Listing all the minimal separators of a 3-connected planar graph

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Abstract

We present an efficient algorithm that lists the minimal separators of a 3-connected planar graph in $O(n)$ per separator.

Key words: minimal separator; planar graphs; enumeration

1 Introduction

In the last ten years, minimal separators have been increasingly studied in graph theory leading to many algorithmic applications [5,9,10,12].

For example, minimal separators are an essential tool to study the treewidth and the minimum fill-in of graphs. In [5], Bodlaender *et al.* conjecture that for a class of graphs with a polynomial number of minimal separators, these problems can be solved in polynomial time. Bouchitté and Todinca introduced the concept of potential maximal clique [2] and showed that, if the number of potential maximal cliques is polynomial, treewidth and minimum fill-in can be solved in polynomial time. They later showed [3] that if a graph has a polynomial number of minimal separators, it has a polynomial number of potential maximal cliques. Those results rely on deep understandings of minimal separators.

Extensive research has been performed to compute the set of the minimal separators of a graph [1,6,7,11]. Berry *et al.* [1] proposed an algorithm of running time $O(nm)$ per separator [1] that uses the concept of generating new minimal

[1] The authors only proved a running time of $O(n^3)$ but the actual bound is $O(nm)$ [8].

separators from a previous minimal separator S by finding the minimal separators contained in $S \cup N(x)$ for $x \in S$. This simple process can generate all the minimal separators of a graph. However, by using this algorithm a minimal separator can be generated many times.

The aim of this article is to address the problem of finding the minimal separators of a 3-connected planar graph G. In order to avoid the problem of recalculation, we define the set $\mathcal{S}_a(S, O)$ of the a, b-minimal separators S' for some b such that the connected component of a in $G \backslash S'$ contains the connected component of a in $G \backslash S$ but avoids the set O. Therefore, it is possible to ensure that a given minimal separator is never computed more than five times.

2 Definitions

Throughout this paper, $G = (V, E)$ is a 3-connected graph without loops with $n = |V|$ and $m = |E|$. For $x \in V$, $N(x) = \{y \mid (x, y) \in E\}$ and for $C \subseteq V$, $N(C) = \{y \notin C \mid \exists x \in C, (x, y) \in E\}$. When the sets A and B are disjoint, their union is denoted by $A \sqcup B$.

A set $S \subseteq V$ is a *separator* if $G \backslash S$ has at least two connected components, an *a, b-separator* if a and b are in different connected components of $G \backslash S$, an *a, b-minimal separator* if no proper subset of S is an a, b-separator. The connected component of a in $G \backslash S$ is $C_a(S)$. The component $C_a(S)$ *is a full connected component* if $N\Big(C_a(S)\Big) = S$. For an a, b-minimal separator S, both $C_a(S)$ and $C_b(S)$ are full. A set S is a *minimal separator* if there exist a and b such that S is an a, b-minimal separator or, which is equivalent, if it has at least two full connected components. An *$a, *$-minimal separator* of a graph $G = (V, E)$ is an a, b-minimal separator of G for some $b \in V$. The set of the $a, *$-minimal separators is denoted by $\mathcal{S}_a$ and the set of the minimal separators of G is denoted by $\mathcal{S}(G)$.

It is possible to order the $a, *$-minimal separators in the following way:

$$S_1 \preccurlyeq_a S_2 \text{ if } C_a(S_1) \subseteq C_a(S_2).$$

The minimal separator S_1 is *closer* to a than S_2. The set of a, b-minimal separators is a lattice for the relation $\preccurlyeq_a$[4] but we only need the following weaker lemma:

Lemma 1 *Let C be a set of vertices of a graph G inducing a connected subgraph of G, a be a vertex of C and b be a vertex of $G \backslash \Big(C \cup N(C)\Big)$.*

The neighbour S of $C_b\big(C \cup N(C)\big)$ is an a,b-minimal separator such that C is a subset of $C_a(S)$ that is closer to a than any a,b-minimal separator S' such that C is a subset of $C_a(S')$.

