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THE NUMBER OF HECKE EIGENVALUES OF SAME SIGNS

Y.-K. LAU & J. WU

ABSTRACT. We give the best possible lower bounds in order of magnitude for the number of positive and negative Hecke eigenvalues. This improves upon a recent work of Kohnen, Lau & Shparlinski. Also, we study an analogous problem for short intervals.

1. INTRODUCTION

Let $k \geq 2$ be an even integer and $N \geq 1$ be squarefree. Among all holomorphic cusp forms of weight k for the congruence subgroup $\Gamma_0(N)$, there are finitely many of them whose Fourier coefficients in the expansion at the cusp ∞ ,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \quad (\Im z > 0),$$

are the Hecke eigenvalues. Up to scalar multiples, these forms are the only simultaneous eigenfunctions of all Hecke operators. We call them the primitive forms, and write $H_k^*(N)$ for the set of all primitive forms of weight k for $\Gamma_0(N)$. One central problem in modular form theory is to study the Hecke eigenvalues $\lambda_f(n)$. (We omit the factor $n^{(k-1)/2}$ to avoid its uneven amplifying effect.) Classically it is known that the arithmetical function $\lambda_f(n)$ is real multiplicative, and verifies Deligne's inequality

$$(1.1) \quad |\lambda_f(n)| \leq d(n)$$

for all $n \geq 1$, where $d(n)$ is the divisor function. Furthermore we have

$$(1.2) \quad \lambda_f(p^\nu) = \lambda_f(p)^\nu \quad \text{and} \quad \lambda_f(p) = \varepsilon_f(p)/\sqrt{p}$$

for all primes $p \mid N$ and integers $\nu \geq 1$, where $\varepsilon_f(p) \in \{\pm 1\}$. (See [5] and [10].) The distribution of the Hecke eigenvalues $\lambda_f(n)$ is delicate. The Lang-Trotter conjecture concerns the frequency of $\lambda_f(p)$ taking a value in the admissible range where p runs over primes. This conjecture is still open but there are progress made on itself or the pertinent questions, for instance, [6], [18], [16], [17], [2], [4], [15], etc. In this regard, various techniques and tools are applied, such as ℓ -adic representations, Chebotarev density theorem, sieve-theoretic arguments, Rankin-Selberg L -functions and the method of \mathcal{B} -free numbers. In [15], Kowalski, Robert & Wu investigated the nonvanishing problem and gave the sharpest upper estimate to-date on the gaps between consecutive nonzero Hecke eigenvalues. Another wide belief is Sato-Tate's conjecture, asserting that $\lambda_f(p)$'s are equidistributed on $[-2, 2]$ with respect to the Sato-Tate measure.

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In this paper, we are concerned with the Hecke eigenvalues of the same sign. Kohnen, Lau & Shparlinski [14, Theorem 1] proved

$$(1.3) \quad \mathcal{N}_f^\pm(x) := \sum_{\substack{n \leq x, (n, N) = 1 \\ \lambda_f(n) \geq 0}} 1 \gg_f \frac{x}{(\log x)^{17}}$$

for $x \geq x_0(f)$.[†] Very recently Wu [21, Corollary] improved this result by reducing the exponent 17 to $1 - 1/\sqrt{3}$, as a simple application of his estimates on power sums of Hecke eigenvalues. The exponent $1 - 1/\sqrt{3}$ can be improved to $2 - 16/(3\pi)$ if one assumes Sato-Tate's conjecture.

Our first result is to remove the logarithmic factor by the \mathcal{B} -free number method, which is the best possible in order of magnitude.

Theorem 1. *Let $f \in \mathbb{H}_k^*(N)$. Then there is a constant x_0 such that the inequality*

$$(1.4) \quad \mathcal{N}_f^\pm(x) \gg_f x$$

holds for all $x \geq x_0$.

Remarks. 1. It is clear from the proof that our method gives the stronger result

$$\sum_{\substack{n \leq x, (n, N) = 1 \\ n \text{ squarefree}, \lambda_f(n) \geq 0}} 1 \gg_f x$$

for every $x \geq x_0(f)$.

2. The method is robust and applies to, for example, modular forms of half-integral weight. We return to this problem in another occasion.

