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Tree-width of hypergraphs and surface duality

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Abstract. In Graph Minor III, Robertson and Seymour conjecture that We prove that given a hypergraph H on a surface of Euler genus k, the tree-width of H^* is at most the maximum of $\mathrm{tw}(H) + 1 + k$ and the maximum size of a hyperedge of H^*.

1 Preliminaries

A *surface* is a connected compact 2-manyfold without boundaries. A surface Σ can be obtained, up to homeomorphism, by adding $k(\Sigma)$ "crosscaps" to the sphere. $k(\Sigma)$ is the *Euler genus* or just *genus* of the surface.

Let Σ be a surface. A graph $G = (V, E)$ on Σ is a drawing of a graph in Σ, i.e. each vertex v is an element of Σ, each edge e is an open curve between two vertices, and edges are pairwise disjoint. We only consider graphs up to homomorphism. A face of G is a connected component of $\Sigma \setminus G$. We denote by $V(G)$, $E(G)$ and $F(G)$ the vertex, edge and face sets of G. We only consider *2-cell* graphs, i.e. graph whose faces are homeomorphic to open discs. The Euler formula links the number of vertices, edges and faces of a graph G to the genus of the surface

$$|V(G)| - |E(G)| + |F(G)| = 2 - k(G).$$

The set $A(G) = V(G) \cup E(G) \cup F(G)$ of *atoms* of G is a partition of Σ. Two Atom x and y of G are *incident* if $x \cap \bar{y}$ or $y \cap \bar{x}$ is non empty, $\bar{z}$ being the closure of z. A *cut-edge* in a graph G on Σ is an edge e separates G, i.e. G intersects at least two connected components of $\Sigma \setminus \bar{e}$. As an example, if a planar graph G has a cut-vertex u, any loop on u that goes "around" a connected component of $G \setminus \{u\}$ is a cut-edge.

Let $G = (V \cup V_E, L)$ be a bipartite graph on Σ. The graph G can be seen as the incidence graph of a hypergraph. For each $v_e \in V_E$, merge v_e and its incident edges into a *hyperedge* e, and call v_e its *center*. Let E be the set of all *hyperedges*. A *hypergraph on* Σ is any such pair $H = (V, E)$. For brevity, we also say *edges* for hyperedges. We extend the notions of cut-edges, 2-cell graphs, atoms and incidence to hypergraphs. Moreover, since they naturally correspond to abstract graphs and hypergraphs, graph and hypergraph on surface inherit terminology

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from them. For example, we denote $|e|$ the number of vertices incident to a hyperedge e, and we denote $\alpha(H)$ the maximum size of an edge of H. Note that a graph on Σ is also a hypergraph on Σ.

The dual of a hypergraph $H = (V, E)$ on Σ is obtained by choosing a vertex v_f for every face f of H. For every edge e of center v_e, we pick up an edge e^* as follows: choose a local orientation of the surface around v_e. This local orientation induces a cyclic order v_1, f_1, v_2, f_2, ..., v_d, f_d of the ends of e and of the faces incident with e (possibly with repetition). The edge e^* is the edge obtained by "rotating" e and whose ends are v_{f_1}, ..., v_{f_d}.

A *tree-decomposition* of a hypergraph H on Σ is a pair $\mathcal{T} = (T, (X_v)_{v\in V(T)})$ with T a tree and $(X_v)_{v\in V(T)}$ a family of *bags* such that:

i. $\bigcup_{v\in V(T)} X_v = H$;
ii. $\forall x,\ y,\ z \in V(T)$ with y on the path from x to z, $X_x \cap X_z \subseteq X_y$.

The *width* of $\mathcal{T}$ is $\mathrm{tw}(\mathcal{T}) = \max\big(|V(X_t)| - 1\ ;\ t \in V(T)\big)$ and the *tree-width* $\mathrm{tw}(H)$ of H is the minimum width of one of its tree-decompositions.

Tree-width was introduced by Robertson and Seymour in connection with graph minors. In [RS84], they conjectured that for a planar graph G, $\mathrm{tw}(G)$ and $\mathrm{tw}(G^*)$ differ by at most one. In an unpublished paper, Lapoire [Lap96] proves a more general result: for any hypergraph H in an orientable surface Σ, $\mathrm{tw}(H^*) \leq \max(\mathrm{tw}(H) + 1 + k(\Sigma), \alpha(H^*) - 1)$. Nevertheless, his proof is rather long and technical. Later, Bouchitté et al. [BMT03] gave an easier proof for planar graphs. Here we generalises Lapoire's result to arbitrary surfaces while being less technical.

