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# A projective formalism applied to topological and probabilistic event structures

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This paper introduces projective systems for topological and probabilistic event structures. The projective formalism is used for studying the domain of configurations of a prime event structure and its space of maximal elements. This is done from both a topological and a probabilistic viewpoint. We give probability measure extension theorems in this framework.

## 1. Introduction

The study of concurrency models rests on a fundamental choice for the semantics of processes. Processes are either seen as sequences (in the *interleaving* semantics) or as partial orders of events (in the *true-concurrency* semantics). On the one hand, most probabilistic concurrency models have been based until now on the interleaving semantics (stochastic Petri nets, probabilistic process algebra, I/O probabilistic automata). On the other hand, domain theory has brought tools from classical topology to the study of partial order based models. This paper brings together elements for studying in a unified fashion both topological and probabilistic aspects of concurrency models under their true-concurrency semantics. It provides, in particular, measure theoretic foundations for recent work on probabilistic true-concurrency models, such as Abbes and Benveniste (2006) and Varacca *et al.* (2004).

Randomisation of processes is usually set up in two steps. We first construct probabilities on partial runs, then use a measure theoretic extension argument to randomise full runs. For Markov chains, this is the usual Kolmogorov extension theorem. Our study is basically motivated by this extension step for true-concurrency models, where the measure theoretic arguments are less standard.

We consider a prime event structure  $\mathcal{E}$ . Two topological spaces are associated with  $\mathcal{E}$ : its domain of configurations  $\mathcal{L}$ ; and the space  $\Omega$  of maximal points of the domain. We introduce projective systems by considering  $\mathcal{L}$  as a projective limit of finite sets. We observe that the projective topology coincides with the Lawson topology on  $\mathcal{L}$ . Consider a probability measure  $\mathbb{P}$  on  $\Omega$  equipped with its Borel  $\sigma$ -algebra. The data  $(\mathcal{E}, \mathbb{P})$  defines a *probabilistic event structure* (Varacca *et al.* 2004; Abbes and Benveniste 2006). The interpretation of  $\mathbb{P}$  is that a finite configuration  $x$  has the probability  $\mathbb{P}(\omega \in \Omega : \omega \supseteq x)$  to occur.

The main contributions of this paper are representations as projective limits of the space  $\Omega$  of maximal configurations, together with an application to the extension of probabilities. Compactness of  $\Omega$  is a necessary and sufficient condition for its representation as a projective limit of *finite* sets – a known general fact usually deduced from Stone representation theory, and with a particular concrete representation in our case. However, the event structures one may encounter in practical situations, such as those given by the unfolding of a finite Petri net, may have a non-compact space  $\Omega$ . Since this does not occur in the sequential case ( $\Omega$  is then the compact space of infinite paths in a regular, locally finite tree), this can be interpreted as a particular feature of concurrency.

This motivates the introduction of the class of *locally finite* event structures, which is a wide class of event structures with the compactness property for  $\Omega$ . It also motivates the introduction for general event structures of a projective decomposition of the space  $\Omega$  through *possibly infinite state spaces*. We show how this decomposition can be combined with the Prokhorov extension theorem to construct probability measures on the space of maximal configurations of event structures. This extension result amounts to an extension of finite probabilities for locally finite event structures.

We obtain as byproducts new proofs of some known results. First, we easily obtain some topological properties of the domain of configurations endowed with the Lawson topology: metrisability, separability and compactness. Second, we obtain a new, direct proof of the extension theorem for continuous valuations to Radon measures, a result that was first given in Lawson (1982). This extension result is the basis of the connection established in Varacca *et al.* (2004) between probabilistic event structures and the probabilistic powerdomain of Jones and Plotkin (1989). As we show, the projective formalism also provides us with tools for studying continuous valuations.

The measure extension results in this paper are used in Abbes and Benveniste (2006) for the actual construction of probabilistic event structures based on the representation by projective systems.

### *Organisation of the paper*

Section 2 introduces topological and probabilistic projective systems. In Section 3, the projective topology on the domain of configurations of an event structure is defined and studied. Section 4 explores how we can represent the space of maximal configurations as a projective limit of finite sets. As an alternative, we propose a projective representation with possibly infinite state spaces. The particular class of locally finite event structures is introduced. Section 5 deals with probabilistic event structures and probability extension theorems.

## **2. Topological and probabilistic projective systems**

In this section we recall some notions on projective systems – general references on this topic are, for instance, Bourbaki (1961) and Gierz *et al.* (2003). We also recall the definition of projective systems of probabilities and the Prokhorov extension theorem – classical references on this topic are Bourbaki (1969) and Schwartz (1973).

2.1. Partial orders and topological projective systems

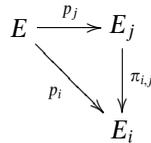
We fix a poset (partially ordered set)  $(I, \leq)$ , which we assume at most countable and directed, that is, any two elements have an upper bound.

We use  $\Delta$  to denote the set of pairs  $(i, j) \in I \times I$  such that  $i \leq j$ . Let  $(E_i)_{i \in I}$  be a collection of sets. A collection of mappings  $(\pi_{i,j})_{(i,j) \in \Delta}$  with  $\pi_{i,j} : E_j \rightarrow E_i$  for all  $(i, j) \in \Delta$  is said to be a *projective system* if  $\pi_{i,i} = \text{Id}_{E_i}$  for all  $i \in I$  and if

$$\forall i, j, k \in I, \quad i \leq j \leq k \Rightarrow \pi_{i,k} = \pi_{i,j} \circ \pi_{j,k}.$$

If each set  $E_i$  is equipped with a partial order (respectively, with a topology), the collection  $(\pi_{i,j})_{i \leq j}$  is said to be a *partial orders projective system* (respectively, a *topological projective system*) if the mappings  $\pi_{i,j}$  are non-decreasing (respectively, continuous).

An *extension* of the projective system  $(\pi_{i,j})_{i \leq j}$  is a set  $E$  together with a collection of mappings  $(p_i)_{i \in I}$ , with  $p_i : E \rightarrow E_i$  for each  $i \in I$ , such that  $p_i = \pi_{i,j} \circ p_j$  for all  $i \leq j$ . This is represented by the following commutative diagram:



If the projective system is a partial orders projective system (respectively, a topological projective system), the extension is said to be *order preserving* (respectively, *topological*) if  $E$  is equipped with a partial order (respectively, with a topology) and if all  $p_i : E \rightarrow E_i$  are order preserving (respectively, continuous).

Let  $(F, (q_i)_{i \in I})$  be another extension of  $(\pi_{i,j})_{i \leq j}$ . An *arrow of extensions* from  $E$  to  $F$  is a mapping  $f : E \rightarrow F$  such that  $p_i = q_i \circ f$  for all  $i \in I$ . The arrow is said to be *order preserving* (respectively, *continuous*) if  $E$  is an order preserving (respectively, a topological) extension, and if the mapping  $E \rightarrow F$  is order preserving (respectively, continuous). Finally, the arrow is said to be an *isomorphism of extensions* (respectively, of order preserving extensions, of topological extensions) if the mapping  $f : E \rightarrow F$  is a bijection (respectively, an isomorphism of partial orders, a homeomorphism) and if  $f^{-1} : F \rightarrow E$  is an arrow of extensions.

A *projective limit* of a projective system  $(\pi_{i,j})_{i \leq j}$  is an extension  $E$  such that for any other extension  $F$ , there is a unique arrow of extensions  $F \rightarrow E$ . If the projective system is assumed to be a partial orders projective system (respectively, a topological projective system), we require that the arrows between extensions are all order preserving (respectively, continuous). If it exists, a projective limit is unique up to a unique extension isomorphism.

2.2. Canonical projective limits

Let  $(E_i)_{i \in I}$ ,  $(\pi_{i,j})_{i \leq j}$  be a projective system and  $X$  denote the product space  $X = \prod_{i \in I} X_i$ . A projective limit of the projective system  $(\pi_{i,j})_{i \leq j}$  is given by the extension  $E$  defined by

$$E = \{(x_i)_{i \in I} \in X : \forall (i, j) \in \Delta \quad x_i = \pi_{i,j}(x_j)\},$$

with mappings  $p_i : E \rightarrow E_i$  defined as the restriction to  $E$  of the projection mappings  $X \rightarrow E_i$ . If the projective system is a partial orders projective system (respectively, a topological projective system), the partial ordering (respectively, the topology) on  $E$  is defined as the restriction to  $E$  of the product partial ordering (respectively, the product topology) on the product space  $X$ .