By coupling (1.3) with Alkan & Zaharescu's result in [1, Theorem 1], it is shown in [14, Theorem 2] (see also [13, Theorem 3.4]) that there are absolute constants $\eta < 1$ and $A > 0$ such that for any $f \in \mathbb{H}_k^*(N)$ the inequality

$$(1.5) \quad \mathcal{N}_f^\pm(x + x^\eta) - \mathcal{N}_f^\pm(x) > 0$$

holds for $x \geq (kN)^A$, but no explicit value of η is evaluated. Apparently it is interesting and important to know how small η can be, in order for a better understanding of the local behaviour. A direct consequence of (1.5) is that $\lambda_f(n)$ has a sign-change in a short interval $[x, x + x^\eta]$ for all sufficiently large x . The sign-change problem was explored in [11], [14], [21] on different aspects. Here we prove that there are plenty of eigenvalues of the same signs in intervals of length about $x^{1/2}$. More precisely, we have the following.

Theorem 2. *Let $f \in \mathbb{H}_k^*(N)$. There is an absolute constant $C > 0$ such that for any $\varepsilon > 0$ and all sufficiently large $x \geq N^2 x_0(k)$, we have*

$$(1.6) \quad \mathcal{N}_f^\pm(x + C_N x^{1/2}) - \mathcal{N}_f^\pm(x) \gg_\varepsilon (Nx)^{1/4 - \varepsilon},$$

where

$$C_N := CN^{1/2} \Psi(N)^3, \quad \Psi(N) := \sum_{d|N} d^{-1/2} \log(2d)$$

and $x_0(k)$ is a suitably large constant depending on k and the implied constant in \gg_ε depends only on ε .

[†]It is worthy to indicate that they gave explicit values for the implied constant in \gg and $x_0(f)$.

The result in Theorem 2 is uniform in the level N , and its method of proof is based on Heath-Brown & Tsang [8]. The exponent of $\Psi(N)$ in C_N can be easily reduced to any number bigger than $3/2$, which however may not be essential as $\Psi(N)$ is already very small - $\log \Psi(N) = o(\sqrt{\log N})$. The range of $x \geq N^2 x_0(k)$ can also be refined to $x \geq N^{1+\varepsilon} k^A$ for some constant $A > 0$, but we save our effort.

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2. PROOF OF THEOREM 1

Let p' be the least prime such that $p' \nmid N$ and $\lambda_f(p') < 0$.[‡] Introduce the set

$$\begin{aligned} \mathcal{B} &= \{p : \lambda_f(p) = 0\} \cup \{p : p \mid N\} \cup \{p'\} \cup \{p^2 : p \nmid p'N \text{ and } \lambda_f(p) \neq 0\} \\ &= \{b_i\}_{i \geq 1} \quad (\text{with increasing order}). \end{aligned}$$

By virtue of Serre's estimate [18, (181)]:

$$|\{p \leq x : \lambda_f(p) = 0\}| \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}}$$

for $x \geq 2$ and any $\delta < \frac{1}{2}$, we infer that

$$\sum_{i \geq 1} 1/b_i < \infty \quad \text{and} \quad (b_i, b_j) = 1 \quad (i \neq j).$$

Let $\mathcal{A} := \{a_i\}_{i \geq 1}$ (with increasing order) be the sequence of all \mathcal{B} -free numbers, i.e. the integers indivisible by any element in \mathcal{B} . According to [7], \mathcal{A} is of positive density

$$(2.1) \quad \lim_{x \rightarrow \infty} \frac{|\mathcal{A} \cap [1, x]|}{x} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{b_i}\right) > 0.$$

From the definition of \mathcal{B} and the multiplicativity of $\lambda_f(n)$, we have $\lambda_f(a) \neq 0$ for all $a \in \mathcal{A}$. Then we partition

$$\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-,$$

where

$$\mathcal{A}^{\pm} := \{a_i \in \mathcal{A} : \lambda_f(a_i) \gtrless 0\}.$$

Without control on the sizes of \mathcal{A}^{\pm} , we construct a set from $\mathcal{A}^+ \cup \mathcal{A}^-$ such that the sign of $\lambda_f(a)$ is switched on the counterpart. Consider

$$\mathcal{N}^{\pm} := \mathcal{A}^{\pm} \cup \{a_i p' : a_i \in \mathcal{A}^{\mp}\}.$$

[‡]According to [11], we have $p' \ll (k^2 N)^{29/60}$.