To avoid technicalities, we suppose that H is connected, contains at least two edges, has no pending vertices (i.e. vertices incident with only one edge) and no cut-edge.

2 P-trees and duality

From now on, $H = (V, E)$ is a hypergraph on a surface Σ. The *border* of a partition μ of E is the set of vertices $\delta(\mu)$ that are incident with edges in at least two parts of μ, and the border of $X \subseteq E$ is the border of the partition $\{X, E\setminus X\}$. A partition $\mu = \{X_1, \ldots, X_p\}$ of E is *connected* if there is a *connecting partition* $\{V_1, X_1, F_1, \ldots, V_p, X_p, F_p\}$ of $A(H) \setminus \delta(\mu)$ so that each $V_i \cup X_i \cup F_i$ is connected in Σ.

A *p-tree* of H is a tree T whose internal nodes have degree three and whose leaves are labelled with the edges of H in a bijective way. Removing an internal node v of T results in a partition μ_v of E. Labelling each internal node v of T with $\delta(\mu_v)$, turns T into a tree-decomposition. The *tree-width* of a p-tree is its *tree-width*, seen as a tree-decomposition. A p-tree is *connected* if all its nodes partitions are connected.

Let $\{A, B\}$ be a connected bipartition of H and $\{V_A, A, F_A, V_B, B, F_B\}$ a corresponding connecting partition. We define a *contracted* hypergraph H/A as follows. Consider the incidence graph $G_H(V \cup V_E, L)$ of H, and identify the edges in A with their centers. By adding edges trough faces in F_A, we can make $G_H[A \cup V_A]$ connected. We then contract $A \cup V_A$ into a single edge center v_A. To make the resulting graph bipartite, we remove all v_A-loops. When removing a loop e incident to only one face F, the new face $F \cup e$ is not a disc but a crosscap. Since the border of $F \cup e$ is a loop, we can "cut" Σ along this loop and replace $F \cup e$ by an open disc while decreasing the genus of the surface. The obtained graph is the bipartite graph of H/A. A connected partition $\{A, B\}$ is *non trivial* if neither H/A nor H/B are equal to H.

We need the following folklore lemma:

Lemma 1. *For any connected bipartition* $\{A, B\}$ *of* H, $\mathrm{tw}(H) \leq \max\bigl(\mathrm{tw}(H/A), \mathrm{tw}(H/B)\bigr)$. *If* $\delta(\{A, B\})$ *belongs to a bag of an optimal tree-decomposition, then* $\mathrm{tw}(H) = \max\bigl(\mathrm{tw}(H/A), \mathrm{tw}(H/B)\bigr)$.

Let S be a set of vertices of H. An *S-bridge* is a minimal subset X of E with the property that $\delta(X) \subseteq S$. There are two kind of S-bridges: singletons containing an edge whose ends all belong to S and sets E_C containing all the edges incident to at least one vertex in C, a connected component of $G \setminus S$. The S-bridges partition E. We define the abstract graph $G_{/S}$ whose vertices are the S-bridges and in which $\{X, Y\}$ is an edge if there is a face incident with both an edge in X and an edge in Y. A key fact is that any bipartition $\{A, B\}$ of $V(G_{/S})$ such that $G_{/S}[A]$ and $G_{/S}[B]$ is connected corresponds to the connected bipartition $\{\cup A, \cup B\}$.

Proposition 1. *There exists a connected p-tree* T *of* H *with* $\mathrm{tw}(T) = \mathrm{tw}(H)$.

Proof. By induction on $|E|$, if $|E| \leq 3$, since H has no cut-edge, the only p-tree is connected and optimal. We can suppose that $|E| \geq 4$. We claim that there exists a connected non trivial bipartition $\{A, B\}$ of E whose border is contained in a bag of an optimal tree-decomposition of H. Two cases arise:

- If the trivial one vertex tree-decomposition whose bag is H is optimal, we consider the graph $G_{/V}$. Since they are in bijection with the edges of H, and since H has no cut edge, $G_{/V}$ has at least four vertices and no cut vertex. There thus exists a bipartition $\{A, B\}$ of $V(G_{/V})$ with $|A|, |B| \geq 2$, $G_{/V}[A]$ and $G_{/V}[B]$ connected which gives a connected non trivial bipartition of E.
- Otherwise, there exists a separator S contained in a bag of an optimal tree-decomposition of H. Let C and D be two connected component of $H \setminus S$, and S_C and S_C their corresponding S-bridges. Since H contains no pending vertex, $|S_C|, |S_D| \geq 2$. Let x and y be the vertices of $G_{/S}$ corresponding to S_C and S_D. Take a spanning tree of $G_{/S}$. Removing an edge between x and y leads to a connected non-trivial bipartition of E, which finishes the proof of the claim.

