Generalized solutions to a non Lipschitz Goursat problem

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Abstract

We study the semilinear wave equation in canonical form with nonLipschitz nonlinearity by using the recent theories of generalized functions. We investigate solutions to the Goursat problem. We turn this non-Lipschitz Goursat problem with irregular data into a biparameter family of problems. The first parameter replaces the problem by a family of Lipschitz problems and the second one regularizes the data. Finally the family of problems is solved in an appropriate biparametric $(\mathcal{C},\mathcal{E},\mathcal{P})$ algebra.

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Key words : algebras of generalized functions, regularization of problems, nonlinear partial differential equations, wave equation, Goursat problem.

1 Introduction

The distribution theory has some limitations when nonlinear problems are considered. The theories of algebras of generalized functions [1], [11], which form at least presheaves of differential algebras, seem to be an efficient tool to overcome these limitations. They have already been used to solve many nonlinear and irregular problems. For example, in the case of singular data and Lipschitz nonlinearity, a method consists in replacing the given problem with a one-parameter family of smooth problems and has been successfully used in [5], [15], [16], [18] among other references. With similar techniques, various type of nonlinearities are considered in [17], [19].

The main purpose of this paper is to establish the existence of a global solution for the non-Lipschitz Goursat problem (P_{form}): $\dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u)$ (for example $F(\cdot,\cdot,u)) = |u|^p$, p integer, $p \geq 2$), in the case of irregular data given along the characteristic curve C: (Ox), and along a monotonic curve γ of equation $x = g(y)$. We want to investigate solutions to this nonlinear problem with distributions or other generalized functions as data. This justifies to search for solutions in algebras containing the space of distributions which are invariant under nonlinear functions, in addition. To do this, we use some regularization processes and cutoff techniques described in the framework of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras of Marti, [12], [13], [14], [15], [16] which are an improvement and generalization of the algebras of J.-F. Colombeau [1], [11]. These algebras are designed to admit multiparametric families of smooth functions as representatives of generalized functions.

The mentioned irregular problem remains in general unsolvable in classical functional spaces. To overcome this difficulty, we replace Problem (P_{form}) by a family of Lipschitz problems [10]. The general process is the following. A first parameter, ε, permits to regularize the non-lipschitz case, a second parameter, ρ, makes the regularization of the data in singular case. By means of these regularizations, we define an associated generalized problem (P_{gen}) and we study his solvability.

The first situation is the case where F has a non Lipschitz non linearity and the data are regular. We replace F with a family of Lipschitz functions (F_ε) given by suitable cutoff techniques which gives rise to a family (P_ε) : $\dfrac{\partial^2 u}{\partial x \partial y} = F_\varepsilon(\cdot,\cdot,u)$ of regularized Lipschitz problems. Then, the classical successive approximation technique permits to obtain, for each ε, a global solution u_ε to this nonlinear wave equation in canonical form [8], [9]. Using the precise estimates given in section 2, we build a $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra, stable under the family (F_ε), in which the class of (u_ε) is the expected solution of the generalized problem (P_{gen}). Thus, we obtain a global generalized solution, when the classical smooth solutions often break down in finite time [20]. With regard to the regularization, we show that this solution depends solely on the class of the cutoff function as a generalized function, not on the particular representative. Moreover, if the initial problem (P_{form}) admits a smooth solution u satisfying appropriate growth estimates on some open subset Ω of $\mathbb{R}^2$, then this solution and the generalized one are equal in a meaning given in Theorem 28.

The second situation is the case of irregular data We replace the problem by a family $\left(P_{(\varepsilon,\rho)}\right)$ of regularized problems. As before, the parameter ε is used to render the problem Lipschitz, and ρ makes it regular. We build a biparametric algebra in which the generalized problem (P_{gen}) is solved. To prove the existence of solution, a biparametric representative is constructed from the existence of smooth solutions for each regularized Lipschitz problem. We also prove that the solution to the regularized problem, on some open subset Ω, is equal to the non-regularized one in a meaning given in Theorem 31.

In the examples we take advantage of our results to give a new approach of the blow-up problem. Using the Hadamard's finite-part and the previous results, we build a generalized solution to some regularized problem which is equal to the classical blow-up solution.

2 Algebras of generalized functions

2.1 The presheaves of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras

2.1.1 Definitions

We refer the reader to [12], [13], [14], [15] for more details.