Clearly $\lambda_f(a) \geq 0$ and $(a, N) = 1$ for all $a \in \mathcal{N}^\pm$ and

$$\mathcal{N}_f^\pm(x) \geq |\mathcal{N}^\pm \cap [1, x]| \geq |\mathcal{A} \cap [1, x/p']|$$

for all $x \geq 1$. The desired result follows with the inequality (2.1).

3. PROOF OF THEOREM 2

The method of proof is based on the investigation of

$$S_f^*(x) := \sum_{n \leq x, (n, N) = 1} \lambda_f(n).$$

Since the L -function associated to f is belonged to the Selberg class and of degree 2, we apply the standard complex analysis to derive truncated Voronoi formulas for $S_f^*(x)$.

Lemma 3.1. *Let $f \in H_k^*(N)$. Then for any $A > 0$ and $\varepsilon > 0$, we have*

$$(3.1) \quad \begin{aligned} S_f^*(x) &= \frac{\eta_f}{\pi\sqrt{2}} (Nx)^{1/4} \sum_{d|N} \frac{(-1)^{\omega(d)} \lambda_f(d)}{d^{1/4}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{dN}} - \frac{\pi}{4}\right) \\ &\quad + O\left(N^{1/2} \left\{1 + \left(\frac{x}{M}\right)^{1/2} + \left(\frac{N}{x}\right)^{1/4}\right\} (Nx)^\varepsilon\right) \end{aligned}$$

uniformly for $1 \leq M \leq x^A$ and $x \geq N^{1+\varepsilon}$, where $\eta_f = \pm 1$ depends on f and the implied O -constant depends on A, ε and k only. The function $\omega(d)$ counts the number of all distinct prime factors of d .

Remark. The case $N = 1$ and $A = 1$ of (3.1) is covered in [12, Theorem 1.1] with $h = k = 1$ therein. Our proof follows closely Section 3.2 of [9], and we first evaluate the case without the constraint $(n, N) = 1$: for any $A > 0$ and $\varepsilon > 0$, we have uniformly in $1 \leq M \leq x^A$,

$$(3.2) \quad \begin{aligned} S_f(x) &:= \sum_{n \leq x} \lambda_f(n) \\ &= \frac{\eta_f (Nx)^{1/4}}{\pi\sqrt{2}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) \\ &\quad + O\left(N^{1/2} \left\{1 + \left(\frac{x}{M}\right)^{1/2} + \left(\frac{N}{x}\right)^{1/4}\right\} (Nx)^\varepsilon\right). \end{aligned}$$

Proof. As usual, denote by $\mu(N)$ the Möbius function. (3.1) follows from (3.2) because

$$(3.3) \quad \begin{aligned} S_f^*(x) &= \sum_{d|N} \mu(d) \sum_{n \leq x/d} \lambda_f(dn) \\ &= \sum_{d|N} (-1)^{\omega(d)} \lambda_f(d) \sum_{n \leq x/d} \lambda_f(n) \end{aligned}$$

by the multiplicativity of $\lambda_f(n)$ and the first equality in (1.2). Note that $x/d \geq x^{\varepsilon/(1+\varepsilon)}$ when $x \geq N^{1+\varepsilon}$ and $d|N$, we can keep the same range of M for all inner sums over n by selecting a suitable A . Inserting (3.2) into (3.3), the main term of (3.1) comes up immediately. The effect of summing the O -terms over $d|N$ is negligible in light of the second formula in (1.2), and hence the result.

To prove (3.2), we consider $M \in \mathbb{N}$ without loss of generality. As usual write

$$L(s, f) := \sum_{n \geq 1} \lambda_f(n) n^{-s} \quad (\Re s > 1).$$

Let $\kappa := 1 + \varepsilon$ and $T > 1$ be a parameter, chosen as

$$(3.4) \quad T^2 = \frac{4\pi^2(M + \frac{1}{2})x}{N}.$$

By the truncated Perron formula (see [20, Corollary II.2.4] with the choice of $\sigma_a = 1$, $\alpha = 2$ and $B(n) = C_\varepsilon n^\varepsilon$), we have

$$(3.5) \quad S_f(x) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} L(s, f) \frac{x^s}{s} ds + O\left(N^{1/2} \left\{ \left(\frac{x}{M}\right)^{1/2} + 1 \right\} (Nx)^\varepsilon\right).$$