Since $\{A, B\}$ is connected, e_A and e_B are respectively not cut-edges in H/A and H/B. By induction, there exists connected p-trees $\mathcal{T}_A$ and $\mathcal{T}_B$ of optimal width of H/A and H/B. By removing the leaves labelled e_A and e_B and adding an edge between their respective neighbour, we obtain from $\mathcal{T}_A \sqcup \mathcal{T}_B$ a p-tree of

H which is connected. Its width is $\max(\mathrm{tw}(\mathcal{T}/A), \mathrm{tw}(\mathcal{T}/B))$ which is equal, by Lemma 1 to $\mathrm{tw}(H)$. $\square$

Because of the natural bijection between $E(H)$ and $E(H^*)$, a p-tree T of H also corresponds to a p-tree T^* of H^*.

Proposition 2. *For any connected p-tree T of H,*

$$\mathrm{tw}(T^*) \leq \max(\mathrm{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1).$$

Proof. Let v be a vertex of T labelled X_v in T and X_v^* in T^*. If v is a leaf, then $X_v^* = \{e^*\}$ and $|X_v^*| - 1 \leq \max(\mathrm{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1)$. Otherwise, let $\{A, B, C\}$ be the E-partition associated to v. The label of v in T and T^* is respectively $X_v = \delta(\{A, B, C\})$ and X_v^*, the set of faces incident with edges in at least two parts among A, B and C.

As for the proof of Proposition 1, since $\{A, B, C\}$ is connected, we may contract A (and B and C). But since we now care about the faces of H, we have to be more careful. We want an upper bound on $|X_v^*|$, we may thus add but not remove faces to X_v^*. So adding edges to make $G_H[A \cup V_A]$ connected is OK, but we cannot remove a loop e on say v_A incident with two faces in X_v^*. Instead, we cut Σ along e and fill the holes with open discs. While doing so, we removed e, we cut v_A in two *siblings*, and we decreased the genus of Σ.

After contracting A, B and C, we obtain a bipartite graph G_v on Σ' that has $|X_v| + 3 + s$ vertices with s the number of siblings, at least $|X_v^*|$ faces and with $k(\Sigma') \leq k(\Sigma) - s$. Since G_v is bipartite and faces in X_v^* are incident with at least 4 edges, $2|E(G_v)| = 4|F_4| + 6|F_6| + \cdots \geq 4|F(G_v)|$ with F_{2k} the set of $2k$-gones faces of G_v, and thus $|E(G_v)| \geq 2|F(G_v)|$. If we apply Euler's formula to G_v on Σ', we obtain: $|X_v| + 3 + s - |E(G_v)| + |F(G_v)| = 2 - k(\Sigma') \geq 2 - k(\Sigma) + s$. Adding this to $|E(G_v)| \geq 2|F(G_v)|$, we get $|X_v| + 1 + k(\Sigma) \geq |F(G_v)| \geq |X_v^*|$ which proves that $|X_v^*| - 1 \leq \max(\mathrm{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1)$, and thus $\mathrm{tw}(T^*) \leq \max(\mathrm{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1)$. $\square$

Let us now prove the main theorem.

Theorem 1. *For any hypergraph H on a surface Σ,*

$$\mathrm{tw}(H^*) \leq \max\big(\mathrm{tw}(H) + 1 + k(\Sigma), \alpha(H^*) - 1\big).$$

Proof. By Proposition 1, let T be a connected p-tree of H such that $\mathrm{tw}(T) = \mathrm{tw}(H)$. By Proposition 2, $\mathrm{tw}(T^*) \leq \max(\mathrm{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1)$. Since $\mathrm{tw}(H^*) \leq \mathrm{tw}(T^*)$, we deduce, $\mathrm{tw}(H^*) \leq \max(\mathrm{tw}(H) + 1 + k(\Sigma), \alpha(H^*) - 1)$. $\square$

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