PROOF. By construction, C is a subset of $C_a(S)$. By definition, the component $C_b(S)$ is full and since S is a subset of $N(C)$, the component $C_a(S)$ is also a full component which implies that S is an a,b-minimal separator.

Let S' be an a,b-minimal separator such that C is a subset of $C_a(S')$. Let p be a path in $C_b(S')$ with b as one of its ends. The vertices of S' are at least at distance 1 of C so the vertices of p are at least at distance 2 of C. Since S is a subset of $N(C)$, $p \cap S = \emptyset$. In other words p is a subset of $C_b(S)$ and $C_b(S') \subseteq C_b(S)$. This last inclusion implies that $C_a(S) \subseteq C_a(S')$ *i.e.* S is closer to a than S'. $\square$

For S an $a,*$-minimal separator and $O \subseteq V$, the set $\mathcal{S}_a(S,O)$ is *the set of the $a,*$-minimal separators S' further from a than S and such that $O \cap C_a(S') = \emptyset$.* If $x \in V$, the set $\mathcal{S}_a^x(S,O)$ is *the set of $S' \in \mathcal{S}_a(S,O)$ such that $x \in C_a(S')$.*

Remark 2 *If $x \in S$, then $\mathcal{S}_a(S,O)$ is the disjoint union*

$$\mathcal{S}_a\big(S, O \cup \{x\}\big) \bigsqcup \mathcal{S}_a^x(S,O).$$

More precisely, if $(S_i)_{i\in I}$ are the elements of $\mathcal{S}_a^x(S,O)$ closest to a, then

$$\mathcal{S}_a(S,O) = \mathcal{S}_a\big(S, O \cup \{x\}\big) \bigsqcup \Big(\bigcup_{i\in I} \mathcal{S}_a(S_i,O)\Big).$$

This gives the skeleton of an algorithm to compute the set $\mathcal{S}_a(S,O)$.

Remark 3 *If S belongs to $\mathcal{S}_a^x(S,O)$, then $\mathcal{S}_a^x(S,O) = \mathcal{S}_a(S,O)$.*

The algorithm is based on remarks 2 and 3. To list $\mathcal{S}_a$, the algorithm computes the sets $\mathcal{S}_a(S,\emptyset)$ for every S closest to a in $\mathcal{S}_a$. During this calculation, it computes $\mathcal{S}_a(S,O)$ with $O \subseteq S$. To do so, it chooses x in $S \backslash O$ and calculates $\mathcal{S}_a^x(S,O)$ and $\mathcal{S}_a(S, O \cup \{x\})$. The set $\mathcal{S}_a^x(S,O)$ is itself a union of $\mathcal{S}_a(S_i,O)$. But to obtain such a decomposition, one needs to find the elements of $\mathcal{S}_a^x(S,O)$ closest to a, which the following proposition does.

Proposition 4 *Let $G = (V,E)$ be a graph, S an $a,*$-minimal separator, $O \subset S$, $x \in S \backslash O$ and $C = C_a(S) \cup \{x\}$*

The elements of $\mathcal{S}_a^x(S,O)$ closest to a are exactly the neighbourhoods of the connected components of $G \backslash \Big\{N(C) \cup C\Big\}$ that contain O and that are maximal for inclusion.

PROOF. Let S_1 be an a,b-minimal separator of $\mathcal{S}_a^x(S,O)$ closest to a. Let S' be the neighbourhood of $C_b\Big(N(C)\cup C\Big)$. By lemma 1, S' is an a,b-minimal separator such that C is a subset of $C_a(S')$ and S' is closer to a that S_1. Moreover, since $C_a(S_1)\cap O=\emptyset$ and S' is closer to a than S_1, $C_a(S')\cap O\subseteq C_a(S_1)\cap O=\emptyset$. Thus S' belongs to $\mathcal{S}_a^x(S,O)$ and is closer to a than S_1. This proves that $S_1=S'$. Since S_1 cannot be a subset of another element of $\mathcal{S}_a^x(S,O)$, S_1 is the neighbourhood of a connected component of $G\backslash\Big\{N(C)\cup C\Big\}$ which is maximal for inclusion.