We use the notation

$$E = \varprojlim E_i$$

to denote canonical projective limits.

For a topological projective system, any topological projective limit is Hausdorff, separable or compact if and only if all  $E_i, i \in I$  are Hausdorff, separable or compact, respectively.

### 2.3. Projective systems of probability measures

If  $(X, \tau)$  is a topological space, we consider the associated Borel  $\sigma$ -algebra  $\mathfrak{F}$ , that is, the  $\sigma$ -algebra generated by  $\tau$ . A *measure*, or *Borel measure*, on the measurable space  $(X, \mathfrak{F})$  is a set function  $\mu : \mathfrak{F} \rightarrow [0, +\infty]$  satisfying  $\mu(\emptyset) = 0$  and countably additive on sequences of pairwise disjoint measurable sets. We consider the class of *bounded Radon measures*:  $\mu$  is *bounded* if  $\mu(X) < \infty$ , and a bounded measure  $\mu$  is *Radon* if, for every measurable subset  $A \in \mathfrak{F}$ , we have (Schwartz 1973, Definition  $R_3$ , page 13):

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ compact}\}.$$

We say that  $\mu$  is a *probability measure* if  $\mu(X) = 1$ .

Let  $(X, \mathfrak{F})$  and  $(X', \mathfrak{F}')$  be two measurable spaces, and let  $f : X \rightarrow X'$  be a measurable mapping. If  $\mu$  is a measure on  $(X, \mathfrak{F})$ , the *image measure* of  $\mu$  under  $f$  is the measure denoted by  $f\mu$ , and defined by  $f\mu(A) = \mu(f^{-1}(A))$  for all  $A \in \mathfrak{F}'$ .

Let  $(E_i)_{i \in I}, (\pi_{i,j})_{i \leq j}$  be a topological projective system. For each  $i \in I$ , let  $\mathfrak{F}_i$  denote the Borel  $\sigma$ -algebra on  $E_i$ , and assume that we are given for each  $i \in I$  a Radon probability measure  $\mu_i$  on  $(E_i, \mathfrak{F}_i)$ . We say that  $(\mu_i)_{i \in I}$  is a *projective system of probability measures* if we have (Schwartz 1973, Section 10, page 74; Bourbaki 1969, Definition 1, page 51)

$$\forall i, j \in I, \quad i \leq j \Rightarrow \mu_i = \pi_{i,j}\mu_j.$$

Let  $E, (p_i)_{i \in I}$  be a topological extension of the projective system,  $\mathfrak{F}$  be the associated Borel  $\sigma$ -algebra and  $\mu$  be a Radon measure on  $(E, \mathfrak{F})$ . The pair  $(E, \mu)$  is said to be a *measure extension* of the projective system of probability measures  $(\mu_i)_{i \in I}$  if  $p_i\mu = \mu_i$  for all  $i \in I$ . The following *extension theorem* is due to Prokhorov: see Schwartz (1973, Theorem 22, page 74 and the Corollary on page 81), or Bourbaki (1969, Theorems 1 and 2, pages 52–53).

**Theorem 2.1 (Prokhorov).** Let  $(E_i)_{i \in I}, (\pi_{i,j})_{i \leq j}$  be a topological projective system, where all spaces  $E_i$  are Hausdorff, and let  $E$  be a topological projective limit of it. Let  $(\mu_i)_{i \in I}$  be an associated projective system of Radon probability measures. Then there exists a

unique Radon measure  $\mu$  defined on the Borel  $\sigma$ -algebra of  $E$  and extension of  $(\mu_i)_{i \in I}$ . This measure  $\mu$  is a probability measure.

### 3. The domain of configurations of a prime event structure as a projective limit

#### 3.1. Prime event structures

Recall from Nielsen *et al.* (1980) that a *prime event structure* is a triple  $(\mathcal{E}, \leq, \#)$ , where  $\mathcal{E}$  is an at most countable set of *events*,  $(\mathcal{E}, \leq)$  is a partial order, and  $\#$  is an irreflexive and symmetric binary relation on  $\mathcal{E}$  called the *conflict* relation satisfying the *inheritance axiom*: for all  $e_1, e_2, e_3 \in \mathcal{E}$ , if  $e_1 \leq e_2$  and  $e_2 \# e_3$ , then  $e_1 \# e_3$ . We assume, moreover, that for every event  $e \in \mathcal{E}$ , the set  $\downarrow e = \{e' \in \mathcal{E} : e' \leq e\}$  is finite. Since in this paper we are only concerned with prime event structures, we will just say *event structure* for short, but always mean *prime* event structures. With a slight abuse of notation, we identify the set  $\mathcal{E}$  and the event structure  $(\mathcal{E}, \leq, \#)$ .

Computation processes associated with  $\mathcal{E}$  are represented by a particular class of subsets of  $\mathcal{E}$ , its configurations. We first define a *prefix* of  $\mathcal{E}$  as a subset  $P \subseteq \mathcal{E}$  such that  $\downarrow e \subseteq P$  for every  $e \in P$ . A *configuration* of  $\mathcal{E}$  is a conflict-free prefix of  $\mathcal{E}$ , that is, a prefix  $x$  such that  $\# \cap (x \times x) = \emptyset$ . We say that two configurations  $x$  and  $x'$  are *compatible* if  $x \cup x'$  is a configuration, otherwise we say that  $x$  and  $x'$  are *incompatible*.

For each prefix  $P$  of  $\mathcal{E}$ ,  $(P, \leq|_P, \#|_P)$  is naturally an event structure, where  $\leq|_P$  and  $\#|_P$  are the restriction of  $\leq$  and of  $\#$  to  $P$ .

**Notations.** We use  $\mathcal{P}_0$  to denote the set of finite prefixes of  $\mathcal{E}$ . Configurations are partially ordered by inclusion; we use  $\mathcal{L}$  to denote the poset of configurations of  $\mathcal{E}$ .  $\mathcal{L}$  is called the *domain of configurations* of  $\mathcal{E}$ . We use  $\mathcal{L}_0$  to denote the poset of *finite* configurations of  $\mathcal{E}$ . For any event  $e \in \mathcal{E}$ ,  $\downarrow e$  is the smallest configuration that contains  $e$ , and  $\downarrow e$  is finite by hypothesis. Finally, for each prefix  $P$  of  $\mathcal{E}$ , we use  $\mathcal{L}_P$  to denote the domain of configurations of  $P$ .

#### 3.2. A natural projective system

Let  $P, P'$  be two prefixes of  $\mathcal{E}$  such that  $P \subseteq P'$ . It is obvious that if  $x$  is a configuration of  $P'$ , then  $x \cap P$  is a configuration of  $P$ . This allows us to define the following two collections of mappings. We use  $\Delta$  to denote the set of pairs  $(P, P') \in \mathcal{P}_0 \times \mathcal{P}_0$  such that  $P \subseteq P'$ . We have:

$$\forall (P, P') \in \Delta, \lambda_{P, P'} : \mathcal{L}_{P'} \rightarrow \mathcal{L}_P, \quad x \in \mathcal{L}_{P'} \mapsto \lambda_{P, P'}(x) = x \cap P. \tag{1}$$

$$\forall P \in \mathcal{P}_0, \lambda_P : \mathcal{L} \rightarrow \mathcal{L}_P, \quad x \in \mathcal{L} \mapsto \lambda_P(x) = x \cap P. \tag{2}$$

We consider the directed poset of indices  $(\mathcal{P}_0, \subseteq)$ . The family  $(\mathcal{L}_P)_{P \in \mathcal{P}_0}$  equipped with the collection of mappings  $(\lambda_{P, P'})_{P \subseteq P'}$  obviously satisfies the axioms of a partial orders projective system (Section 2.1). Moreover, the data  $(\mathcal{L}, (\lambda_P)_{P \in \mathcal{P}_0})$  is an order preserving extension of the projective system  $(\lambda_{P, P'})_{P \subseteq P'}$ . Actually, we have the following theorem.