Take

- Λ a set of indices;
- A a solid subring of the ring $\mathbb{K}^{\Lambda}$, ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), that is A has the following stability property: whenever $(|s_\lambda|)_\lambda \leq (r_\lambda)_\lambda$ (i.e. for any λ, $|s_\lambda| \leq r_\lambda$) for any pair $((s_\lambda)_\lambda,(r_\lambda)_\lambda) \in \mathbb{K}^{\Lambda}\times|A|$, it follows that $(s_\lambda)_\lambda \in A$, with $|A|=\{(|r_\lambda|)_\lambda : (r_\lambda)_\lambda \in A\}$;
- I_A an solid ideal of A with the same property;
- $\mathcal{E}$ a sheaf of $\mathbb{K}$-topological algebras on a topological space X, such that for any open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega)=(p_i)_{i\in I(\Omega)}$ of seminorms satisfying

  $$\forall i\in I(\Omega),\ \exists(j,k,C)\in I(\Omega)\times I(\Omega)\times\mathbb{R}_+^*,\ \forall f,g\in\mathcal{E}(\Omega): p_i(fg)\leq Cp_j(f)p_k(g).$$

  Assume that

- For any two open subsets Ω_1, Ω_2 of X such that $\Omega_1\subset\Omega_2$, we have $I(\Omega_1)\subset I(\Omega_2)$ and if ρ_1^2 is the restriction operator $\mathcal{E}(\Omega_2)\to\mathcal{E}(\Omega_1)$, then, for each $p_i\in\mathcal{P}(\Omega_1)$, the seminorm $\widetilde{p_i}=p_i\circ\rho_1^2$ extends p_i to $\mathcal{P}(\Omega_2)$;
- For any family $\mathcal{F}=(\Omega_h)_{h\in H}$ of open subsets of X if $\Omega=\cup_{h\in H}\Omega_h$, then, for each $p_i\in\mathcal{P}(\Omega)$, $i\in I(\Omega)$, there exists a finite subfamily $\Omega_1,\dots,\Omega_{n(i)}$ of $\mathcal{F}$ and corresponding seminorms $p_1\in\mathcal{P}(\Omega_1),\dots,p_{n(i)}\in\mathcal{P}(\Omega_{n(i)})$, such that, for each $u\in\mathcal{E}(\Omega)$,

  $$p_i(u)\leq p_1\left(u_{|\Omega_1}\right)+\dots+p_{n(i)}(u_{|\Omega_{n(i)}}).$$

  Set

  $$\begin{aligned}\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega)&=\{(u_\lambda)_\lambda\in[\mathcal{E}(\Omega)]^{\Lambda}:\forall i\in I(\Omega),\ ((p_i(u_\lambda))_\lambda\in|A|\},\\ \mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega)&=\{(u_\lambda)_\lambda\in[\mathcal{E}(\Omega)]^{\Lambda}:\forall i\in I(\Omega),\ (p_i(u_\lambda))_\lambda\in|I_A|\},\\ \mathcal{C}&=A/I_A.\end{aligned}$$

  One can prove that $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ is a sheaf of subalgebras of the sheaf $\mathcal{E}^{\Lambda}$ and $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$ is a sheaf of ideals of $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ [13]. Moreover, the constant sheaf $\mathcal{X}_{(A,\mathbb{K},|.|)}/\mathcal{N}_{(I_A,\mathbb{K},|.|)}$ is exactly the sheaf $\mathcal{C}=A/I_A$.

Definition 1 *We call presheaf of* $(\mathcal{C},\mathcal{E},\mathcal{P})$*-algebra the factor presheaf of algebras over the ring* $\mathcal{C}=A/I_A$

$$\mathcal{A}=\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}.$$

We denote by $[u_\lambda]$ *the class in* $\mathcal{A}(\Omega)$ *defined by the representative* $(u_\lambda)_{\lambda\in\Lambda}\in\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$.

2.1.2 Overgenerated rings

See [7]. *Let* $B_p=\left\{(r_{n,\lambda})_\lambda\in(\mathbb{R}_+^*)^\Lambda : n=1,...,p\right\}$ *and* B *be the subset of* $(\mathbb{R}_+^*)^\Lambda$ obtained as rational functions with coefficients in $\mathbb{R}_+^*$, of elements in B_p as variables. Define

$$A=\left\{(a_\lambda)_\lambda\in\mathbb{K}^\Lambda \mid \exists\,(b_\lambda)_\lambda\in B,\exists\lambda_0\in\Lambda,\forall\lambda\prec\lambda_0 : |a_\lambda|\le b_\lambda\right\}.$$

Definition 2 *In the above situation, we say that* A *is* overgenerated *by* B_p *(and it is easy to see that* A *is a solid subring of* $\mathbb{K}^\Lambda$*). If* I_A *is some solid ideal of* A*, we also say that* $\mathcal{C}=A/I_A$ *is* overgenerated *by* B_p*.*

Example 3 *For example, as a "canonical" ideal of* A*, we can take*

$$I_A=\left\{(a_\lambda)_\lambda\in\mathbb{K}^\Lambda \mid \forall\,(b_\lambda)_\lambda\in B,\exists\lambda_0\in\Lambda,\forall\lambda\prec\lambda_0 : |a_\lambda|\le b_\lambda\right\}.$$

Remark 4 *We can see that with this definition* B *is stable by inverse.*

2.1.3 Relationship with distribution theory

Let Ω an open subset of $\mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ can be embedded into $\mathcal{A}(\Omega)$. If $(\varphi_\lambda)_{\lambda\in(0,1]}$ is a family of mollifiers $\varphi_\lambda(x)=\frac{1}{\lambda^n}\varphi\left(\frac{x}{\lambda}\right)$, $x\in\mathbb{R}^n$, $\int\varphi(x)\,dx=1$ and if $T\in\mathcal{D}'(\mathbb{R}^n)$, the convolution product family $(T*\varphi_\lambda)_\lambda$ is a family of smooth functions slowly increasing in $\frac{1}{\lambda}$. So we shall choose the subring A overgenerated by some B_p of $(\mathbb{R}_+^*)^{(0,1]}$ containing the family $(\lambda)_\lambda$, [3], [18].