Conversely, let S_1 be a neighbourhood of a connected component D of $G\backslash\Big\{N(C)\cup C\}$ that contains O and that is maximal for inclusion. By lemma 1, S_1 is an element of $\mathcal{S}_a^x(S,O)$ that is closer to a than any a,b-minimal separator of $\mathcal{S}_a^x(S,O)$ with b in D. So if S_2 is an a,b-minimal separator of $\mathcal{S}_a^x(S,O)$ strictly closer to a than S_1, S_1 is not an a,b-minimal separator. Suppose for a contradiction that such an a,b-minimal separator exists. It follows from the first part of the proof that such an a,b-minimal separator S_2 closest to a is the neighbourhood of $C_b\Big(N(C)\cup C\Big)$. The set S_2 is an element of $\mathcal{S}_a^x(S,O)$ that is closer to a than S_1 and S_1 is a subset of S_2 (because $S_1\setminus S_2\subseteq C_a(S_2)\setminus C_a(S_1)$ and S_2 is closer to a then S_1) and therefore S_1 is a strict subset of S_2 contradicting the fact that S_1 is maximal for inclusion. $\square$

Proposition 4 gives us a way to find the minimal elements of $\mathcal{S}_a^x(S,O)$, for example by using a graph search to compute the neighbourhoods of the connected components of $G\backslash\{N(C)\cup C\}$ and then choosing among the minimal separators found the ones that contain O and that are maximal by inclusion. Using the skeleton of remark 2, we can construct an algorithm to compute the set $\mathcal{S}_a(S,O)$ that may look like:

Algorithm 1 `_calc3_`

```
begin
    if S\O = ∅ then
        return({S})
    else
        let x ∈ S\O
        𝒮 ←_calc3_(G, a, S, O ∪ {x})

        for each S_i in find_closest_elements(G, a, x, S, O)
            𝒮 ← 𝒮∪_calc3_(G, a, S_i, O)
    return(𝒮)
end
```

However several problems need to be solved.

i. We do not know whether the sets $\mathcal{S}_a(S_i, O)$ are disjoint or not. If not, a minimal separator could be computed many times, which would lead to a bad complexity.
ii. To implement the function `find_closest_elements`, proposition 4 states that we can start with a graph search of G.

   But if $\mathcal{S}_a(S, O) = \{S\}$, the recursive calls to the algorithm will try to find an element of $\mathcal{S}_a^x(S, O)$ closest to a for every $x \in S \backslash O$. Each call to `find_min_elements` costs at least $O(m)$ and finally, we would have spent at least $O(nm)$ to realise that $\mathcal{S}_a(S, O) = \{S\}$.

Proposition 6 in section 3.1 ensures that for 3-connected planar graphs, problem (i) is true, *i.e.* if S_1 and S_2 are two minimal elements of $\mathcal{S}_a^x(S, O)$, the sets $\mathcal{S}_a(S_1, O)$ and $\mathcal{S}_a(S_2, O)$ are disjoint. Section 3.3 then shows how to determine whether $\mathcal{S}_a^x(S, O)$ is empty or not in an overall $O(n)$.

3 Planar graphs

In this section, we will consider 3-connected planar graphs without loops.

Let Σ be the plane. A *plane graph* $G_\Sigma = (V_\Sigma, E_\Sigma)$ is a graph drawn on the plane, that is $V_\Sigma \subset \Sigma$ and each $e \in E_\Sigma$ is a simple curve of Σ between two vertices of V_Σ in such a way that the interiors of two distinct edges do not meet. We will denote by $\widetilde{G}_\Sigma$ the drawing of G_Σ. A *planar graph* is the abstract graph of a plane graph. We will consider plane graphs up to a topological homeomorphism.

A *face* of G_Σ is a connected component of $\Sigma \backslash \widetilde{G}_\Sigma$.

3.1 Minimal separators of 3-connected planar graphs

Proposition 5 *In a 3-connected planar graph, minimal separators are minimal for inclusion.*

PROOF. Suppose that $S \subset S'$ are two minimal separators of a 3-connected planar graph.