**Theorem 3.1.** The order preserving extension  $\mathcal{L}$  is a projective limit of the projective system  $(\lambda_{P,P'})_{P \subseteq P'}$ .

*Proof.* Let  $X$  be the canonical projective limit of  $(\pi_{P,P'})_{P \subseteq P'}$ .  $X$  is defined as follows:

$$X = \{(z_P)_{P \in \mathcal{P}_0} : \forall (P, P') \in \Delta, z_P = \lambda_{P,P'}(z_{P'})\}.$$

We show that  $\Phi : \mathcal{L} \rightarrow X$  defined by  $\Phi(x) = (x_P)_{P \in \mathcal{P}_0}$ , with  $x_P = \lambda_P(x)$  for each  $P \in \mathcal{P}_0$ , is an isomorphism of order preserving extensions.  $\Phi$  is clearly an order preserving extension arrow.

$\Phi$  is a bijection. Indeed, the inverse mapping is given by

$$\forall y \in X, \quad \text{with } y = (y_P)_{P \in \mathcal{P}_0}, \quad \Phi^{-1}(y) = \bigcup_{P \in \mathcal{P}_0} y_P.$$

Now  $\Phi^{-1}$  is also an order preserving extension arrow, so  $\Phi$  is an isomorphism of order preserving extensions, as claimed. □

### 3.3. The projective topology on the domain of configurations

Since each  $\mathcal{L}_P$  is finite for  $P \in \mathcal{P}_0$ ,  $\mathcal{L}_P$  is equipped with the discrete topology, and any projective limit is naturally endowed with the projective topology. In particular, since  $\mathcal{L}$  is a projective limit according to Theorem 3.1, we define by this way the *projective topology* on the domain of configurations  $\mathcal{L}$ , making the canonical bijection  $\Phi : \mathcal{L} \rightarrow X$  a homeomorphism. It follows from this definition that the projective topology is the weakest topology on  $\mathcal{L}$  that makes all the  $\lambda_P : \mathcal{L} \rightarrow \mathcal{L}_P$  continuous for  $P$  ranging over  $\mathcal{P}_0$ . A basis of open sets of the projective topology on  $\mathcal{L}$  is given by the collection of sets of the form

$$\lambda_P^{-1}(x),$$

with  $P$  ranging over  $\mathcal{P}_0$  and  $x$  ranging over  $\mathcal{L}_P$ .

**Remark.** The convergence in the projective topology is addressed in a very intuitive way as follows. Let  $(x_n)_{n \geq 0}$  be a sequence of  $\mathcal{L}$ , and let  $x \in \mathcal{L}$ . Then  $\lim_{n \rightarrow \infty} x_n = x$  in the projective topology if and only if

$$\forall P \in \mathcal{P}_0, \quad \exists N \geq 0, \quad \forall n \text{ integer}, \quad n \geq N \implies x_n \cap P = x \cap P.$$

### 3.4. Other definitions of the projective topology

The metrics for infinite traces constructed for instance in Kwiatowska (1990) and Katoen *et al.* (2001) are metrics for the projective topology. Indeed, the construction reproduces the classical construction of a metric on a product space. Restricting the metric obtained in this way to the projective limit defines the projective topology.

We shall also relate the projective topology to other topologies from domain theory. See Gierz *et al.* (2003) for the standard definitions of a DCPO  $L$  (Directed complete poset), and of the Scott topology on a DCPO  $L$ . The *lower topology* is defined as the weakest

topology such that the sets  $\uparrow x = \{y \in L : y \geq x\}$  for  $x$  ranging over  $L$  are closed. Finally, the *Lawson topology* is the join of the Scott and the lower topology. The domain of configurations  $\mathcal{L}$  of an event structure is obviously a DCPO (the supremum of a directed set of configurations is given by their union), so these definitions apply to  $\mathcal{L}$ .

In order to relate them to the projective topology, we note the following. Recall that, if  $X$  and  $Y$  are two partial orders, a pair  $(f, g)$  of mappings  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  is called a *projection embedding pair* if they are order preserving and if, moreover,  $f \circ g \leq \text{Id}_Y$ , and  $g \circ f = \text{Id}_X$ . In this case, one is determined by the other. Mapping  $f$  is called the lower adjoint, and  $g$  is called the upper adjoint.

Here it is obvious that for  $P, P' \in \mathcal{P}_0$  with  $P \subseteq P'$ , we have  $\lambda_{P, P'} : \mathcal{L}_{P'} \rightarrow \mathcal{L}_P$  is the upper adjoint of a projection embedding pair, whose lower adjoint  $i_{P, P'} : \mathcal{L}_P \rightarrow \mathcal{L}_{P'}$  is defined by  $i_{P, P'}(x) = x$  for all  $x \in \mathcal{L}_P$ . Therefore, the projective system  $(\lambda_{P, P'})_{P \in \mathcal{P}'}$  is said to be of *finite type*, and its projective limit is called a *bifinite domain* – see Gierz *et al.* (2003) and Abbes and Keimel (2006). Hence, according to Theorem 3.1, the domain of configurations  $\mathcal{L}$  is a bifinite domain. It follows from the results of Abbes and Keimel (2006) that the projective topology on  $\mathcal{L}$  coincides with the Lawson topology. In particular, this is an easy way to deduce that the Lawson topology makes  $\mathcal{L}$  a compact, separable and metrisable space.

### 3.5. Application: extension of continuous valuations

The notion of valuation goes back to Birkhoff (1940), which also refers to a 1900 paper by Dedekind. Let  $(X, \tau)$  be a topological space and  $v : \tau \rightarrow [0, +\infty]$  be a function. Function  $v$  is said to be a *valuation* on  $(X, \tau)$  if  $v$  satisfies the following properties:

- 1  $v(\emptyset) = 0$
- 2  $\forall U, V \in \tau, U \subseteq V \Rightarrow v(U) \leq v(V)$  (monotony)
- 3  $\forall U, V \in \tau, v(U \cup V) + v(U \cap V) = v(U) + v(V)$  (modularity).

A first problem is to extend  $v$  to a finitely additive measure on the algebra of sets generated by  $\tau$ ; this was solved in Horn and Tarski (1948). A second problem, assuming that  $v$  is also *continuous* (see below), is to extend  $v$  from the algebra of sets to a Borel measure on the Borel  $\sigma$ -algebra generated by  $\tau$ . Several authors have studied this problem: in particular, see Norberg and Verdaat (1997), Lawson (1982), Alvarez-Manilla *et al.* (2000), Keimel and Lawson (2005) and Abbes and Keimel (2006).

A proof of the extension result from continuous valuations to Radon measures can be achieved with projective systems in the case of bifinite domains as in Abbes and Keimel (2006). Here the proof we present is even more direct for event structures.

A valuation  $v$  on a topological space  $(X, \tau)$  is said to be *bounded by  $a$*  if  $v(X) \leq a$ , and *continuous* (Lawson 1982) if for any subset  $D \subseteq \tau$  directed with respect to the inclusion  $\subseteq$ , we have  $v(\bigcup_{V \in D} V) = \sup_{V \in D} v(V)$ .

**Theorem 3.2.** Let  $\mathcal{E}$  be an event structure and  $\mathcal{L}$  denote the domain of configurations of  $\mathcal{E}$ . For any continuous and bounded valuation  $v$  defined on the Scott-open sets of  $\mathcal{L}$ ,  $v$  has a unique extension to a Radon measure defined on the  $\sigma$ -algebra  $\mathfrak{S}$  generated on  $\mathcal{L}$  by the Scott topology.

We shall use the following result, which is cited in Alvarez-Manilla *et al.* (2000), and is a direct consequence of the key results of Horn and Tarski (1948).

**Lemma 3.3.** A bounded valuation  $v$  has a unique extension to a finitely additive and bounded measure on the algebra of sets generated by the Scott topology.

