separator that is not minimal for inclusion, which would contradict proposition 5. Another graph search can be used to tag all the vertices. This costs \( O(n + m) \).

The for loop tests the neighbours of \( v_l \) to check if they are forbidden. Since the vertex \( v_l \) is always different, this costs at most \( O(m) \).
In a planar graph, the number $m$ of edges satisfies $0 \leq m \leq 3n - 6$, so the time spent on each minimal separator is $O(n)$, which gives an overall time complexity of $O(n|S_a(S,O)|)$. □

The following algorithm uses the function `calc3_aux` to compute the set of all minimal separators of a planar graph $G$.

**Algorithm 3 all_min_sep3**

**input:**
- $G$ a 3-connected planar graph

**output:**
- the set of $a,\ast$-minimal separators of $G$

**begin**

$S \leftarrow \emptyset$

**find** $a \in V$ with $d(a) < 6$

**for each** minimal separator $S \subseteq N(a)$

$S \leftarrow S \cup \text{calc3}_\text{aux}(G, a, S, \emptyset)$

**for each** $y \in N(a)$

**for each** $a,\ast$-minimal separator $S \subseteq N(y)$

$S \leftarrow S \cup \text{calc3}_\text{aux}(G, y, S, \emptyset)$

**return**($S$)

**end**

**Theorem 18** Algorithm all_min_sep3 computes the set of the minimal separators of a 3-connected planar graph in time $O(n|S(G)|)$

**PROOF.** Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex $a$, $S \in S(G)$ either belongs to $S_a$ or runs through $a$. In the second case, it is a $b,\ast$-minimal separator for a neighbour $b$ of $a$.

Moreover, there exists a vertex $a$ of degree at most five in a planar graph. Let $b_1, \ldots, b_p$ be its neighbours.

By computing $S_a \cup \left( \bigcup_{i \in [1..p]} S_{b_i} \right)$, a minimal separator can be calculated no more than five times, which gives the claimed complexity. □
5 Conclusion

This article confirms the feeling of Berry et al. [1]. In their conclusion, they note that their algorithm may compute a minimal separator up to $n$ times and that this could be improved. This is exactly what we have gained for 3-connected planar graphs. Our algorithm can be modified to list the minimal separators of an arbitrary planar graph. We also feel that there could be a better general algorithm to compute the minimal separators of a graph.

This article gives another proof that planar graphs and their minimal separators in particular are peculiar. We feel that topological properties such as proposition 9 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

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References