We shift the line of integration horizontally to $\Re s = -\varepsilon$, the main term gives

$$(3.6) \quad \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} L(s, f) \frac{x^s}{s} ds = L(0, f) + \frac{1}{2\pi i} \int_{\mathcal{L}} L(s, f) \frac{x^s}{s} ds,$$

where \mathcal{L} is the contour joining the points $\kappa \pm iT$ and $-\varepsilon \pm iT$. Using the convexity bound

$$L(\sigma + it, f) \ll (\sqrt{N}(k + |t|))^{\max\{0, 1 - \sigma\} + \varepsilon} \quad (-\varepsilon \leq \sigma \leq \kappa),$$

the integrals over the horizontal segments and the term $L(0, f)$ can be absorbed in $O((NTx)^\varepsilon(N^{1/2} + T^{-1}x))$. The O -constant depends on k and ε , and in the sequel, such a dependence in implied constants will be tacitly allowed.

To handle the integral over the vertical segment $\mathcal{L}_v := [-\varepsilon - iT, -\varepsilon + iT]$, we invoke the functional equation

$$\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s, f) = i^k \eta_f \left(\frac{\sqrt{N}}{2\pi}\right)^{1-s} \Gamma\left(1 - s + \frac{k-1}{2}\right) L(1 - s, f)$$

where $\eta_f := \mu(N)\lambda_f(N)\sqrt{N} \in \{\pm 1\}$ (see [10, p.375] with an obvious change of notation). Then we deduce that

$$(3.7) \quad \frac{1}{2\pi i} \int_{\mathcal{L}_v} L(s, f) \frac{x^s}{s} ds = i^k \eta_f \sum_{n \geq 1} \frac{\lambda_f(n)}{n} I_{\mathcal{L}_v}(nx),$$

where

$$I_{\mathcal{L}_v}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v} \left(\frac{4\pi^2}{N}\right)^{s-1/2} \frac{\Gamma(1 - s + (k-1)/2)}{\Gamma(s + (k-1)/2)} \frac{y^s}{s} ds.$$

The quotient of the two gamma factors is

$$|t|^{1-2\sigma} e^{-2i(t \log |t| - t) + i \operatorname{sgn}(t) \pi(k-1)/2} \{1 + O(t^{-1})\}$$

for bounded σ and any $|t| \geq 1$, where the implied constant depends on σ and k . Together with the second mean value theorem for integrals (see [20], Theorem I.0.3), we obtain

$$(3.8) \quad \begin{aligned} I_{\mathcal{L}_v}(nx) &\ll N^{1/2} \left(\frac{N}{nx}\right)^\varepsilon \left(\left| \int_1^T t^{2\varepsilon} e^{-ig(t)} dt \right| + T^{2\varepsilon} \right) \\ &\ll N^{1/2} \left(\frac{NT^2}{nx}\right)^\varepsilon \left(\left| \int_a^b e^{-ig(t)} dt \right| + 1 \right) \end{aligned}$$

for some $1 \leq a \leq b \leq T$, where $g(t) := t \log(Nt^2/(4\pi^2 nx)) - 2t$. In view of (3.4), we have

$$g'(t) = -\log(4\pi^2 nx/(Nt^2)) < 0 \quad \text{and} \quad |g'(t)| \geq |\log(n/(M + \frac{1}{2}))|$$

for $n \geq M + 1$ and $1 \leq t \leq T$. Using (1.1) and [20, Theorem I.6.2], we infer that

$$(3.9) \quad \begin{aligned} \sum_{n>M} \frac{\lambda_f(n)}{n} I_{\mathcal{L}_v}(nx) &\ll N^{1/2} \left(\frac{NT^2}{x}\right)^\varepsilon \sum_{n>M} \frac{d(n)}{n^{1+\varepsilon}} \left(\left| \log \frac{n}{M + \frac{1}{2}} \right|^{-1} + 1 \right) \\ &\ll N^{1/2} \left(\frac{NT^2}{x}\right)^\varepsilon \left\{ \sum_{M < n \leq 2M} \frac{d(n)(M + \frac{1}{2})}{n^{1+\varepsilon} |n - M - \frac{1}{2}|} + \frac{1}{M^{\varepsilon/2}} \right\} \\ &\ll N^{1/2} \left(\frac{NT^2}{\sqrt{Mx}}\right)^\varepsilon \\ &\ll N^{1/2} (Nx)^\varepsilon. \end{aligned}$$