Let a, b, c and d be vertices such that S' is an a, b-minimal separator and S is a c,d-minimal separator. Since S is not an a, b-minimal separator, either $C_c(S')$ or $C_d(S')$ is disjoint with $C_a(S')$ and $C_b(S')$. Suppose that $C_c(S')$ is such a component. In this case, $C_c(S)$ and $N\Big(C_c(S)\Big)$ are respectively equal to $C_c(S')$ and S.

But then G admits $K_{3,3}$ as a minor for if we contract $C_a(S')$, $C_b(S')$ and $C_c(S')$ into the vertices a', b' and c', all these vertices have S in their neighbourhood and since G is 3-connected, $|S| \geq 3$. This contradicts the fact that G is planar. $\square$

Proposition 6 *Let $G = (V, E)$ be a 3-connected planar graph, a a vertex of G, S an $a, *$-minimal separator, O a subset of S and x a vertex of $S \backslash O$.*

If S_1 and S_2 are two distinct elements of $\mathcal{S}_a^x(S, O)$ that are closest to a, then

$$\mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O) = \emptyset.$$

PROOF. Let C be $C_a(S) \cup \{x\}$ and suppose for a contradiction that S_3 is a minimal separator of $\mathcal{S}_a(S_1, O) \cap \mathcal{S}_a(S_2, O)$ with S_1 and S_2 two distinct elements of $\mathcal{S}_a^x(S, O)$ closest to a. Let b be a vertex such that S_3 is an a, b-minimal separator.

Since S_3 is further from a than S_1 and S_2, both S_1 and S_2 are a, b-separators. There exists an a, b-minimal separator S' included in S_1. By proposition 5, a minimal separator of G is minimal for inclusion which proves that $S_1 = S'$ and S_1 is an a, b-minimal separator. By lemma 1, the neighbourhood S_4 of $C_b\Big(N(C) \cup C\Big)$ is an a, b-minimal separator such that C is a subset of $C_a(S_4)$ that is closer to a than S_1. So $C_a(S_4) \cap O \subseteq C_a(S_1) \cap O = \emptyset$, and S_4 is an element of $\mathcal{S}_a^x(S, O)$ that is closer to a than S_1. Similarly, S_2 is an a, b-minimal separator and S_4 is closer to a than S_2 which contradicts the fact that S_1 and S_2 are two distinct elements of $\mathcal{S}_a^x(S, O)$ closest to a. $\square$

3.2 The intermediate graph

Definition 7 *Let $G_\Sigma = (V_\Sigma, E_\Sigma)$ be a 3-connected plane graph. Let F be the set of its faces. In each face $f \in F$ pick up one point v_f. Let R_F be the set $\{v_f \mid f \in F\}$. The intermediate graph $G_I = (V_I, E_I)$ is a plane graph whose vertex set is $V_I = V_\Sigma \cup R_F$. We place an edge between a vertex $v \in V$ and $v_f \in R_F$ if and only if the vertex v is incident to the face f.*

For G' a subgraph of G_I, the set $\widetilde{G'} \cap V_\Sigma$ will be denoted by $V(G')$.

Proposition 8 *Let μ be a cycle of G_I such that the curve $\widetilde{\mu}$ separates at least two vertices a and b of V_Σ.*

The set $V(\mu)$ is an a, b-separator of G_Σ.

PROOF. Let p be a path in G_Σ from a to b. Since a and b are not in the

same connected component of $\Sigma\backslash\widetilde{\mu}$, $\widetilde{p}$ intersect $\widetilde{\mu}$. By construction, $p\cap\mu \subseteq V_\Sigma$. This implies that every path from a to b meets $V(\mu)$ and so $V(\mu)$ is an a,b-separator. $\square$

Proposition 9 *Let S be an a,b-minimal separator of G. There exists a simple cycle μ of G_I such that the Jordan curve defined by μ separates the vertices of $C_a(S)$ and $C_b(S)$ and such that $V(\mu) = S$.*

PROOF. Let C be the connected component of a in $G\backslash S$. Let us contract C into a supervertex v_C to build the graph $G_{/C}$. There is a cycle $\mu_{/C}$ in $(G_{/C})_I$ such that $V(\mu_{/C})$ is the neighbourhood of v_C in $G_{/C}$. Therefore, the neighbourhood of C in G_I has the structure of a cycle μ.