*Proof of Theorem 3.2.* We use the notation  $\uparrow x = \{y \in \mathcal{L} : y \sqsupseteq x\}$  for any configuration  $x$  – recall that  $\uparrow x$  is Scott open for any  $x \in \mathcal{L}_0$ . Let  $\mathfrak{S}$  denote the  $\sigma$ -algebra generated by the Scott topology on  $\mathcal{L}$ . Note first that  $\mathfrak{S}$  coincides with the projective  $\sigma$ -algebra on  $\mathcal{L}$ , say  $\mathfrak{S}'$ , generated by the projective topology on  $\mathcal{L}$ . Indeed, we have seen in Section 3.4 that the Lawson topology coincides with the projective topology. In particular, Scott-open sets belong to  $\mathfrak{S}'$ , hence  $\mathfrak{S} \subseteq \mathfrak{S}'$ . For the converse inclusion, it is enough to show that every subset with the form  $U = \lambda_P^{-1}(x)$  with  $P \in \mathcal{P}_0$  and  $x \in \mathcal{L}_P$  belongs to  $\mathfrak{S}$ . This follows from the following form of  $U$ :

$$U = \uparrow x \setminus \bigcup_{y \in \mathcal{L}_P, y \sqsupseteq x, y \neq x} \uparrow y. \tag{3}$$

Now consider the extension  $\mu$  of  $v$  given by Lemma 3.3 on the algebra of sets  $\mathcal{A}$  generated by the Scott topology. For each finite prefix  $P \in \mathcal{P}_0$ , and for each element  $x \in \mathcal{L}_P$ , it follows from (3) that  $\lambda_P^{-1}(x) \in \mathcal{A}$ . We may thus define a finite measure  $\mu_P$  on  $\mathcal{L}_P$  as follows:

$$\forall x \in \mathcal{L}_P, \quad \mu_P(x) = \mu(\lambda_P^{-1}(x)).$$

It follows from the identity  $\lambda_P = \lambda_{P,Q} \circ \lambda_Q$  for  $P, Q \in \mathcal{P}_0$  with  $P \sqsubseteq Q$  that  $(\mu_P)_{P \in \mathcal{P}_0}$  is a projective system of measures on  $(\mathcal{L}_P)_{P \in \mathcal{P}_0}$ , which is trivially bounded and Radon. Since, according to Theorem 3.1,  $\mathcal{L}$  is canonically homeomorphic to the projective limit  $\varprojlim \mathcal{L}_P$ , the Prokhorov theorem (Theorem 2.1) insures the existence of a measure  $\bar{\mu}$  on  $\mathfrak{S}$  such that  $\lambda_P \bar{\mu} = \mu_P$  for every  $P \in \mathcal{P}_0$ . To show that  $\bar{\mu}$  extends  $v$  as expected, we first show that  $\bar{\mu}(\uparrow x) = v(\uparrow x)$  for every finite configuration  $x$ . Indeed, let  $x \in \mathcal{L}_0$ , and let  $P$  be the finite prefix  $P = x$ . Then we have  $\uparrow x = \lambda_P^{-1}(x)$ , which is a Scott-open set, from which we get

$$\bar{\mu}(\uparrow x) = \bar{\mu}(\lambda_P^{-1}(x)) = \mu_P(x) = v(\lambda_P^{-1}(x)) = v(\uparrow x),$$

as claimed. By induction, and using the modularity of both the valuation  $v$  and the measure  $\bar{\mu}$ , this implies that for every finite union  $\uparrow x_1 \cup \dots \cup \uparrow x_n$ , we also have

$$\bar{\mu}(\uparrow x_1 \cup \dots \cup \uparrow x_n) = v(\uparrow x_1 \cup \dots \cup \uparrow x_n). \tag{4}$$

Now let  $U$  be a non-void Scott-open set. Choose  $(P_k)_{k \geq 0}$  to be a non-decreasing sequence of finite prefixes of  $\mathcal{E}$  such that  $\bigcup_k P_k = \mathcal{E}$ , and for every  $k \geq 0$  set

$$J_k = U \cap \mathcal{L}_{P_k}, \quad V_k = \bigcup_{x \in J_k} \uparrow x.$$

By (4) we have  $v(V_k) = \bar{\mu}(V_k)$  for every  $k \geq 0$ . Since  $U$  is Scott-open, we have that  $\bigcup_k V_k = U$ . The union is non-decreasing, so we apply the continuities of both the valuation  $v$  and the measure  $\bar{\mu}$  to obtain

$$\bar{\mu}(U) = \lim_{k \rightarrow \infty} \bar{\mu}(V_k) = \lim_{k \rightarrow \infty} v(V_k) = v(U).$$

This shows the existence of the extension  $\bar{\mu}$ . The uniqueness follows from uniqueness in the Prokhorov theorem and in Lemma 3.3.  $\square$

#### 4. The space of maximal configurations

We now study the space  $\Omega$  of maximal configurations of an event structure  $\mathcal{E}$ . This space is of great interest in particular because of its probabilistic interpretation, which we will investigate in Section 5. We relate the concurrency properties of  $\mathcal{E}$  with the topological properties of  $\Omega$ . In particular, we exhibit the class of locally finite event structures.

##### 4.1. The topological space of maximal configurations

We consider, as above, an event structure  $\mathcal{E}$ , and we use  $\mathcal{L}$  to denote the domain of configurations of  $\mathcal{E}$ . We say that a configuration  $\omega \in \mathcal{L}$  is *maximal* if  $\omega$  is a maximal element of  $(\mathcal{L}, \subseteq)$ , that is,  $\forall x \in \mathcal{L}, x \supseteq \omega \Rightarrow x = \omega$ . We use  $\Omega$  to denote the set of maximal configurations of  $\mathcal{E}$ , and equip  $\Omega$  with the restriction of the projective topology on  $\mathcal{L}$ .

Recall that  $\mathcal{L}_0$  denotes the poset of finite configurations of  $\mathcal{E}$ .

**Definition 4.1 (Shadows and finite shadows).** For any  $x \in \mathcal{L}$ , we define the *shadow* of  $x$ , denoted by  $\mathcal{S}(x)$ , as the following subset of  $\Omega$ :

$$\mathcal{S}(x) = \{\omega \in \Omega : \omega \supseteq x\}.$$

We say that  $\mathcal{S}(x)$  is a *finite shadow* if  $x$  is finite (although  $\mathcal{S}(x)$  is not finite in general).

Applying Zorn's lemma, one sees that every shadow is non-empty, and, in particular,  $\Omega = \mathcal{S}(\emptyset)$  is non-empty.

**Lemma 4.2.** The collection of finite shadows  $\mathcal{S}(x)$ , with  $x$  ranging over  $\mathcal{L}_0$ , is a basis, at most countable, of open sets of  $\Omega$ .

*Proof.* Observe first that for any two shadows  $S_1$  and  $S_2$ , the intersection  $S_1 \cap S_2$  is either empty or is a shadow itself. Moreover, any finite shadow  $\mathcal{S}(x)$  can be written  $\mathcal{S}(x) = \Omega \cap \lambda_P^{-1}(x)$ , where  $P = x$ , hence every finite shadow is open in  $\Omega$ . Therefore, it is enough to prove that any elementary neighbourhood  $U$  of some point  $\omega_0 \in \Omega$  of the form  $U = \lambda_P^{-1}(u) \cap \Omega$  with  $P \in \mathcal{P}_0$  and  $u \in \mathcal{L}_P$  contains a neighbourhood of  $\omega_0$  of the form  $\mathcal{S}(x)$ , with  $x \in \mathcal{L}_0$ .