2.1.4 The association process

We assume that Λ is left-filtering for a given partial order relation $\prec$. We denote by Ω an open subset of X, E a given sheaf of topological $\mathbb{K}$-vector spaces containing $\mathcal{E}$ as a subsheaf, a a given map from Λ to $\mathbb{K}$ such that $(a(\lambda))_\lambda=(a_\lambda)_\lambda$ is an element of A. We also assume that

$$\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega)\subset\left\{(u_\lambda)_\lambda\in\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) : \lim_{E(\Omega),\Lambda}u_\lambda=0\right\}.$$

Definition 5 *We say that* $u = [u_\lambda]$ *and* $v = [v_\lambda] \in \mathcal{E}(\Omega)$ *are a-E associated if*

$$\lim_{E(\Omega),\Lambda} a_\lambda (u_\lambda - v_\lambda) = 0.$$

That is to say, for each neighborhood V *of* 0 *for the* E*-topology, there exists* $\lambda_0 \in \Lambda$ *such that* $\lambda \prec \lambda_0 \Longrightarrow a_\lambda (u_\lambda - v_\lambda) \in V$. *We write*

$$u \underset{E(\Omega)}{\overset{a}{\sim}} v.$$

Remark 6 *We can also define an association process between* $u = [u_\lambda]$ *and* $T \in \mathcal{E}(\Omega)$ *by writing simply*

$$u \sim T \iff \lim_{E(\Omega),\Lambda} u_\lambda = T.$$

Taking $E = \mathcal{D}'$, $\mathcal{E} = \mathrm{C}^\infty$, $\Lambda = (0,1]$, *we recover the association process defined in the literature (J.-F. Colombeau , [1]).*

2.2 $\mathcal{D}'$-singular support

Assume that

$$\mathcal{N}_{\mathcal{D}'}^{\mathcal{A}}(\Omega) = \left\{ (u_\lambda)_\lambda \in \mathcal{X}(\Omega) : \lim_{\lambda \to 0} u_\lambda = 0 \text{ in } \mathcal{D}'(\Omega) \right\} \supset \mathcal{N}(\Omega).$$

Set

$$\mathcal{D}'_{\mathcal{A}}(\Omega) = \left\{ [u_\lambda] \in \mathcal{A}(\Omega) : \exists T \in \mathcal{D}'(\Omega),\ \lim_{\lambda \to 0} (u_\lambda) = T \text{ in } \mathcal{D}'(\Omega) \right\}.$$

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is clearly well defined because the limit is independent of the chosen representative; indeed, if $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$ we have $\lim\limits_{\substack{\lambda \to 0 \\ \mathcal{D}'(\mathbb{R})}} i_\lambda = 0$.

$\mathcal{D}'_{\mathcal{A}}(\Omega)$ is an $\mathbb{R}$-vector subspace of $\mathcal{A}(\Omega)$. Therefore we can consider the set $\mathcal{O}_{D'_{\mathcal{A}}}$ of all x having a neighborhood V on which u is associated to a distribution:

$$\mathcal{O}_{D'_{\mathcal{A}}}(u) = \left\{ x \in \Omega : \exists V \in \mathcal{V}(x),\ \ u|_V \in \mathcal{D}'_{\mathcal{A}}(V) \right\},$$

$\mathcal{V}(x)$ being the set of all neighborhoods of x.

Definition 7 *The* $\mathcal{D}'$*-singular support of* $u \in \mathcal{A}(\Omega)$, *denoted* $\operatorname{singsupp}_{\mathcal{D}'}(u) = S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u)$, *is the set*

$$S^{\mathcal{A}}_{\mathcal{D}'_{\mathcal{A}}}(u) = \Omega \backslash \mathcal{O}_{D'_{\mathcal{A}}}(u).$$