Suppose $\widetilde{\mu}$ is not a Jordan curve, the border μ' of the connected component of b in $\Sigma\backslash\widetilde{\mu}$ is a strict sub-lace of $\widetilde{\mu}$ which separates a and b. However, proposition 8 shows that $V(\mu')$ which is a strict subset of S is an a,b-separator. This contradicts the fact that S is a a,b-minimal separator. $\square$

Proposition 9 shows that the minimal separators of a 3-connected planar graph correspond to cycles of the intermediate graph. Thus, when a set corresponds to no cycle of the G_I, it is not a minimal separator. However, this is not a characterisation of the minimal separators of a 3-connected planar graph for some cycles of G_I correspond to no minimal separator of G.

There are several ways to find an exact criterion for minimal separators. The following section presents a criterion that is well suited to our purpose.

3.3 Ordered separators

Definition 10 *An ordered separator of G is a sequence of distinct vertices*

$$(v_0, \ldots, v_{p-1}),\ (p > 2)$$

such that

i. there exists a face to which v_i and $v_{i+1\,[p]}$ are incident;
ii. v_i and v_j are incident to a common face only if $i = j+1\,[p]$ or $j = i+1\,[p]$;
iii. if $p = 3$, no face is incident to v_i, $v_{i+1\,[p]}$ and $v_{i+2\,[p]}$.

The notation $i\,[p]$ means i modulo p.

A set $S = \{v_0, \ldots, v_{p-1}\}$ is an ordered separator if there exists a permutation σ such that $(v_{\sigma(0)}, \ldots, v_{\sigma(p-1)})$ is an ordered separator.

If $S = (v_0, \ldots, v_{p-1})$ is an ordered separator of G, then S is naturally associated to the set $\{v_0, \ldots, v_{p-1}\}$. We will use an ordered separator either as a sequence or as the corresponding set.

Lemma 11 *Every minimal separator S of G is ordered.*

PROOF. Let S be an a, b-minimal separator of G.

Proposition 9 states that there exists a simple cycle of G_I

$$\mu = (v_0, f_0, \ldots, v_{p-1}, f_{p-1})$$

such that $V(\mu) = S$.

Let us prove that $T = (v_0, \ldots, v_{p-1})$ is an ordered separator corresponding to S.

i. The construction of T ensures that v_i and v_{i+1} are incident to a common face (f_i).
ii. Suppose that v_i et v_j are incident to a common face f and that $i + 1 \neq j\,[p]$ and $j + 1 \neq i\,[p]$.
$\mu_1 = (v_i, f_i, v_{i+1}, f_{i+1}, \ldots, v_j, f)$ and $\mu_2 = (v_j, f_j, v_{j+1}, f_{j+1}, \ldots, v_i, f)$ are laces of G_I. Moreover, since either μ_1 or μ_2 separates a and b, there exists an a, b-separator strictly included in S which is absurd.
iii. Suppose that $p = 3$ and that v_0, v_1 et v_2 are all incident to a common face f. If we add a vertex f to G connected to the vertices v_0, v_1 and v_2, the graph remains planar which is absurd because this graph has $K_{3,3}$ as a minor. Indeed, the connected component of a, the connected component of b and the vertex f are all incident to v_0, v_1 and v_2 which builds up a $K_{3,3}$.

The sequence T is an ordered separator corresponding to S. $\square$

Conversely,

Lemma 12 *Every ordered separator of G is a minimal separator of G.*

PROOF. Let $S = (v_0, \ldots, v_{p-1})$ be an ordered separator of G.

First, S is a separator. Otherwise, $G \backslash S$ would be connected or empty. In both cases, all the vertices of S would be incident to a common face.

