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Tree-width of hypergraphs and surface duality

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Abstract. In Graph Minor III, Robertson and Seymour conjecture that We prove that given a hypergraph $H$ on a surface of Euler genus $k$, the tree-width of $H^*$ is at most the maximum of $\text{tw}(H) + 1 + k$ and the maximum size of a hyperedge of $H^*$.

1 Preliminaries

A surface is a connected compact 2-manifold without boundaries. A surface $\Sigma$ can be obtained, up to homeomorphism, by adding $k(\Sigma)$ "crosscaps" to the sphere. $k(\Sigma)$ is the Euler genus or just genus of the surface.

Let $\Sigma$ be a surface. A graph $G=(V,E)$ on $\Sigma$ is a drawing of a graph in $\Sigma$, i.e. each vertex $v$ is an element of $\Sigma$, each edge $e$ is an open curve between two vertices, and edges are pairwise disjoint. We only consider graphs up to homomorphism. A face of $G$ is a connected component of $\Sigma \setminus G$. We denote by $V(G)$, $E(G)$ and $F(G)$ the vertex, edge and face sets of $G$. We only consider 2-cell graphs, i.e. graph whose faces are homeomorphic to open discs. The Euler formula links the number of vertices, edges and faces of a graph $G$ to the genus of the surface

$$|V(G)| - |E(G)| + |F(G)| = 2 - k(G).$$

The set $A(G) = V(G) \cup E(G) \cup F(G)$ of atoms of $G$ is a partition of $\Sigma$. Two Atom $x$ and $y$ of $G$ are incident if $x \cap \bar{y}$ or $y \cap \bar{x}$ is non empty, $\bar{z}$ being the closure of $z$. A cut-edge in a graph $G$ on $\Sigma$ is an edge $e$ separates $G$, i.e. $G$ intersects at least two connected components of $\Sigma \setminus \bar{e}$. As an example, if a planar graph $G$ has a cut-vertex $u$, any loop on $u$ that goes “around” a connected component of $G \setminus \{u\}$ is a cut-edge.

Let $G = (V \cup V_E, L)$ be a bipartite graph on $\Sigma$. The graph $G$ can be seen as the incidence graph of a hypergraph. For each $v_e \in V_E$, merge $v_e$ and its incident edges into a hyperedge $e$, and call $v_e$ its center. Let $E$ be the set of all hyperedges. A hypergraph on $\Sigma$ is any such pair $H = (V, E)$. For brevity, we also say edges for hyperedges. We extend the notions of cut-edges, 2-cell graphs, atoms and incidence to hypergraphs. Moreover, since they naturally correspond to abstract graphs and hypergraphs, graph and hypergraph on surface inherit terminology

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from them. For example, we denote $|e|$ the number of vertices incident to a hyperedge $e$, and we denote $\alpha(H)$ the maximum size of an edge of $H$. Note that a graph on $\Sigma$ is also a hypergraph on $\Sigma$.

The dual of a hypergraph $H = (V, E)$ on $\Sigma$ is obtained by choosing a vertex $v_f$ for every face $f$ of $H$. For every edge $e$ of center $v_e$, we pick up an edge $e^*$ as follows: choose a local orientation of the surface around $v_e$. This local orientation induces a cyclic order $v_1, f_1, v_2, f_2, \ldots, v_d, f_d$ of the ends of $e$ and of the faces incident with $e$ (possibly with repetition). The edge $e^*$ is the edge obtained by “rotating” $e$ and whose ends are $v_{f_1}, \ldots, v_{f_d}$.

A tree-decomposition of a hypergraph $H$ on $\Sigma$ is a pair $T = (T, (X_v)_{v \in V(T)})$ with $T$ a tree and $(X_v)_{v \in V(T)}$ a family of bags such that:

i. $\bigcup_{v \in V(T)} X_v = H$;
ii. $\forall x, y, z \in V(T)$ with $y$ on the path from $x$ to $z$, $X_x \cap X_z \subseteq X_y$.

The width of $T$ is $\text{tw}(T) = \max\{|V(X_t)| - 1 : t \in V(T)\}$ and the tree-width $\text{tw}(H)$ of $H$ is the minimum width of one of its tree-decompositions.

Tree-width was introduced by Robertson and Seymour in connection with graph minors. In [RS84], they conjectured that for a planar graph $G$, $\text{tw}(G)$ and $\text{tw}(G^*)$ differ by at most one. In an unpublished paper, Lapoire [Lap06] proves a more general result: for any hypergraph $H$ in an orientable surface $\Sigma$, $\text{tw}(H^*) \leq \max(\text{tw}(H) + 1 + k(\Sigma), \alpha(H^*) - 1)$. Nevertheless, his proof is rather long and technical. Later, Bouchitté et al. [BMT03] gave an easier proof for planar graphs. Here we generalises Lapoire’s result to arbitrary surfaces while being less technical.