For $n \leq M$, we extend the segment of integration \mathcal{L}_v to an infinite line \mathcal{L}_v^* in order to apply Lemma 1 in [3]. Write

$$\mathcal{L}_v^\pm := \left[\frac{1}{2} + \varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm i\infty\right), \quad \mathcal{L}_h^\pm := [-\varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm iT]$$

and define \mathcal{L}_v^* to be the positively oriented contour consisting of \mathcal{L}_v , \mathcal{L}_v^\pm and \mathcal{L}_h^\pm . The contribution over the horizontal segments \mathcal{L}_h^\pm is

$$\begin{aligned} I_{\mathcal{L}_h^\pm}(nx) &\ll \int_{-\varepsilon}^{1/2-\varepsilon} \left(\frac{4\pi^2}{N}\right)^{\sigma-1/2} T^{1-2\sigma} \frac{(nx)^\sigma}{T} d\sigma \\ &\ll N^{1/2} \int_{-\varepsilon}^{1/2-\varepsilon} \left(\frac{nx}{NT^2}\right)^\sigma d\sigma \\ &\ll N^{1/2} (Nx)^\varepsilon. \end{aligned}$$

As in (3.8), for $n \leq M$ we get that

$$\begin{aligned} I_{\mathcal{L}_v^\pm}(nx) &\ll N^{1/2} \left(\frac{nx}{N}\right)^{1/2+\varepsilon} \left(\int_T^\infty t^{-1-2\varepsilon} e^{-ig(t)} dt + \frac{1}{T^{1+2\varepsilon}} \right) \\ &\ll N^{1/2} \left(\frac{nx}{NT^2}\right)^{1/2+\varepsilon} \left(\left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right) \\ &\ll N^{1/2} \left(\left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right). \end{aligned}$$

So

$$(3.10) \quad \sum_{n \leq M} \frac{\lambda_f(n)}{n} (I_{\mathcal{L}_v^\pm}(nx) + I_{\mathcal{L}_h^\pm}(nx)) \ll \sum_{n \leq M} \frac{d(n)}{n} (|I_{\mathcal{L}_v^\pm}(nx)| + |I_{\mathcal{L}_h^\pm}(nx)|) \\ \ll N^{1/2} (Nx)^\varepsilon.$$

Now all the poles of the integrand in

$$I_{\mathcal{L}_v^*}(y) = \frac{\sqrt{N}}{2\pi} \frac{1}{2\pi i} \int_{\mathcal{L}_v^*} \frac{\Gamma(1-s + (k-1)/2)\Gamma(s)}{\Gamma(s + (k-1)/2)\Gamma(1+s)} \left(\frac{4\pi^2 y}{N}\right)^s ds$$

lie on the right of the contour \mathcal{L}_v^* . After a change of variable s into $1-s$, we see that

$$I_{\mathcal{L}_v^*}(y) = \frac{\sqrt{N}}{2\pi} I_0\left(\frac{4\pi^2 y}{N}\right),$$

with

$$I_0(t) := \frac{1}{2\pi i} \int_{\mathcal{L}_\varepsilon} \frac{\Gamma(s + (k-1)/2)\Gamma(1-s)}{\Gamma(1-s + (k-1)/2)\Gamma(2-s)} t^{1-s} ds.$$

Here \mathcal{L}_ε consists of the line $s = \frac{1}{2} - \varepsilon + i\tau$ with $|\tau| \geq T$, together with three sides of the rectangle whose vertices are $\frac{1}{2} - \varepsilon - iT$, $1 + \varepsilon - iT$, $1 + \varepsilon + iT$ and $\frac{1}{2} - \varepsilon + iT$. Clearly our I_0 is a particular case of I_ρ defined in [3, Lemma 1], corresponding to the choice of parameters $\rho = 0$, $\delta = A = 1$, $\omega = 1$, $h = 2$, $k_0 = -(2k+1)/4$. It hence follows that

$$(3.11) \quad I_{\mathcal{L}_v^*}(nx) = \frac{i^k (nNx)^{1/4}}{\pi\sqrt{2}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) + O\left(\frac{N^{3/4+\varepsilon}}{(nx)^{1/4}}\right),$$