Let S' be a minimal separator included in S. By lemma 11, S' is ordered and since condition ii forbids an ordered separator to have a strictly included ordered separator, $S' = S$. The ordered separator S is a minimal separator. $\square$

From lemmata 11 and 12, we have the following proposition:

Proposition 13 *A set $S \subseteq V$ is a minimal separator of a 3-connected planar graph $G = (V, E)$ if and only if it corresponds to an ordered separator of G.*

At this point, we have a characterisation of the minimal separators of a 3-connected planar graph. Let us see how it enables us to find out whether $\mathcal{S}_a^x(S, O)$ is empty or not ($O \subseteq S$ and $x \in S \backslash O$).

Proposition 14 *Let $S = (v_0, \ldots, v_{p-1})$ be an ordered $a, *$-separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of S.*

If there exists a face that is incident to both $y \in N(v_{i+1}) \backslash C_a(S)$ and v_j with $0 < j < i$, then $\mathcal{S}_a^{v_{i+1}}(S, O)$ is empty.

PROOF. Let b be such that S is an a, b-minimal separator and suppose that $y \in N(v_{i+1})$ and v_j with $0 < j < i$ are both incident to a face f. Since S is an ordered separator, there exists a cycle $(v_0, f_0, \ldots, v_k, f_k)$ of G_I corresponding to a Jordan curve $\widetilde{\mu}$. Let Σ_b be the connected component of $\Sigma \backslash \widetilde{\mu}$ that contains b. Since y and v_j are incident to f, there exists a path (v_{i+1}, g, y, f, v_j) that corresponds to a curve $\widetilde{\nu}$ that cuts Σ_b in two parts Σ_b^1 and Σ_b^2 whose borders are $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ respectively. Since $0 < j < i$, neither $V(\widetilde{\mu}_1)$ nor $V(\widetilde{\mu}_2)$ contains O.

Suppose that S' is an element of $\mathcal{S}_a^{v_{i+1}}(S, O)$ closest to a. Let c be such that S' is an a, c-minimal separator. The vertex c belongs to Σ_b. We may suppose that c belongs to Σ_b^1. By proposition 4, S' is the neighbourhood of $C_c\Big(S \cup N(v_{i+1})\Big)$ *i.e.* $S' = V(\widetilde{\mu}_1)$, but O is not a subset of S' which is absurd. $\square$

Conversely,

Proposition 15 *Let $S = (v_0, \ldots, v_{p-1})$ be an ordered $a, *$-separator of a 3-connected planar graph $G = (V, E)$ and $O = (v_0, \ldots, v_i)$, $(i < p - 1)$ be an initial sequence of S.*

If there is no face incident to both $y \in N(v_{i+1}) \backslash C_a(S)$ and v_j $(0 < j < i)$, then there is an ordered separator in $S \cup N(v_{i+1}) \backslash C_a(S)$ that contains O.

PROOF. The neighbours $(y_1, \ldots, y_l)$ of v_{i+1} taken in clockwise order are such that y_i and y_{i+1} are incident to a common face. Moreover, since v_{i+1} and v_i are both incident to a face f_1 and since v_{i+1} and v_{i+2} are both incident to a face f_2, there is a sequence $P = (v_i, x_1, \ldots, x_k, v_0)$ such that there exists a

face incident to any two consecutive vertices of P and such that P uses only vertices of $N(v_{i+1})\backslash C_a(S)$ and $v_{i+2}, \ldots, v_{p-1}$. One such sequence is

$$(v_i, y_j, y_{j+1}, \ldots, y_k, v_{i+2}, \ldots, v_{p-1}, v_0).$$

Let P be such a sequence between v_i and v_0 of minimal length. Together with $(v_1, \ldots, v_{i-1})$, P forms an ordered separator of G as required. $\square$

4 The algorithm

The properties of the previous section allow us to build up an algorithm to compute the set $\mathcal{S}_a(S, O)$ with $O \subseteq S$.