To avoid technicalities, we suppose that $H$ is connected, contains at least two edges, has no pending vertices (i.e. vertices incident with only one edge) and no cut-edge.

2 P-trees and duality

From now on, $H = (V, E)$ is a hypergraph on a surface $\Sigma$. The border of a partition $\mu$ of $E$ is the set of vertices $\delta(\mu)$ that are incident with edges in at least two parts of $\mu$, and the border of $X \subseteq E$ is the border of the partition $\{X, E \setminus X\}$. A partition $\mu = \{X_1, \ldots, X_p\}$ of $E$ is connected if there is a connecting partition $\{V_1, X_1, F_1, \ldots, V_p, X_p, F_p\}$ of $A(H) \setminus \delta(\mu)$ so that each $V_i \cup X_i \cup F_i$ is connected in $\Sigma$.

A p-tree of $H$ is a tree $T$ whose internal nodes have degree three and whose leaves are labelled with the edges of $H$ in a bijective way. Removing an internal node $v$ of $T$ results in a partition $\mu_v$ of $E$. Labelling each internal node $v$ of $T$ with $\delta(\mu_v)$, turns $T$ into a tree-decomposition. The tree-width of a p-tree is its tree-width, seen as a tree-decomposition. A p-tree is connected if all its nodes partitions are connected.

Let $\{A, B\}$ be a connected bipartition of $H$ and $\{V_A, A, F_A, V_B, B, F_B\}$ a corresponding connecting partition. We define a contracted hypergraph $H/A$ as follows. Consider the incidence graph $G_H(V \cup V_E, L)$ of $H$, and identify the
edges in $A$ with their centers. By adding edges trough faces in $F_A$, we can make $G_H[A \cup V_A]$ connected. We then contract $A \cup V_A$ into a single edge center $v_A$. To make the resulting graph bipartite, we remove all $v_A$-loops. When removing a loop $e$ incident to only one face $F$, the new face $F \cup e$ is not a disc but a crosscap. Since the border of $F \cup e$ is a loop, we can “cut” $\Sigma$ along this loop and replace $F \cup e$ by an open disc while decreasing the genus of the surface. The obtained graph is the bipartite graph of $H/A$. A connected partition $\{A, B\}$ is non trivial if neither $H/A$ nor $H/B$ are equal to $H$.

We need the following folklore lemma:

**Lemma 1.** For any connected bipartition $\{A, B\}$ of $H$, $\text{tw}(H) \leq \max(\text{tw}(H/A), \text{tw}(H/B))$. If $\delta(\{A, B\})$ belongs to a bag of an optimal tree-decomposition, then $\text{tw}(H) = \max(\text{tw}(H/A), \text{tw}(H/B))$.

Let $S$ be a set of vertices of $H$. An $S$-bridge is a minimal subset $X$ of $E$ with the property that $\delta(X) \subseteq S$. There are two kind of $S$-bridges: singletons containing an edge whose ends all belong to $S$ and sets $E_C$ containing all the edges incident to at least one vertex in $C$, a connected component of $G \setminus S$. The $S$-bridges partition $E$. We define the abstract graph $G_{/S}$ whose vertices are the $S$-bridges and in which $\{X, Y\}$ is an edge if there is a face incident with both an edge in $X$ and an edge in $Y$. A key fact is that any bipartition $\{A, B\}$ of $V(G_{/S})$ such that $G_{/S}[A]$ and $G_{/S}[B]$ is connected corresponds to the connected bipartition $\{\cup A, \cup B\}$.

**Proposition 1.** There exists a connected p-tree $T$ of $H$ with $\text{tw}(T) = \text{tw}(H)$.

**Proof.** By induction on $|E|$, if $|E| \leq 3$, since $H$ has no cut-edge, the only p-tree is connected and optimal. We can suppose that $|E| \geq 4$. We claim that there exists a connected non trivial bipartition $\{A, B\}$ of $E$ whose border is contained in a bag of an optimal tree-decomposition of $H$. Two cases arise:

- If the trivial one vertex tree-decomposition whose bag is $H$ is optimal, we consider the graph $G_{/V}$. Since they are in bijection with the edges of $H$, and since $H$ has no cut edge, $G_{/V}$ has at least four vertices and no cut vertex. There thus exists a bipartition $\{A, B\}$ of $V(G_{/V})$ with $|A|, |B| \geq 2$, $G_{/V}[A]$ and $G_{/V}[B]$ connected which gives a connected non trivial bipartition of $E$.
- Otherwise, there exists a separator $S$ contained in a bag of an optimal tree-decomposition of $H$. Let $C$ and $D$ be two connected component of $H \setminus S$, and $S_C$ and $S_D$ their corresponding $S$-bridges. Since $H$ contains no pending vertex, $|S_C|, |S_D| \geq 2$. Let $x$ and $y$ be the vertices of $G_{/S}$ corresponding to $S_C$ and $S_D$. Take a spanning tree of $G_{/S}$. Removing an edge between $x$ and $y$ leads to a connected non-trivial bipartition of $E$, which finishes the proof of the claim.