2.3 Algebraic framework for our problem

Set $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ for $d = 1, 2$, $E = \mathcal{D}'$ and Λ a set of indices, $\lambda \in \Lambda$. For any open set Ω, in $\mathbb{R}^d$, $\mathcal{E}(\Omega)$ is endowed with the $\mathcal{P}(\Omega)$ topology of uniform convergence of all derivatives on compact subsets of Ω. This topology may be defined by the family of the seminorms

$$P_{K,l}(u_\lambda) = \sup_{|\alpha|\leq l} P_{K,\alpha}(u_\lambda) \text{ with } P_{K,\alpha}(u_\lambda) = \sup_{x\in K} |D^\alpha u_\lambda(x)|, \ \ K \Subset \Omega$$

and $D^\alpha = \dfrac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial z_1^{\alpha_1}\ldots\partial z_d^{\alpha_d}}$ for $z = (z_1, \ldots, z_d) \in \Omega$, $l \in \mathbb{N}$, $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$. Let A be a subring of the ring $\mathbb{R}^\Lambda$ of family of reals with the usual laws. We consider a solid ideal I_A of A. Then we have

$$\mathcal{X}(\Omega) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N}, \ (P_{K,l}(u_\lambda))_\lambda \in |A|\},$$
$$\mathcal{N}(\Omega) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N}, \ (P_{K,l}(u_\lambda))_\lambda \in |I_A|\},$$
$$\mathcal{A}(\Omega) = \mathcal{X}(\Omega)/\mathcal{N}(\Omega).$$

The generalized derivation $D^\alpha : u(= [u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$ provides $\mathcal{A}(\Omega)$ with a differential algebraic structure.

Example 8 *Set* $\Lambda = (0, 1]$. *Consider*

$$A = \mathbb{R}_M^\Lambda = \left\{(m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \exists p \in \mathbb{R}_+^*, \ \exists C \in \mathbb{R}_+^*, \ \exists \mu \in \ (0, 1], \ \forall \lambda \in \ (0, \mu], \ \ |m_\lambda| \leq C\lambda^{-p}\right\}$$

and the ideal

$$I_A = \left\{(m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \forall q \in \mathbb{R}_+^*, \ \exists D \in \mathbb{R}_+^*, \ \exists \mu \in \ (0, 1], \ \forall \lambda \in \ (0, \mu], \ \ |m_\varepsilon| \leq D\lambda^q\right\}.$$

Set $\mathcal{E}_M(\Omega) = \mathcal{X}(\Omega)$. *The sheaf of factor algebras* $\mathcal{G}(\cdot) = \mathcal{E}_M(\cdot)/\mathcal{N}(\cdot)$ *is called the sheaf of simplified Colombeau algebras.* $\mathcal{A}\left(\mathbb{R}^d\right) = \mathcal{G}\left(\mathbb{R}^d\right)$ *is the simplified Colombeau algebra of generalized functions.*

We have the analogue of theorem 1.2.3. of [11] for $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras. We suppose here that Λ is left filtering and give this proposition for $\mathcal{A}\left(\mathbb{R}^2\right)$, although it is valid in more general situations.

Proposition 9 *Assume that the set* B*, introduced in Definition 2, is stable by inverse and that there exists* $(a_\lambda)_\lambda \in B$ *with* $\lim_\Lambda a_\lambda = 0$. *Consider* $(u_\lambda)_\lambda \in \mathcal{X}(\mathbb{R}^2)$ *such that*

$$\forall K \Subset \mathbb{R}^2, \ \ (P_{K,0}(u_\lambda))_\lambda \in |I_A|.$$

Then $(u_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$.

We refer the reader to [7], [4] for a detailed proof.

Definition 10 *Let* Ω *be an open subset of* $\mathbb{R}^2$, $\Omega' = \Omega \times \mathbb{R} \subset \mathbb{R}^3$. *Assume that* $\Lambda = \Lambda_1 \times \Lambda_2$, $\lambda = (\varepsilon, \upsilon) \in \Lambda_1 \times \Lambda_2$. *Let* $F_\varepsilon \in \mathrm{C}^\infty(\Omega', \mathbb{R})$. *We say that the algebra* $\mathcal{A}(\Omega)$ *is* stable under the family $(F_\varepsilon)_\varepsilon$ *if the following two conditions are satisfied:*

- *For each* $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$ *and* $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$, *there is a positive finite sequence* C_0,..., C_l, *such that*

$$P_{K,l}(F_\varepsilon(\cdot,\cdot,u_\lambda)) \leq \sum_{i=0}^{l} C_i P^i_{K,l}(u_\lambda).$$

- *For each* $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$, $(v_\lambda)_\lambda$ *and* $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$, *there is a positive finite sequence* D_1,..., D_l *such that*

$$P_{K,l}(F_\varepsilon(\cdot,\cdot,v_\lambda) - F_\varepsilon(\cdot,\cdot,u_\lambda)) \leq \sum_{j=1}^{l} D_j P^j_{K,l}(v_\lambda - u_\lambda).$$

Remark 11 *If* $\mathcal{A}(\Omega)$ *is stable under* $(F_\varepsilon)_\varepsilon$ *then, for all* $(u_\lambda)_\lambda \in \mathcal{X}(\Omega)$ *and* $(i_\lambda)_\lambda \in \mathcal{N}(\Omega)$, *we have* $(F_\varepsilon(\cdot,\cdot,u_\lambda))_\lambda \in \mathcal{X}(\Omega)$; $(F_\varepsilon(\cdot,\cdot,u_\lambda + i_\lambda) - F_\varepsilon(\cdot,\cdot,u_\lambda))_\lambda \in \mathcal{N}(\Omega)$.