The value of e'_0 in Lemma 1 of [3] is $1/\sqrt{\pi}$ by direct computation. We conclude

$$(3.12) \quad \sum_{n \leq M} \frac{\lambda_f(n)}{n} I_{\mathcal{L}_v^*}(nx) = \frac{i^k (Nx)^{1/4}}{\pi\sqrt{2}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{N}} - \frac{\pi}{4}\right) \\ + O\left(N^{1/2} \left\{ \left(\frac{N}{x}\right)^{1/4} + 1 \right\} (Nx)^\varepsilon\right),$$

from (3.10) and (3.11), and finally the asymptotic formula (3.2) by (3.5)-(3.7), (3.9) and (3.12). \square

Following Theorem 1 of [8], we have the next lemma.

Lemma 3.2. *Let $f \in H_k^*(N)$. There exist positive absolute constants C, c_1, c_2 such that for all sufficiently large $X \geq N^2 X_0(k)$, we can find $x_1, x_2 \in [X, X + C_N X^{1/2}]$ for which*

$$S_f^*(x_1) > c_1 (NX)^{1/4} \quad \text{and} \quad S_f^*(x_2) < -c_2 (NX)^{1/4},$$

where $C_N := CN^{1/2}\Psi(N)^3$ and $X_0(k)$ is a constant depending only on k . The same result also holds for $S_f(x)$.

Proof. Define

$$K_\tau(u) := (1 - |u|)(1 + \tau \cos(4\pi\alpha u)),$$

where $\tau = 1$ or -1 and α is a (large) parameter, both chosen at our disposal. Consider the following integral

$$r_\beta = r_\beta(\alpha, \tau, t) := \int_{-1}^1 K_\tau(u) \cos\left(4\pi(t + \alpha u)\sqrt{\beta} - \frac{\pi}{4}\right) du,$$

where $t \in \mathbb{N}$ and $\beta > 0$. Because

$$w(\xi) := \int_{-1}^1 (1 - |u|)e^{i2\pi\xi u} du = \begin{cases} 1 & \text{if } \xi = 0, \\ O(\min(1, \xi^{-2})) & \text{if } \xi \neq 0, \end{cases}$$

we can write, with the notation $\alpha_\beta := 2\alpha\sqrt{\beta}$ and $\alpha_\beta^\pm := 2\alpha(\sqrt{\beta} \pm 1)$,

$$\begin{aligned} (3.13) \quad r_\beta &= \int_{-1}^1 (1 - |u|) \left(1 + \tau \frac{e^{i4\pi\alpha u} + e^{-i4\pi\alpha u}}{2}\right) \Re e^{i\{4\pi(t+\alpha u)\sqrt{\beta} - \pi/4\}} du \\ &= \Re e^{i(4\pi t\sqrt{\beta} - \pi/4)} \int_{-1}^1 (1 - |u|) \left(e^{i2\pi\alpha_\beta u} + \frac{\tau}{2}e^{i2\pi\alpha_\beta^+ u} + \frac{\tau}{2}e^{i2\pi\alpha_\beta^- u}\right) du \\ &= \left(w(\alpha_\beta) + \frac{\tau}{2}w(\alpha_\beta^+) + \frac{\tau}{2}w(\alpha_\beta^-)\right) \cos\left(4\pi t\sqrt{\beta} - \frac{\pi}{4}\right) \\ &= \delta_{\beta=1} \frac{\tau}{2\sqrt{2}} + O\left(\min\left(1, \frac{1}{\alpha^2\beta}\right) + \delta_{\beta \neq 1} \min\left(1, \frac{1}{(\alpha_\beta^-)^2}\right)\right), \end{aligned}$$

where the O -constant is absolute,

$$\delta_{\beta=1} := \begin{cases} 1 & \text{if } \beta = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta_{\beta \neq 1} := 1 - \delta_{\beta=1}.$$

The last error term in (3.13) appears only when $\beta \neq 1$.