Since $\{A, B\}$ is connected, $e_A$ and $e_B$ are respectively not cut-edges in $H/A$ and $H/B$. By induction, there exists connected p-trees $T_A$ and $T_B$ of optimal width of $H/A$ and $H/B$. By removing the leaves labelled $e_A$ and $e_B$ and adding an edge between their respective neighbour, we obtain from $T_A \cup T_B$ a p-tree of
H which is connected. Its width is \( \max(\text{tw}(T/A), \text{tw}(T/B)) \) which is equal, by Lemma \[\text{Lemma} \] to \( \text{tw}(H) \).

Because of the natural bijection between \( E(H) \) and \( E(H^*) \), a p-tree \( T \) of \( H \) also corresponds to a p-tree \( T^* \) of \( H^* \).

**Proposition 2.** For any connected p-tree \( T \) of \( H \),

\[
\text{tw}(T^*) \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1).
\]

**Proof.** Let \( v \) be a vertex of \( T \) labelled \( X_v \) in \( T \) and \( X_v^* \) in \( T^* \). If \( v \) is a leaf, then \( X_v^* = \{ e^* \} \) and \( |X_v^*| - 1 \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1) \). Otherwise, let \( \{ A, B, C \} \) be the \( E \)-partition associated to \( v \). The label of \( v \) in \( T \) and \( T^* \) is respectively \( X_v = \delta(\{ A, B, C \}) \) and \( X_v^* \), the set of faces incident with edges in at least two parts among \( A, B \) and \( C \).

As for the proof of Proposition \[\text{Proposition} \], since \( \{ A, B, C \} \) is connected, we may contract \( A \) (and \( B \) and \( C \)). But since we now care about the faces of \( H \), we have to be more careful. We want an upper bound on \( |X_v^*| \), we may thus add but not remove faces to \( X_v^* \). So adding edges to make \( G_H[A \cup V_A] \) connected is OK, but we cannot remove a loop \( e \) on say \( v_A \) incident with two faces in \( X_v^* \). Instead, we cut \( \Sigma \) along \( e \) and fill the holes with open discs. While doing so, we removed \( e \), we cut \( v_A \) in two siblings, and we decreased the genus of \( \Sigma \).

After contracting \( A, B \) and \( C \), we obtain a bipartite graph \( G_v \) on \( \Sigma' \) that has \( |X_v| + 3 + s \) vertices with \( s \) the number of siblings, at least \( |X_v^*| \) faces and with \( k(\Sigma') \leq k(\Sigma) - s \). Since \( G_v \) is bipartite and faces in \( X_v^* \) are incident with at least 4 edges, \( 2|E(G_v)| = 4|F_4| + 6|F_6| + \cdots \geq 4|F(G_v)| \) with \( F_{2k} \) the set of \( 2k \)-gones faces of \( G_v \), and thus \( |E(G_v)| \geq 2|F(G_v)| \). If we apply Euler’s formula to \( G_v \) on \( \Sigma' \), we obtain: \( |X_v| + 3 + s - |E(G_v)| + |F(G_v)| = 2 - k(\Sigma') \geq 2 - k(\Sigma) + s \). Adding this to \( |E(G_v)| \geq 2|F(G_v)| \), we get \( |X_v| + 1 + k(\Sigma) \geq |F(G_v)| \geq |X_v^*| \) which proves that \( |X_v^*| - 1 \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1) \), and thus \( \text{tw}(T^*) \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1) \).

Let us now prove the main theorem.

**Theorem 1.** For any hypergraph \( H \) on a surface \( \Sigma \),

\[
\text{tw}(H^*) \leq \max(\text{tw}(H) + 1 + k(\Sigma), \alpha(H^*) - 1).
\]

**Proof.** By Proposition \[\text{Proposition} \], let \( T \) be a connected p-tree of \( H \) such that \( \text{tw}(T) = \text{tw}(H) \). By Proposition \[\text{Proposition} \] \( \text{tw}(T^*) \leq \max(\text{tw}(T) + 1 + k(\Sigma), \alpha(H^*) - 1) \). Since \( \text{tw}(H^*) \leq \text{tw}(T^*) \), we deduce, \( \text{tw}(H^*) \leq \max(\text{tw}(H) + 1 + k(\Sigma), \alpha(H^*) - 1) \).

**References**