2.3.1 Generalized operator associated to a stability property

For each $f \in \mathrm{C}^\infty(\mathbb{R}^2)$ we define

$$H_\lambda(f) = F_\varepsilon(\cdot,\cdot,f) : \mathrm{C}^\infty(\mathbb{R}^2) \to \mathrm{C}^\infty(\mathbb{R}^2),\ f \mapsto ((x,y) \mapsto F_\varepsilon(x,y,f(x,y))).$$

Clearly, the family $(H_\lambda)_\lambda$ maps $\left(\mathrm{C}^\infty(\mathbb{R}^2)\right)^\Lambda$ into $\left(\mathrm{C}^\infty(\mathbb{R}^2)\right)^\Lambda$ and allows to define a map from $\mathcal{A}(\mathbb{R}^2)$ into $\mathcal{A}(\mathbb{R}^2)$. For $u = [u_\lambda] \in \mathcal{A}(\mathbb{R}^2)$, $([F_\varepsilon(.,.,u_\lambda)])$ is a well defined element of $\mathcal{A}(\mathbb{R}^2)$ (i.e. not depending on the representative $(u_\lambda)_\lambda$ of u). This leads to the following:

Definition 12 *If* $\mathcal{A}(\mathbb{R}^2)$ *if stable under* $(F_\varepsilon)_\varepsilon$, *the operator*

$$\mathcal{F} : \mathcal{A}(\mathbb{R}^2) \to \mathcal{A}(\mathbb{R}^2),\quad u = [u_\lambda] \mapsto [F_\varepsilon(.,.,u_\lambda)] = [H_\lambda(u_\lambda)]$$

is called the generalized operator associated to the family $(F_\varepsilon)_\varepsilon$. *See [7].*

Definition 13 *Let* $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$ *and* $f_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})$, *we define* $F_\varepsilon(x,y,z) = F(x,y,zf_\varepsilon(z))$. *The family* $(F_\varepsilon)_\varepsilon$ *is called the* family associated to F *via* the family $(f_\varepsilon)_\varepsilon$. *If* $\mathcal{A}(\mathbb{R}^2)$ *if stable under* $(F_\varepsilon)_\varepsilon$, *the operator*

$$\mathcal{F} : \mathcal{A}(\mathbb{R}^2) \to \mathcal{A}(\mathbb{R}^2),\quad u = [u_\lambda] \mapsto [F_\varepsilon(.,.,u_\lambda)] = [H_\lambda(u_\lambda)]$$

is called the generalized operator associated to F *via* the family $(f_\varepsilon)_\varepsilon$.

2.3.2 Generalized restriction mappings

Assume that $\Lambda = \Lambda_1 \times \Lambda_2$, $\lambda = (\varepsilon,\rho) \in \Lambda_1 \times \Lambda_2$. Set $g \in \mathrm{C}^\infty(\mathbb{R})$. For each $f \in \mathrm{C}^\infty(\mathbb{R}^2)$ set

$$L_\lambda(f) : \mathrm{C}^\infty(\mathbb{R}) \to \mathrm{C}^\infty(\mathbb{R}),\ g \mapsto (y \mapsto f(g(y),y));$$

$$R_\lambda(f) : \mathrm{C}^\infty(\mathbb{R}) \to \mathrm{C}^\infty(\mathbb{R}),\ g \mapsto (x \mapsto f(x,g(x))).$$

The families $(L_\lambda)_\lambda$, $(R_\lambda)_\lambda$ map $\left(\mathrm{C}^\infty(\mathbb{R}^2)\right)^\Lambda$ into $(\mathrm{C}^\infty(\mathbb{R}))^\Lambda$.

Definition 14 *The smooth function g is* compatible with first side restriction *(resp.* second-restriction*) if*

$$\forall \left(u_\lambda\right)_\lambda \in \mathcal{X}(\mathbb{R}^2),\ \left(u_\lambda\left(g(\cdot),\cdot\right)\right)_\lambda \in \mathcal{X}(\mathbb{R})\ ;\ \forall \left(i_\lambda\right)_\lambda \in \mathcal{N}(\mathbb{R}^2),\ \left(i_\lambda\left(g(\cdot),\cdot\right)\right)_\lambda \in \mathcal{N}(\mathbb{R}),$$

$$\textit{(resp. } \forall \left(u_\lambda\right)_\lambda \in \mathcal{X}(\mathbb{R}^2),\ \left(u_\lambda\left(\cdot,g(\cdot)\right)\right)_\lambda \in \mathcal{X}(\mathbb{R})\ ;\ \forall \left(i_\lambda\right)_\lambda \in \mathcal{N}(\mathbb{R}^2),\ \left(i_\lambda\left(\cdot,g(\cdot)\right)\right)_\lambda \in \mathcal{N}(\mathbb{R})\textit{)}.$$