We fix such an  $\omega_0$  and  $U = \lambda_P^{-1}(u) \cap \Omega$  with  $\omega_0 \in U$ . For each event  $e \in P \setminus u$ ,  $e$  is in conflict with at least one event of  $\omega_0$ , say  $f_e$ , otherwise  $\omega_0 \cup \downarrow e$  would be a configuration strictly containing  $\omega_0$ , and thus  $\omega_0$  would not be maximal. Since  $P$  is finite, we consider a finite prefix  $Q$  that contains both  $P$  and all event  $f_e$ , for  $e$  ranging over  $P \setminus u$ , and the finite configuration  $x = \omega_0 \cap Q$ . Then we claim that we have  $\omega_0 \in \mathcal{S}(x) \subseteq U$ . Indeed,  $\omega_0 \in \mathcal{S}(x)$  comes from the construction of  $x$ . Let  $\omega \in \mathcal{S}(x)$ . Then  $\omega \supseteq x$  since  $x \supseteq u$ , and thus  $\lambda_P(\omega) \supseteq u$ . Finally, assume that  $\lambda_P(\omega) \subseteq u$  does not hold. Then there is an event  $e \in P \setminus u$  such that  $\omega \ni e$ . But then,  $\omega$  contains both events  $e$  and  $f_e$ , although they are in conflict, which contradicts the assumption that  $\omega$  is conflict-free, and shows that

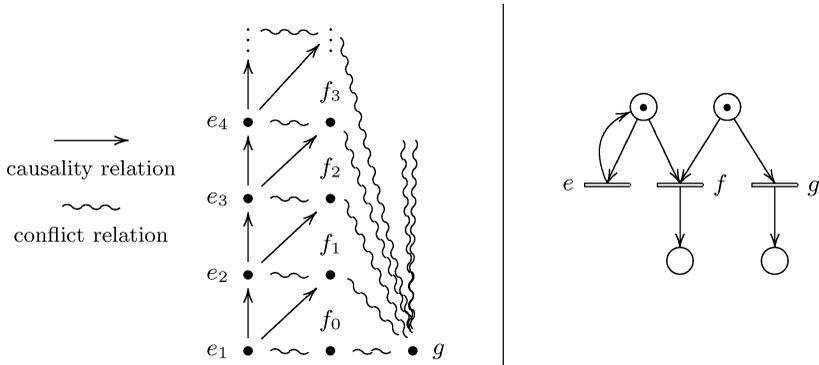


Figure 1. The diagram on the right shows a Petri net unfolding to the event structure depicted on the left. The associated space  $\Omega$  is non-compact.

$\lambda_P(\omega) = u$ . Finally, we have shown that  $\omega_0 \in \mathcal{S}(x) \subseteq U$ , as we claimed, and the proof is complete. □

#### 4.2. A non-compact example

Before we investigate the representations of  $\Omega$  as a projective limit in detail, we will analyse a simple example to make the problems encountered more concrete. The examples we will look at use the notion of the *unfolding* of a Petri net to an event structure – see Nielsen *et al.* (1980).

Consider the event structure  $\mathcal{E}$  depicted on the left of Figure 1. The event structure  $\mathcal{E}$  is the unfolding of the Petri net depicted on the right. The space  $\Omega$  of maximal configurations of  $\mathcal{E}$  consists of the following elements:

$$\omega_\infty = \{g, e_1, e_2, \dots\}, \quad \omega_n = \{e_1, \dots, e_n, f_n\}, \quad \forall n \geq 0.$$

Clearly, every finite and maximal configuration  $\omega$  is isolated in  $\Omega$  (that is,  $\{\omega\}$  is an open set). Hence all  $\omega_n$  are isolated. We have  $\{\omega_\infty\} = \mathcal{S}(\{g\})$ , so  $\omega_\infty$  also is isolated. An infinite set whose elements are all isolated cannot be compact. Hence  $\Omega$  is not compact. In other words, the element  $\omega_\infty$  is *not* the Alexandrov point at infinity of the sequence  $\{\omega_n, n \geq 0\}$ . As a consequence,  $\Omega$  cannot be described as a projective limit of finite sets.

#### 4.3. Using finite projective systems for $\Omega$

This subsection investigates the topological conditions insuring that ‘ $\Omega$  is the projective limit of its traces in finite prefixes’. For a finite prefix  $P \in \mathcal{P}_0$ , we define the following subset of  $\mathcal{L}_P$ :

$$\Gamma_P = \{\omega \cap P : \omega \in \Omega\}.$$

Note that  $\Gamma_P$  does not necessarily identify with the set  $\Omega_P$  of maximal configurations of  $P$ . We will return to this question in Section 4.4. We use  $\pi_P$  and  $\pi_{P,P'}$ , with  $P, P' \in \mathcal{P}_0$

and  $P \subseteq P'$ , to denote the mappings

$$\pi_P : \Omega \rightarrow \Gamma_P \quad \text{and} \quad \pi_{P,P'} : \Gamma_{P'} \rightarrow \Gamma_P$$

defined by  $\pi_P(\omega) = \omega \cap P$  and  $\pi_{P,P'}(x) = x \cap P$ . Notice that  $\pi_P$  and  $\pi_{P,P'}$  are nothing but the restrictions of the mappings  $\lambda_P$  and  $\lambda_{P,P'}$  (cf. Section 3.2):

$$\pi_P = \lambda_P|_{\Omega}, \quad \pi_{P,P'} = \lambda_{P,P'}|_{\Gamma_{P'}}.$$

From this it follows, on the one hand, that the collection  $(\pi_{P,P'})_{P \subseteq P'}$  is a projective system of finite sets, which is trivially topological when equipping the sets  $\Gamma_P$  with the discrete topologies; we use  $\Gamma = \varprojlim \Gamma_P$  to denote the canonical projective limit. On the other hand,  $\Omega$  is a topological extension of  $(\pi_{P,P'})_{P \subseteq P'}$ , from which we get an arrow of topological extensions  $\Omega \rightarrow \Gamma$ . There is, moreover, a continuous injection  $\Gamma \hookrightarrow X$ , where  $X = \varprojlim \mathcal{L}_P$ , conjugated to the continuous injection  $\Omega \hookrightarrow \mathcal{L}$  according to the following commutative diagram:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\Phi} & X \\ \uparrow & & \uparrow \\ \Omega & \longrightarrow & \Gamma \end{array}$$

However,  $\Omega$  might not be the projective limit of  $(\Gamma_P)_{P \in \mathcal{P}_0}$ , as shown by Theorem 4.4 below together with the example of Section 4.2.

**Lemma 4.3.** Let  $\overline{\Omega}$  denote the topological closure of  $\Omega$  in  $\mathcal{L}$ . The arrow of topological extensions  $\Omega \rightarrow \Gamma$  extends uniquely to an isomorphism of extensions  $\overline{\Omega} \rightarrow \Gamma$ .

*Proof.* Let  $\Phi : \mathcal{L} \rightarrow X$  denote the canonical homeomorphism of Theorem 3.1, with  $X = \varprojlim \mathcal{L}_P$ . We first show that  $\Phi^{-1}(\Gamma) = \overline{\Omega}$ . As an intersection of closed subsets of  $X$ ,

$$\Gamma = \bigcap_{P \in \mathcal{P}_0} \lambda_P^{-1}(\Gamma_P),$$

with the continuous mappings  $\lambda_P : \mathcal{L} \rightarrow \mathcal{L}_P$ ,  $\Gamma$  is a closed subset of  $X$ . Since  $\Phi$  is continuous, and with  $\Phi(\Omega) \subseteq \Gamma$ , this implies  $\Phi(\overline{\Omega}) \subseteq \Gamma$  and thus  $\overline{\Omega} \subseteq \Phi^{-1}(\Gamma)$ .

To show the converse inclusion, we pick an element  $x \in \Phi^{-1}(\Gamma)$  and show that  $U \cap \Omega \neq \emptyset$  for any elementary neighbourhood  $U$  of  $x$  in  $\mathcal{L}$ . Since  $\Phi(x) \in \Gamma$ , there is an element  $(x_P)_{P \in \mathcal{P}_0} \in \Gamma$  such that  $x \cap P = x_P$  for all  $P \in \mathcal{P}_0$ . For each  $P \in \mathcal{P}_0$ , we pick  $\omega_P \in \Omega$  such that  $\omega_P \cap P = x_P$ . Let  $U$  be an elementary neighbourhood of  $x$  in  $\mathcal{L}$ ;  $U$  has the form  $U = \lambda_Q^{-1}(z)$  where  $Q$  is a finite prefix and  $z \in \mathcal{L}_Q$ . We then have  $\lambda_Q(x) = z = x_Q = \omega_Q \cap Q = \lambda_Q(\omega_Q)$ . Hence  $\omega_Q \in U$ , which shows that  $U \cap \Omega \neq \emptyset$ . Since this holds for any  $U$ , we get that  $x \in \overline{\Omega}$ . So far we have shown that  $\overline{\Omega} = \Phi^{-1}(\Gamma)$ , as claimed.

Thus the restriction of  $\Phi$  to  $\overline{\Omega}$  defines a continuous extension  $\Phi|_{\overline{\Omega}} : \overline{\Omega} \rightarrow \Gamma$ . Since  $\overline{\Omega}$  is compact as a closed subset of the Hausdorff compact space  $\mathcal{L}$ , the mapping  $\Phi|_{\overline{\Omega}}$  is actually a homeomorphism that extends the topological extension  $\Omega \rightarrow \Gamma$ . Uniqueness is obvious. □

**Theorem 4.4.** The space  $\Omega$  is compact if and only if the arrow of extensions  $\Omega \rightarrow \Gamma$  is an isomorphism of topological extensions.