Algorithm 2 `calc3_aux`

input:
G a 3-connected planar graph
a a vertex of G
$S = (v_0, \ldots, v_{p-1})$ an ordered separator such that $a \notin S$
$O = (v_0, \ldots, v_i)$ with $i \leq p-1$ a subset of S
The vertices that have an incident face in common with v_l ($l \neq 0$) are tagged l unless they can be tagged j ($1 \leq j \leq l-1$).
These vertices are the forbidden vertices.
The vertices of $C_a(S)$ are also tagged "$C_a(S)$".
output:
$\mathcal{S}_a(S, O)$ the set of a, b-minimal separators S' further from a than S such that $C_a(S') \cap O = \emptyset$.

begin
  if $i = p-1$ **then**
    return$(\{S\})$
  else
    $x \leftarrow v_{i+1}$
    $\mathcal{S} \leftarrow$`calc3_aux`$(G, a, S, (v_0, \ldots, v_i, x))$

    for each $y \in N(x)$ *not tagged "$C_a(S)$"*
      if y *is tagged* $j < i$ **then**
        return$(\mathcal{S})$
    for each S' *in* `find_closest_elements`(G, a, x, S, O)
      tag the vertices according to S'
      $\mathcal{S} \leftarrow \mathcal{S} \cup$ `calc3_aux`$(G, a, S', (v_0, \ldots, v_i))$
end

Proposition 16 *The algorithm* `calc3_aux` *is correct. It computes the set*

$\mathcal{S}_a(S,O)$ *of a 3-connected planar graph.*

PROOF. The algorithm is an application of remark 2 and proposition 13, 14 and 15. □

Proposition 17 *The algorithm can be implemented to compute the set* $\mathcal{S}_a(S,O)$ *in time* $O(n|\mathcal{S}_a(S,O)|)$.

PROOF. The algorithm `_calc_3_aux` is a recursive version of the **for** loop below:

for l **from** $i+1$ **to** $p-1$
  `empty`←`FALSE`
  for each $y \in N(v_l)$ not tagged "$C_a(S)$"
    if y is tagged $j < l-1$ **then**
      `empty`←`TRUE`
  if not `empty` **then**
    for each S' in `find_closest_elements`$(G, a, v_l, S, (v_0, \ldots, l-1))$
      tag the vertices according to S'
      $\mathcal{S} \leftarrow \mathcal{S} \cup$ `calc3_aux`$(G, a, S', (v_0, \ldots, l))$
return$(\mathcal{S})$

For each minimal separator S, the algorithm performs the following operations:

i. the function `find_closest_elements` produces S;
ii. the vertices of G are tagged;
iii. the **for** loop is executed in the recursive call to `calc3_aux`
iv. S is returned.

The function `find_closest_elements` can be implemented in linear time. Computing the neighbourhoods of the connected component of $G \backslash \left\{ N(C) \cup C \right\}$ that contain O can clearly be done in linear time with a graph search, but not computing those that are maximal for inclusion. However, since the graph is 3-connected planar, anyone of these neighbourhoods is necessarily maximal for inclusion, because if some neighbourhood S was a strict subset of some other neighbourhood S' then S' would be a minimal separator that is not minimal for inclusion, which would contradict proposition 5. Another graph search can be used to tag all the vertices. This costs $O(n+m)$.

The **for** loop tests the neighbours of v_l to check if they are forbidden. Since the vertex v_l is always different, this costs at most $O(m)$.

In a planar graph, the number m of edges satisfies $0 \leq m \leq 3n - 6$, so the time spent on each minimal separator is $O(n)$, which gives an overall time complexity of $O(n|\mathcal{S}_a(S, O)|)$. $\square$

The following algorithm uses the function `calc3_aux` to compute the set of all minimal separators of a planar graph G.