Clearly, if $u = [u_\lambda] \in \mathcal{A}(\mathbb{R}^2)$ then $[u_\lambda\left(g(\cdot),\cdot\right)]$ (resp. $[u_\lambda\left(\cdot,g(\cdot)\right)]$) is a well defined element of $\mathcal{A}(\mathbb{R})$ (i.e. not depending on the representative of u.) This leads to the following:

Definition 15 *If the smooth function g is compatible with first side restriction (resp. second side restriction), the mapping*

$$\mathcal{L}_g : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}\right), \quad u = [u_\lambda] \mapsto [u_\lambda\left(g(\cdot),\cdot\right)] = [L_\lambda\left(u_\lambda\right)]$$

$$\textit{(resp. } \mathcal{R}_g : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}\right), \quad u = [u_\lambda] \mapsto [u_\lambda\left(\cdot,g(\cdot)\right)] = [R_\lambda\left(u_\lambda\right)]\textit{)}$$

is called the generalized first side restriction *(resp.* second side restriction*)* mapping *associated to the function g.*

Remark 16 *The previous process generalizes the standard one defining the restriction of the generalized function $u = [u_\lambda] \in \mathcal{A}\left(\mathbb{R}^2\right)$ to the manifold $\{x = g\left(y\right)\}$ (resp. $\{y = g\left(x\right)\}$).*

Proposition 17 *If function g is c-bounded (for each $K \Subset \mathbb{R}$ it exists $K' \Subset \mathbb{R}$ such that $g(K) \subset K'$) then the function g is compatible is compatible with first side restriction (resp. second side restriction).*

Take $\left(u_\lambda\right)_\lambda$ (resp. $\left(i_\lambda\right)_\lambda$) in $\mathcal{X}(\mathbb{R}^2)$ (resp. $\mathcal{N}(\mathbb{R}^2)$) and set $v_\lambda\left(y\right) = u_\lambda\left(g(y),y\right)$. We have

$$\begin{aligned} &p_{K,0}\left(v_\lambda\right) \leq p_{K'\times K,0}\left(u_\lambda\right) \\ &P_{K,1}\left(v_\lambda\right) \leq p_{K'\times K,(1,0)}\left(u_\lambda\right) p_{K,1}\left(g\right) + p_{K'\times K,(0,1)}\left(u_\lambda\right). \end{aligned}$$

By induction we can see that for each $K \Subset \mathbb{R}$, *and each* $l \in \mathbb{N}$, $p_{K,l}\left(v_\lambda\right)$ is estimated by sums or products of terms like $p_{K'\times K,(n,m)}\left(u_\lambda\right)$ for $n+m \leq l$, or $p_{K,k}\left(g\right)$ for $k \leq l$, then $p_{K,l}\left(v_\lambda\right)$ is in $|A|$. Similarly, setting $j_\lambda\left(t\right) = i_\lambda\left(g(y),y\right)$ leads to $p_{K,l}\left(j_\lambda\right) \in |I_A|$. Then $\left(u_\lambda\left(g(\cdot),\cdot\right)\right)_\lambda$ (resp. $i_\lambda\left(g(\cdot),\cdot\right)$) belongs to $\mathcal{X}(\mathbb{R})$ (resp. $\mathcal{N}(\mathbb{R})$).

2.4 A generalized differential problem associated to the classical one

Our goal is to give a meaning to the differential Goursat problem formally written as

$$(P_{form}) \left\{ \begin{array}{l} \dfrac{\partial^2}{\partial x \partial y} u = F(\cdot,\cdot,u), \\ u|_{(Ox)} = \varphi, \\ u|_\gamma = \psi, \end{array} \right.$$

where F a nonlinear function of its arguments may be non Lipschitz, γ the manifold $\{y = g(x)\}$, ψ, φ are data may be as irregular as distributions. We don't have a classical surrounding in which we can pose (and a fortiori solve) the problem. Set $\theta \in \mathrm{C}^\infty(\mathbb{R})$ define by $\theta(x) = 0$. Let $(\phi_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$. In the sequel, by means of regularizing processes we will define an associated problem to (P_{form}).

$$(P_{gen}) \begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = \mathcal{F}(u) \\ \mathcal{R}_\theta(u) = \varphi, \\ \mathcal{L}_g(u) = \psi \end{cases}$$

where u is searched in some convenient algebra $\mathcal{A}(\mathbb{R}^2)$, $\mathcal{F}$ the generalized operator associated to F *via* the family $(f_\varepsilon)_\varepsilon$, $\mathcal{R}_\theta$ and $\mathcal{L}_g$ are defined as previously, ψ, φ being some given element in $\mathcal{A}(\mathbb{R})$