*Proof.* If the arrow  $\Omega \rightarrow \Gamma$  is an isomorphism, it is a homeomorphism, and thus  $\Omega$  is compact. Conversely, if  $\Omega$  is compact, it is closed in  $\mathcal{L}$ , and thus  $\overline{\Omega} = \Omega$ . By Lemma 4.3, we then have an isomorphism of topological extensions  $\Omega \rightarrow \Gamma$ .  $\square$

4.4. *Stopping prefixes: a general projective representation for  $\Omega$*

Since  $\Omega$  cannot be in all cases the projective limit of finite sets, as shown by the example of Section 4.2, we are brought to introduce an alternative to finite prefixes and the associated sets  $\Gamma_P$ . For each prefix  $P$ , we use  $\Omega_P$  to denote the set of maximal configurations of  $P$ . Although we clearly have  $\Omega_P \subseteq \Gamma_P$ , nothing guarantees that the converse inclusion holds. Indeed, consider the following simple example:  $\mathcal{E} = \{a, b\}$ , with  $a \# b$ , and  $P = \{a\}$ ,  $\omega = \{b\}$ . Then  $\omega \cap P = \emptyset \in \Gamma_P$ , but  $\emptyset$  is not maximal in  $P$ , which leads us to the following definition.

**Definition 4.5 (Intrinsic prefixes and stopping prefixes).** Let  $P$  be a prefix of an event structure  $\mathcal{E}$ . We use  $\Omega_P$  to denote the set of maximal configurations of  $P$ . We say that  $P$  is *intrinsic* to  $\mathcal{E}$  if  $\Omega_P = \Gamma_P$ .

We recall that the *minimal conflict* relation on  $\mathcal{E}$ , denoted by  $\#_\mu$ , is defined by

$$\forall (e_1, e_2) \in \mathcal{E} \times \mathcal{E}, \quad e_1 \#_\mu e_2 \Leftrightarrow (\downarrow e_1 \times \downarrow e_2) \cap \# = \{(e_1, e_2), (e_2, e_1)\}.$$

We say that a prefix  $B$  is a *stopping prefix* if  $B$  is  $\#_\mu$ -closed, that is, if it satisfies

$$\forall e_1 \in B, \forall e_2 \in \mathcal{E}, \quad e_1 \#_\mu e_2 \Rightarrow e_2 \in B.$$

**Lemma 4.6.**

- 1 If  $x, x'$  are two incompatible configurations, there are events  $e \in x$  and  $e' \in x'$  such that  $e \#_\mu e'$ .
- 2 Any stopping prefix of  $\mathcal{E}$  is intrinsic to  $\mathcal{E}$ .

*Proof.*

- 1 This part is an easy proof based on the finiteness of the predecessors.
- 2 Let  $B$  be a stopping prefix of  $\mathcal{E}$ , let  $\omega \in \Omega$  and let  $\omega_B = \omega \cap B$ . Assume that  $\omega_B$  is not maximal in  $B$ . Then there is an event  $e$  in  $B$  satisfying  $e \notin \omega_B$ ,  $(\omega_B \cup \downarrow e) \in \mathcal{L}$ . In particular,  $e \notin \omega$ , and since  $\omega$  is maximal, this implies that  $\downarrow e$  and  $\omega$  are incompatible. According to Point 1 above, there are events  $e_1 \in \downarrow e$  and  $e_2 \in \omega$  such that  $e_1 \#_\mu e_2$ . Since  $e_1 \in \downarrow e$ , and since  $e$  belongs to prefix  $B$ ,  $e_1$  also belongs to  $B$ . Since  $B$  is  $\#_\mu$ -closed, and since  $e_1 \#_\mu e_2$ , this implies that  $e_2 \in B$ , and thus  $e_2 \in \omega_B$ . But then configuration  $\omega_B \cup \downarrow e$  contains the conflict  $e_1 \#_\mu e_2$ , which is a contradiction.  $\square$

Clearly, stopping prefixes form a complete lattice. Therefore, and since  $\mathcal{E}$  is a stopping prefix, for every event  $e \in \mathcal{E}$ , there is a unique smallest stopping prefix that contains  $e$ , which we denote by  $B(e)$ , and introduce the following definition.

**Definition 4.7.** A stopping prefix  $B$  is said to be *elementary* if  $B$  is a finite union (may be empty) of stopping prefixes with the form  $B(e)$ . We use  $\mathcal{D}$  to denote the directed poset of elementary stopping prefixes.

For each  $B \in \mathcal{D}$ ,  $\Omega_B$  is equipped with the restriction of the projective topology on  $\mathcal{L}$  to  $\Omega_B$ . Applying Lemma 4.2 to any  $B \in \mathcal{D}$  seen as an event structure, the topology on  $\Omega_B$  is generated by the collection

$$\{\xi \in \Omega_B : \xi \supseteq x\}, \tag{5}$$

for  $x$  ranging over the finite configurations of  $B$ . According to Lemma 4.6, every  $B \in \mathcal{D}$  is intrinsic, from which we get a mapping  $\pi_B : \Omega \rightarrow \Omega_B$ . For the same reason, we also have for each pair  $B, B' \in \mathcal{D}$  with  $B \subseteq B'$ , a mapping  $\pi_{B,B'} : \Omega_{B'} \rightarrow \Omega_B$ ,  $\xi \in \Omega_{B'} \mapsto \xi \cap B \in \Omega_B$ . Clearly, the family  $(\Omega_B)_{B \in \mathcal{D}}$  forms a topological projective system with respect to the collection of mappings  $(\pi_{B,B'})_{B \subseteq B'}$ , for  $B, B' \in \mathcal{D}$  – continuity of mappings  $\pi_{B,B'}$  is easily seen, for instance, by using the basis of open sets given by (5). For the same reasons,  $(\Omega, (\pi_B)_{B \in \mathcal{D}})$  is a topological extension of the projective system.

**Theorem 4.8.** The topological extension  $(\Omega, (\pi_B)_{B \in \mathcal{D}})$  is a projective limit of the topological projective system  $(\Omega_B)_{B \in \mathcal{D}}$ .

*Proof.* Set  $\Lambda = \varprojlim (\Omega_B)_{B \in \mathcal{D}}$ , the canonical projective limit, and consider the arrow of extensions  $\phi : \Omega \rightarrow \Lambda$  given by  $\phi(\omega) = (\pi_B(\omega))_{B \in \mathcal{D}}$  for  $\omega \in \Omega$ . We show that  $\phi$  is an isomorphism of topological extensions.

We first describe the inverse mapping of  $\phi$ . For  $(z_B)_{B \in \mathcal{D}}$  an element of  $\Lambda$ , define  $z = \bigcup_{B \in \mathcal{D}} z_B$ . Then  $z$  is clearly a configuration of  $\mathcal{E}$ . We show that  $z$  is maximal. If  $z$  is not maximal, there is an event  $e$  such that  $e \notin z$  and  $z \cup \{e\}$  is a configuration. Let  $B = B(e)$ . Then  $B \in \mathcal{D}$ , so there is an element  $\xi \in \Omega$  such that  $\xi \cap B = z_B$ . Then  $z_B$  is compatible with  $\downarrow e$ , and it follows from Lemma 4.6 Point 2 that  $z_B$  is maximal in  $B$ . Therefore  $e \in z_B$ , which contradicts the assumption that  $e \notin z$ . Hence  $z \in \Omega$  as claimed. This defines a mapping  $\psi : \Lambda \rightarrow \Omega$ , which is obviously an arrow of extensions, inverse of  $\phi$ .

The arrow of extensions  $\phi$  is continuous, by virtue of the universal property of the projective limit  $\Lambda$ , since all mappings  $\pi_B : \Omega \rightarrow \Omega_B$  are continuous.