For all $X \geq N^2 X_0(k)$ (whose value will be specified below), we write $T = (X/N)^{1/2}$ and $t = [T] + 1 \in \mathbb{N}$, and consider the convolution

$$J_\tau = \int_{-1}^1 F_f(t + \alpha u) K_\tau(u) du,$$

where

$$F_f(t + \alpha u) := \frac{\pi\sqrt{2} S_f^*(N(t + \alpha u)^2)}{\eta_f \sqrt{N(t + \alpha u)}}.$$

By Lemma 3.1 with $M = NT^2 = X$, we deduce that

$$F_f(t + \alpha u) = \sum_{d|N} \frac{(-1)^{\omega(d)} \lambda_f(d)}{d^{1/4}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi(t + \alpha u)\sqrt{\frac{n}{d}} - \frac{\pi}{4}\right) + O_k\left(\frac{1}{T^{1/4}}\right),$$

and

$$(3.14) \quad J_\tau = \sum_{d|N} \frac{(-1)^{\omega(d)} \lambda_f(d)}{d^{1/4}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} r_{n/d} + O_k\left(\frac{1}{T^{1/4}}\right)$$

by (1.2).

Next we estimate the contribution of the O -term in (3.13) to J_τ . Using (1.2) and (1.1) again, its contribution to J_τ is

$$(3.15) \quad \ll \sum_{d|N} \frac{1}{d^{3/4}} \left\{ \sum_{\substack{n \leq M \\ n \neq d}} \frac{d(n)}{n^{3/4}} R'_{d,n}(\alpha) + \sum_{\substack{n \leq M \\ n \neq d}} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \right\},$$

where

$$R'_{d,n}(\alpha) := \min \left(1, \frac{d}{\alpha^2 n} \right), \quad R''_{d,n}(\alpha) := \min \left(1, \frac{d}{\alpha^2 |\sqrt{n} - \sqrt{d}|^2} \right).$$

Consider the second sum in the curly braces. We separate n into

$$n \leq \alpha_- d, \quad \alpha_- d < n < \alpha_+ d \quad \text{or} \quad \alpha_+ d \leq n$$

where $\alpha_\pm := (1 - \alpha^{-1/2})^{\mp 2}$, and $R''_{d,n}(\alpha)$ is $\leq 1/\alpha$, 1 or $d/(\alpha n)$ accordingly. Therefore,

$$\sum_{\substack{n \leq M \\ n \neq d}} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \leq \frac{1}{\alpha} \sum_{n \leq \alpha_- d} \frac{d(n)}{n^{3/4}} + \sum_{\substack{\alpha_- d < n < \alpha_+ d \\ n \neq d}} \frac{d(n)}{n^{3/4}} + \frac{d}{\alpha} \sum_{n > \alpha_+ d} \frac{d(n)}{n^{7/4}}.$$

Obviously the first and last terms on the right-hand side are $\ll \alpha^{-1} d^{1/4} \log(2d)$. Note that $n \asymp d$ in the second sum. So, by using Shiu's Theorem 2 in [19] it follows

$$\begin{aligned} \sum_{\substack{\alpha_- d < n < \alpha_+ d \\ n \neq d}} \frac{d(n)}{n^{3/4}} &\ll d^{-3/4} \sum_{\substack{\alpha_- d < n < \alpha_+ d \\ n \neq d}} d(n) \\ &\ll \alpha^{-1/2} d^{1/4} \log(2d) \end{aligned}$$

if $d > \alpha$. Otherwise (i.e. $d \leq \alpha$), pulling out $d(n) \ll n^\varepsilon \ll d^\varepsilon \ll \alpha^\varepsilon$, we have

$$\begin{aligned} \sum_{\substack{\alpha_- d < n < \alpha_+ d \\ n \neq d}} d(n) n^{-3/4} &\ll \alpha^\varepsilon d^{-3/4} \sum_{\substack{\alpha_- d < n < \alpha_+ d \\ n \neq d}} 1 \\ &\ll \alpha^\varepsilon d^{-3/4} \alpha^{-1/2} d \\ &\ll \alpha^{-1/3} d^{1/4} \log(2d). \end{aligned}$$

(We can assume that $(\alpha_+ - \alpha_-)d \geq \alpha^{-1/2}d \geq c'$ for a small constant c' , otherwise the last sum is empty.) Hence

$$\sum_{\substack{n \leq M \\ n \neq d}} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \ll \alpha^{-1/3} d^{1/4} \log(2d).$$