In terms of representatives, and thanks to the stability and restriction hypothesis, solving (P_{gen}) amounts to find a family $(u_\lambda)_\lambda \in \mathcal{X}(\mathbb{R}^2)$ such that

$$\begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) - F_\varepsilon(x,y,u_\lambda(x,y)) = i_\lambda(x,y) \\ u_\lambda(x,0) - \varphi_\lambda(x) = j_\lambda(x), \\ u_\lambda(g(y),y) - \psi_\lambda(y) = l_\lambda(y) \end{cases}$$

where $(i_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$, $(j_\lambda)_\lambda$, $(l_\lambda)_\lambda \in \mathcal{N}(\mathbb{R})$. Suppose we can find $u_\lambda \in \mathrm{C}^\infty(\mathbb{R}^2)$ verifying

$$(P_\lambda) \begin{cases} \dfrac{\partial^2 u_\lambda}{\partial x \partial y}(x,y) = F_\varepsilon(x,y,u_\lambda(x,y)) \\ u_\lambda(x,0) = \varphi_\lambda(x), \\ u_\lambda(g(y),y) = \psi_\lambda(y) \end{cases}$$

then, if we can prove that $(u_\lambda)_\lambda \in \mathcal{X}(\mathbb{R}^2)$, $u = [u_\lambda]$ is a solution of (P_{gen}). Let $(h_\varepsilon)_\varepsilon \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_1}$. If $v = [v_\lambda]$ is another solution of (P_{gen}) obtain by the family $(H_\varepsilon)_\varepsilon$ associated to F via the family $(h_\varepsilon)_\varepsilon$, this implies

$$\begin{cases} \dfrac{\partial^2 (v_\lambda - u_\lambda)}{\partial x \partial y}(x,y) - (H_\varepsilon(x,y,v_\lambda(x,y)) - F_\varepsilon(x,y,u_\lambda(x,y))) = a_\lambda(x,y) \\ v_\lambda(x,0) - u_\lambda(x,0) = b_\lambda(x), \\ v_\lambda(g(y),y) - u_\lambda(g(y),y) = c_\lambda(y) \end{cases}$$

where $(a_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$ and $(b_\lambda)_\lambda$, $(c_\lambda)_\lambda \in \mathcal{N}(\mathbb{R})$. We have to prove that $(v_\lambda - u_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$ if we intend to prove that the solution of (P_{gen}) in the algebra $\mathcal{A}(\mathbb{R}^2)$ does not depend on the representative of class $[f_\varepsilon]$ in a subalgebra of $\mathcal{A}(\mathbb{R})$.

3 Estimates for a parametrized regular problem

We study the following Goursat problem

$$(P_{form})\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u), \\ u|_{(Ox)} = \varphi, \\ u|_{\gamma} = \psi, \end{cases}$$

where γ is the curve of equation $x = g(y)$, φ and ψ are the Goursat data which will be specified later. The function F may be non Lipschitz (in u).

We are going to replace (P_{form}) with a family $(P_{\varepsilon,\rho})$ of regularized problems

$$(P_{\varepsilon,\rho})\begin{cases} \dfrac{\partial^2 u_{\varepsilon,\rho}}{\partial x \partial y}(x,y) = F_\varepsilon(x,y,u_{\varepsilon,\rho}(x,y)), \\ u_{\varepsilon,\rho}(x,0) = \varphi_\rho(x), \\ u_{\varepsilon,\rho}((g\left(y\right),y)) = \psi_\rho(y), \end{cases} \tag{1}$$

where F_ε is Lipschitz and φ_ρ and ψ_ρ regular. In the following sections, we shall describe how construct these functions and give an algebraic interpretation of the results. But first, we are going to prove that $(P_{\varepsilon,\rho})$ has a unique smooth solution under the following assumption

$$(H_{\varepsilon,\rho})\begin{cases} \text{a)} \quad g \in \mathrm{C}^\infty(\mathbb{R}),\ g' \geq 0,\ g(\mathbb{R}) = \mathbb{R} \\ \text{b)} \quad F_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R}),\ \forall K \Subset \mathbb{R}^2,\ \sup\limits_{(x,y)\in K;z\in\mathbb{R}} |\partial_z F_\varepsilon(x,y,z)| = m_{K,\varepsilon} < +\infty \\ \text{c)} \quad \varphi_\rho \text{ and } \psi_\rho \in \mathrm{C}^\infty(\mathbb{R}). \end{cases} \tag{H}$$