It remains only to show that  $\phi^{-1}$  is continuous, that is, that  $\phi(V)$  is open for every open set  $V \subseteq \Omega$ . According to Lemma 4.2, there is no loss of generality if we restrict  $V$  to be of the form  $V = \mathcal{S}(x)$  with  $x$  a finite configuration of  $\mathcal{E}$ . Consider the following elementary stopping prefix:

$$C = \bigcup_{e \in x} B(e).$$

Then we have  $\phi(V) = \{z = (z_B)_{B \in \mathcal{D}} : z_C \supseteq x\}$ , which shows that  $\phi(V)$  is open, as claimed. This completes the proof. □

**Remark.** At the cost of introducing possibly infinite state spaces, we have given a projective representation of  $\Omega$ . The particularity of concurrency is that, even for unfoldings of *finite* state machine such as finite Petri nets, we may have to consider these infinite state spaces. See Section 5.4 for an example.

#### 4.5. Locally finite and confusion-free event structures

A case of particular interest is when the directed poset  $\mathcal{D}$  of elementary stopping prefixes coincides with the lattice of *finite stopping prefixes*, which leads us to the following definition.

**Definition 4.9 (Locally finite event structures).** An event structure  $\mathcal{E}$  is said to be *locally finite* if, for every event  $e \in \mathcal{E}$ , there is a finite stopping prefix  $B$  that contains  $e$ .

**Proposition 4.10.** If  $\mathcal{E}$  is locally finite, then  $\Omega$  is compact.

*Proof.* If  $\mathcal{E}$  is locally finite, the decomposition of Theorem 4.8 only involves finite sets. The compactness of  $\Omega$  is then immediate.  $\square$

**Remark.** The compactness of  $\Omega$  for locally finite event structures can be shown directly, without using Theorem 4.8. Indeed, we have the following result: *if every event  $e$  belongs to a finite intrinsic prefix of  $\mathcal{E}$ , then  $\Omega$  is compact* (this is left as an exercise: hint, show that  $\Omega$  is closed in the domain  $\mathcal{L}$ ). Using Lemma 4.6 Point 2, this implies that  $\Omega$  is compact if  $\mathcal{E}$  is locally finite.

The following definition has a clear computational meaning.

**Definition 4.11 (Preregular event structures).** We say an event structure  $\mathcal{E}$  is *preregular* if for each finite configuration  $x$  of  $\mathcal{E}$  there are finitely many events *enabled* at  $x$ , that is, events  $e$  such that  $e \notin x$  and  $x \cup \{e\} \in \mathcal{L}_0$ .

As a particular kind of locally finite event structures, we find the class of *confusion-free* and *preregular* event structures (Nielsen *et al.* 1980), which are analogous to the concrete domains of Kahn and Plotkin (1978). We leave the reader to prove that if  $\mathcal{E}$  is a confusion-free event structure, the following conditions are equivalent:

- (a)  $\mathcal{E}$  is locally finite,
- (b)  $\Omega$  is compact,
- (c)  $\mathcal{E}$  is preregular.

On the other hand, the examples shown in Figures 2 and 3 show that for general event structures compactness of  $\Omega$  does not imply the local finiteness of  $\mathcal{E}$ , even if  $\mathcal{E}$  is preregular (hint for the compactness of  $\Omega$  in both examples:  $\Omega$  is closed in  $\mathcal{L}$ ).

## 5. Extension of probability measures

We now apply the previous results to the theory of probabilistic event structures.

### 5.1. Probabilistic event structures

**Definition 5.1 (Projective  $\sigma$ -algebra and probabilistic event structures).** Let  $\mathcal{E}$  be an event structure. The *projective  $\sigma$ -algebra* on  $\Omega$  is the smallest  $\sigma$ -algebra on  $\Omega$  that contains all the finite shadows. A *probabilistic event structure* is a pair  $(\mathcal{E}, \mathbb{P})$ , where  $\mathcal{E}$  is an event structure and  $\mathbb{P}$  is a probability measure on the measurable space  $(\Omega, \mathfrak{F})$ , where  $\mathfrak{F}$  is the projective  $\sigma$ -algebra on  $\Omega$ .

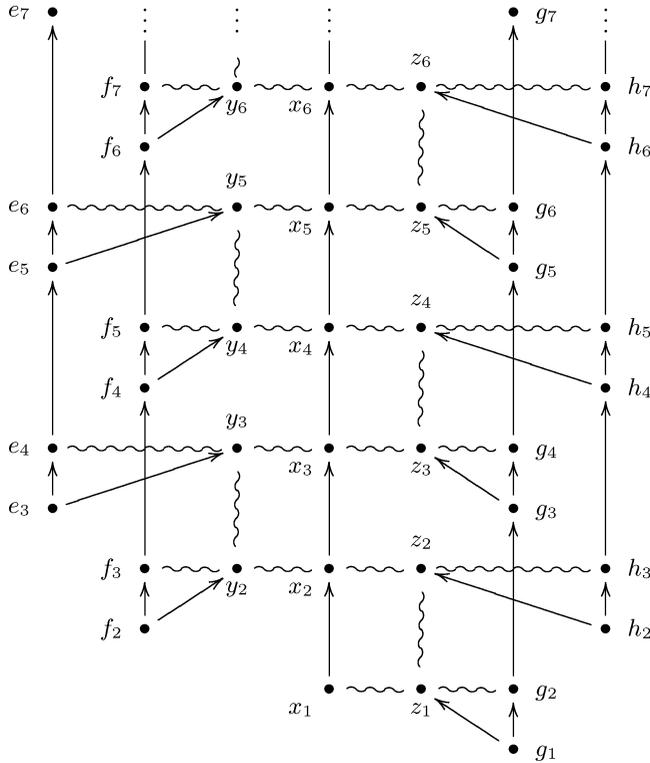


Figure 2. A pre-regular event structure that is non-locally finite with  $\Omega$  still compact.

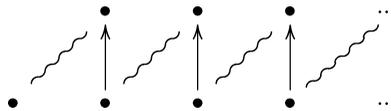


Figure 3. A non-locally finite event structure with  $\Omega$  still compact.

It follows from Lemma 4.2 that the projective  $\sigma$ -algebra of Definition 5.1 coincides with the Borel  $\sigma$ -algebra on  $\Omega$  generated by the induced Scott topology, and with the Borel  $\sigma$ -algebra generated by the projective topology.

5.2. A first approach to the extension of probabilities through finite probabilities

In this section we stick to the idea of giving an extension result to define a probability  $\mathbb{P}$  on  $\Omega$  from finite probabilities. Let  $(\mu_P)_{P \in \mathcal{P}_0}$  be a projective system of measures defined on the projective system  $(\Gamma_P)_{P \in \mathcal{P}_0}$  (see Section 4.3). Since the  $\Gamma_P$  are finite, the probabilities  $\mu_P$  are trivially Radon, and thus extend to a Radon probability measure  $\mathbb{Q}$  on the projective limit  $\Gamma = \varprojlim \Gamma_P$ . According to Lemma 4.3, there is a homeomorphism  $\bar{\Omega} \rightarrow \Gamma$ . We identify  $\Gamma$  and  $\bar{\Omega}$  so that  $\mathbb{Q}$  is defined on  $\bar{\Omega}$ . Finally, we consider the set  $\partial\Omega$  defined by  $\partial\Omega = \bar{\Omega} \setminus \Omega$ . With this notation, we have the following result.

**Theorem 5.2.** Let  $\mathcal{E}$  be an event structure, let  $(\mu_P)_{P \in \mathcal{P}_0}$  be a projective system of probability measures on  $(\Gamma_P)_{P \in \mathcal{P}_0}$  and let  $\mathbb{Q}$  be the projective limit of  $(\mu_P)_{P \in \mathcal{P}_0}$  on  $\overline{\Omega}$ . A necessary and sufficient condition for the existence of a probability Radon measure  $\mathbb{P}$  on  $\Omega$  such that

$$\forall P \in \mathcal{P}_0, \quad \pi_P \mathbb{P} = \mu_P \tag{6}$$

is that  $\mathbb{Q}(\partial\Omega) = 0$ . In this case  $\mathbb{P}$  is unique, and given by the restriction of  $\mathbb{Q}$  to  $\Omega$ .