Algorithm 3 `all_min_sep3`

input:
  G a 3-connected planar graph
output:
  the set of $a, *$-minimal separators of G

begin
  $\mathcal{S} \leftarrow \emptyset$
  find $a \in V$ `with` $d(a) < 6$
  for each *minimal separator* $S \subseteq N(a)$
    $\mathcal{S} \leftarrow \mathcal{S} \cup$ `calc3_aux`$(G, a, S, \emptyset)$
  for each $y \in N(a)$
    for each *$a, *$-minimal separator* $S \subseteq N(y)$
      $\mathcal{S} \leftarrow \mathcal{S} \cup$ `calc3_aux`$(G, y, S, \emptyset)$
  return$(\mathcal{S})$
end

Theorem 18 *Algorithm* `all_min_sep3` *computes the set of the minimal separators of a 3-connected planar graph in time $O(n|\mathcal{S}(G)|)$*

PROOF. Since in a 3-connected planar graph minimal separators are minimal for inclusion, given a vertex a, $S \in \mathcal{S}(G)$ either belongs to $\mathcal{S}_a$ or runs through a. In the second case, it is a $b,*$-minimal separator for a neighbour b of a.

Moreover, there exists a vertex a of degree at most five in a planar graph. Let $b_1, \ldots, b_p$ be its neighbours.

By computing $\mathcal{S}_a \bigcup \left(\bigcup_{i \in [1..p]} \mathcal{S}_{b_i}\right)$, a minimal separator can be calculated no more than five times, which gives the claimed complexity. $\square$

5 Conclusion

This article confirms the feeling of Berry *et al.* [1]. In their conclusion, they note that their algorithm may compute a minimal separator up to n times and that this could be improved. This is exactly what we have gained for 3-connected planar graphs. Our algorithm can be modified to list the minimal separators of an arbitrary planar graph. We also feel that there could be a better general algorithm to compute the minimal separators of a graph.

This article gives another proof that planar graphs and their minimal separators in particular are peculiar. We feel that topological properties such as proposition 9 are yet to be found and that such properties are the key to compute the treewidth of planar graphs.

Acknowledgement

We thank Vincent Bouchitté and Ioan Todinca for the fruitful discussions we have had on this topic.

References

[1] A. Berry, J.P. Bordat, and O. Cogis. Generating all the minimal separators of a graph. *International Journal of Foundations of Computer Science*, 11:397–404, 2000.

[2] V. Bouchitté and I. Todinca. Minimal triangulations for graphs with "few" minimal separators. In *Proceedings 6th Annual European Symposium on Algorithms (ESA'98)*, volume 1461 of *Lecture Notes in Computer Science*, pages 344–355. Springer-Verlag, 1998.

[3] V. Bouchitté and I. Todinca. Listing all potential maximal cliques of a graph. *Theoretical Computer Science*, 276(1-2):212–323, 2001.

[4] F. Escalante. Schnittverbände in graphen. *Abhandlungen aus dem Mathematischen Seminar des Universität Hamburg*, 38:199–220, 1972.

[5] T. Kloks, H.L. Bodlaender, H. Müller, and D. Kratsch. Computing treewidth and minimum fill-in: all you need are the minimal separators. In *Proceedings First Annual European Symposium on Algorithms (ESA'93)*, volume 726 of *Lecture Notes in Computer Science*, pages 260–271. Springer-Verlag, 1993.

[6] T. Kloks and D. Kratsch. Finding all minimal separators of a graph. In *Proceedings 11th Annual Symposium on Theoretical Aspects of Computer*

Science (STACS'94), volume 775 of *Lecture Notes in Computer Science*, pages 759–768. Springer-Verlag, 1994.

[7] T. Kloks and D. Kratsch. Listing all minimal separators of a graph. *SIAM J. Comput.*, 27(3):605–613, 1998.

[8] F. Mazoit. *Décomposition algorithmique des graphes.* PhD thesis, École normale supérieure de Lyon, 2004.

[9] A. Parra. *Structural and Algorithmic Aspects of Chordal Graph Embeddings.* PhD thesis, Technische Universität Berlin, 1996.

[10] A. Parra and P. Scheffler. Characterizations and algorithmic applications of chordal graph embeddings. *Discrete Appl. Math.*, 79(1-3):171–188, 1997.

[11] H. Shen and W. Liang. Efficient enumeration of all minimal separators in a graph. *Theoretical Computer Science*, 180:169–180, 1997.

[12] I. Todinca. *Aspects algorithmiques des triangulations minimales des graphes.* PhD thesis, École Normale Supérieure de Lyon, 1999.