The first sum in the bracket of (3.15) can be treated in the same fashion (even more easily). Thus, (3.15) is bound by

$$\ll \alpha^{-1/3} \sum_{d|N} \frac{\log(2d)}{d^{1/2}} =: \alpha^{-1/3} \Psi(N).$$

We conclude from (3.14) with (3.13) and (1.2) that

$$J_\tau = \frac{\tau}{2\sqrt{2}} \sum_{d|N} \frac{(-1)^{\omega(d)}}{d^2} + O\left(\frac{\Psi(N)}{\alpha^{1/3}}\right) + O_k\left(\frac{1}{T^{1/4}}\right),$$

where the implied constant is absolute in the first O -term, but depends on k in the second. Noticing that

$$\sum_{d|N} \frac{(-1)^{\omega(d)}}{d^2} = \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \geq \frac{6}{\pi^2}$$

and $T \geq \sqrt{NX_0(k)}$, we take $\alpha = C\Psi(N)^3$ with a large absolute constant C and a large $X_0(k)$ so that both O -terms $O(\alpha^{-1/3}\Psi(N))$ and $O_k(T^{-1/4})$ are $\leq \cos(\pi/4)/\pi^2 = 1/(\pi^2\sqrt{2})$. Therefore

$$J_{-1} < -1/(\pi^2\sqrt{2}) \quad \text{and} \quad J_1 > 1/(\pi^2\sqrt{2}).$$

With the nonnegativity of $K_\tau(u)$ and the estimate

$$1 - (2\pi\alpha)^{-2} \leq \int_{-1}^1 K_\tau(u) du \leq 2 \quad (\tau = \pm 1),$$

we have

$$2F_f(t + \alpha\eta_+) \geq 1/(\pi^2\sqrt{2}) \quad \text{and} \quad (1 - (2\pi\alpha)^{-2})F_f(t + \alpha\eta_-) \leq -1/(\pi^2\sqrt{2})$$

for some $\eta_+, \eta_- \in [-1, 1]$. Let $C_N = CN^{1/2}\Psi(N)^3$. As

$$X - 3C_N\sqrt{X} \leq N(t + \alpha\eta_\pm)^2 \leq X + 3C_N\sqrt{X},$$

our assertion follows from the definition of F_f and replacing $X - 3C_N\sqrt{X}$ by X . \square

Now we are ready to prove Theorem 2.

We exploit the consecutive sign changes of $S_f^*(x)$. Let $x \geq N^2X_0(k)$ where $X_0(k)$ takes the value as in Lemma 3.2. We apply Lemma 3.2 to the intervals $[x, x + C_Nx^{1/2}]$ and $[y, y + C_Ny^{1/2}]$ where $y = x + C_Nx^{1/2}$. Over each of the intervals, $S_f^*(x)$ attains in magnitude $(Nx)^{1/4}$ in both positive and negative directions. Hence, we can find three points $x < x_1 < x_2 < x_3 < x + 3C_Nx^{1/2}$ such that $S_f^*(x_i)$ ($i = 1, 2, 3$) takes alternate signs and their absolute values are $\gg (Nx)^{1/4}$. (Note that $2\sqrt{x} \geq \sqrt{x + C_N\sqrt{x}}$.) It follows that the two differences

$$S_f^*(x_2) - S_f^*(x_1) = \sum_{\substack{x_1 < n \leq x_2 \\ (n, N)=1}} \lambda_f(n)$$

and

$$S_f^*(x_3) - S_f^*(x_2) = \sum_{\substack{x_2 < n \leq x_3 \\ (n, N)=1}} \lambda_f(n)$$

have absolute values $\gg (Nx)^{1/4}$ but are of opposite signs. This implies (1.6), since for example, if

$$\sum_{\substack{a < n < b \\ (n, N)=1}} \lambda_f(n) < -c'(Nx)^{1/4}$$

for some constant $c' > 0$ and $b \ll x$, then we have

$$\begin{aligned} c'(Nx)^{1/4} &< \sum_{\substack{a < n < b, (n, N) = 1 \\ \lambda_f(n) < 0}} (-\lambda_f(n)) \\ &\ll x^\varepsilon \sum_{\substack{a < n < b, (n, N) = 1 \\ \lambda_f(n) < 0}} 1. \end{aligned}$$

This completes the proof of Theorem 2. \square

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