*Proof.* Assume there is a Radon probability  $\mathbb{P}$  on  $\Omega$  satisfying (6). Then for any compact subset  $A$  of  $\Omega$ , we have

$$A = \bigcap_{P \in \mathcal{P}_0} \pi_P^{-1}(\pi_P(A)),$$

which is a countable filtered intersection. Therefore, and using (6), we get

$$\mathbb{P}(A) = \inf_{P \in \mathcal{P}_0} \mu_P(\pi_P(A)) = \mathbb{Q}(A).$$

Since this holds for any compact subset of  $\Omega$ , and since both  $\mathbb{P}$  and  $\mathbb{Q}$  are Radon, this implies that  $\mathbb{P}$  coincides with the restriction  $\mathbb{Q}|_{\Omega}$ . But both  $\mathbb{P}$  and  $\mathbb{Q}$  are probability measures. Therefore  $\mathbb{Q}(\partial\Omega) = 1 - \mathbb{Q}(\Omega) = 1 - \mathbb{P}(\Omega) = 0$ .

Conversely, if  $\mathbb{Q}(\partial\Omega) = 0$ , it is clear that  $\mathbb{P}$  defined as the restriction  $\mathbb{P} = \mathbb{Q}|_{\Omega}$  is a Radon probability satisfying (6). The fact that  $\mathbb{P}$  is unique follows the same argument as above. □

5.3. *Extensions of probabilities for general event structures: second approach, using infinite state spaces*

If we allow possibly infinite state spaces, a simpler extension result can be formulated. Applications of the following theorem are given in Abbes and Benveniste (2006) for locally finite event structures.

**Theorem 5.3.** Let  $(\mu_B)_{B \in \mathcal{D}}$  be a projective system of Radon probability measures defined on the projective system  $(\Omega_B)_{B \in \mathcal{D}}$ . Then there is a unique Radon probability measure  $\mathbb{P}$  on  $(\Omega, \mathfrak{F})$  such that

$$\forall B \in \mathcal{D}, \quad \pi_B \mathbb{P} = \mu_B.$$

In particular, if  $\mathcal{E}$  is locally finite, every projective system of (finite) probabilities on  $(\Omega_B)_{B \in \mathcal{D}}$  extends to a unique Radon probability measure on  $\Omega$ .

*Proof.* Since  $\mathcal{D}$  admits a cofinal sequence, this is an immediate consequence of Theorem 4.8 and of the Prokhorov theorem (Theorem 2.1). □

5.4. *An example*

Theorem 5.2 suffers from two defects when it comes to practical applications. First, it is formulated through the use of the sets  $\Gamma_P$ , which are the trace of  $\Omega$  in a finite prefix  $P$ . Determining  $\Gamma_P$  might not be easy since  $\Gamma_P$  depends on both  $P$  and the surrounding event structure  $\mathcal{E}$ . Second, it is not easy to check the condition  $\mathbb{Q}(\partial\Omega) = 0$  either.

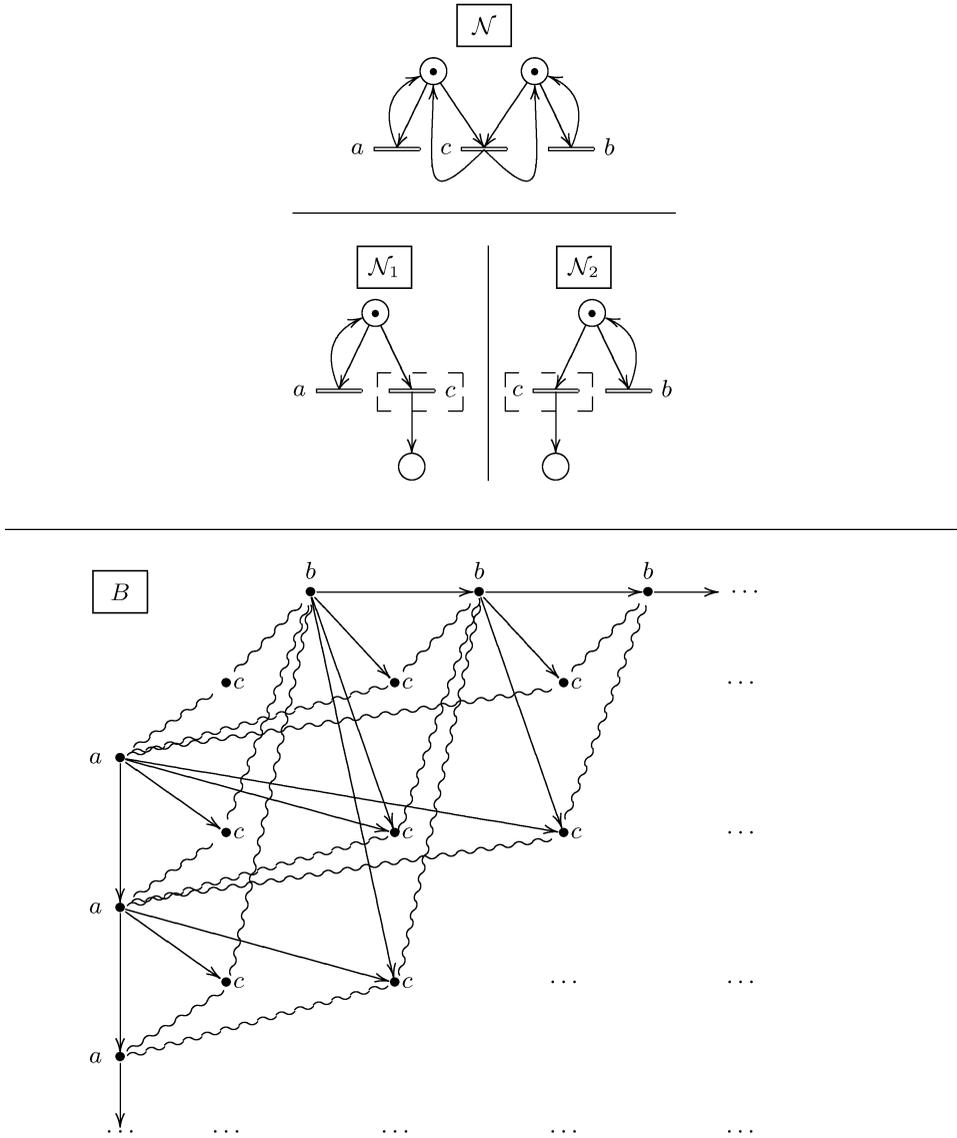


Figure 4. The top diagram shows a Petri net  $\mathcal{N}$ : the ‘live double loop’. The bottom diagram shows an (infinite) elementary stopping prefix  $B$  of the unfolding of  $\mathcal{N}$ . The middle diagrams show two Petri nets whose synchronisation product unfolds to  $B$ .

So Theorem 5.3 appears more practical, although it works with possibly infinite spaces, as shown in the following example. Consider the ‘live double loop’  $\mathcal{N}$  shown at the top of Figure 4. The unfolding of  $\mathcal{N}$  is obtained as follows: start from the event structure  $B$  shown at the bottom of Figure 4. After each event labelled by  $c$ , add a fresh copy of  $B$ , whose minimal nodes are  $\succeq$ -related to the latter  $c$  event; then repeat the process

infinitely. The unfolding  $\mathcal{E}$  is thus a regular tree with basis  $B$ , which is an elementary (infinite) stopping prefix. Theorem 5.3 reduces the construction of a probability  $\mathbb{P}$  on  $\Omega$  to the construction of a probability  $\mathbb{P}_B$  on  $\Omega_B$ .

Note that in this example the space  $\Omega$  of maximal configurations is *not* locally compact since no element of  $\Omega$  has a compact neighbourhood.

## 6. Conclusion

This paper has introduced projective systems for studying the true-concurrency model of event structures from a topological point of view and with probabilistic applications. We have underlined the role of compactness for the space  $\Omega$  of maximal configurations as a necessary and sufficient condition for  $\Omega$  to be a projective limit of finite sets. Alternatively,  $\Omega$  can always be described as the projective limit of its traces in elementary stopping prefixes. The Prokhorov extension theorem applies, and provides extension theorems for probabilistic event structures.

The extension theorem for locally finite event structures was shown in Abbes and Benveniste (2006) to be operational as a basis for the construction of true-concurrent random processes. In the case of infinite state spaces, the construction of atomic probabilities seems to be connected with the construction of synchronous products of event structures and Petri nets. For a general theory of random true-concurrent processes with communicating channels, such results certainly need to be explored deeply.

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