Following [8], one can prove that $(P_{\varepsilon,\rho})$ is equivalent to the integral formulation

$$\left(P'_{\varepsilon,\rho}\right) : u_{\varepsilon,\rho}(x,y) = u_{0,\varepsilon,\rho}(x,y) + \iint_{D(x,y,g_\eta)} F_\varepsilon(\xi,\zeta,u_{\varepsilon,\rho}(\xi,\zeta))d\xi d\zeta, \tag{2}$$

where $u_{0,\varepsilon,\rho}(x,y) = \psi_\rho(y) + \varphi_\rho(x) - \varphi_\rho(g(y))$, with

$$D(x,y,g) = \begin{cases} \{(\xi,\eta) :\ g(y) \leq \xi \leq x,\ 0 \leq \eta \leq y\} \text{ if } g(y) \leq x \text{ and } 0 \leq y, \\ \{(\xi,\eta) : x \leq \xi \leq g(y),\ 0 \leq \eta \leq y\} \text{ if } g(y) \geq x \text{ and } 0 \leq y, \\ \{(\xi,\eta) : x \leq \xi \leq g(y),\ y \leq \eta \leq 0\} \text{ if } g(y) \geq x \text{ and } y \leq 0, \\ \{(\xi,\eta) :\ g(y) \leq \xi \leq x,\ y \leq \eta \leq 0\} \text{ if } g(y) \leq x \text{ and } y \leq 0. \end{cases}$$

Theorem 18 *Under Assumption $(H_{\varepsilon,\rho})$, Problem $(P_{\varepsilon,\rho})$ has a unique solution in $C^\infty(\mathbb{R}^2)$.*

We refer the redear to [8], [10] for a detailed proof. The main idea consists in a Picard's procedure to define a sequence of successive approximations.

$$u_{n,\varepsilon,\rho}(x,y) = u_{0,\varepsilon,\rho}(x,y) + \iint_{D(x,y,g)} F_\varepsilon(\xi,\zeta,u_{n-1,\varepsilon,\rho}(\xi,\zeta))d\xi d\zeta.$$

From the assumptions, putting $v_{n,\varepsilon,\rho} = u_{n,\varepsilon,\rho} - u_{n-1,\varepsilon,\rho}$, we can prove that

$$\|v_{n,\varepsilon,\rho}\|_{\infty,K_{\lambda,\eta}} \leq \frac{\Phi_{\lambda,\varepsilon,\rho}}{m_{\lambda,\varepsilon}} \frac{[m_{\lambda,\varepsilon}\,(g(\lambda)-g(-\lambda))\,\lambda)]^n}{n!}$$

when $K_\lambda = [g(-\lambda), g(\lambda)] \times [-\lambda, \lambda]$, with $m_{\lambda,\varepsilon} = \sup\limits_{(x,y)\in K_\lambda;\ t\in\mathbb{R}} \left|\frac{\partial F_\varepsilon}{\partial z}(x,y,t)\right|$ and

$$\Phi_{\lambda,\varepsilon,\rho} = \|F_\varepsilon(\cdot,\cdot,0)\|_{\infty,K_\lambda} + m_{\lambda,\varepsilon}\,\|u_{0,\varepsilon,\rho}\|_{\infty,K_\lambda}.$$

Finally the sequence $u_{n,\varepsilon,\rho}$ converges uniformly on any compact set to

$$u_{\varepsilon,\rho} = u_{0,\varepsilon,\rho} + \sum_{n\geq 1} v_{n,\varepsilon,\rho}$$

which verifies $\left(P'_{\varepsilon,\rho}\right)$. Gronwall's lemma gives the uniqueness of $u_{\varepsilon,\rho}$. Moreover, we have the estimate

$$\|u\|_{\infty,K} \leq \|u_{\varepsilon,\rho}\|_{\infty,K_\lambda} \leq \|u_{0,\varepsilon,\rho}\|_{\infty,K_{\lambda,}} + \frac{\Phi_{\lambda,\varepsilon,\rho}}{m_{\lambda,\varepsilon}} \exp[m_{\lambda,\varepsilon}\,(g(\lambda)-g(-\lambda))\,\lambda)]. \tag{3}$$

4 Case of regular data

We study the non Lipschitz Goursat problem (P_{form}) when the data are given along the characteristic curve C: (Ox), and along a monotonic curve γ of equation $x = g\,(y)$.

4.1 Cut off procedure

Let ε a parameter belonging to the interval $(0,1]$; let $(r_\varepsilon)_\varepsilon$ be in $\mathbb{R}_*^{(0,1]}$ such that $r_\varepsilon > 0$ and $\lim\limits_{\varepsilon\to 0} r_\varepsilon = +\infty$. Consider a family of smooth one-variable functions $(f_\varepsilon)_\varepsilon$ such that

$$\sup_{z\in[-r_\varepsilon,r_\varepsilon]} |f_\varepsilon(z)| = 1,\ f_\varepsilon(z) = \begin{cases} 0, \text{ if } \ |z| \geq r_\varepsilon \\ 1, \text{ if } \ -r_\varepsilon + 1 \leq z \leq r_\varepsilon - 1 \end{cases}, \tag{A1}$$

and $\dfrac{\partial^n f_\varepsilon}{\partial z^n}$ is bounded on $[-r_\varepsilon, r_\varepsilon]$ for any integer n, $n > 0$. Set

$$\sup_{z\in[-r_\varepsilon,r_\varepsilon]} \left|\frac{\partial^n f_\varepsilon}{\partial z^n}(z)\right| = M_n.$$

Let $\phi_\varepsilon(z) = z f_\varepsilon(z)$. We approximate the function F by $(x,y,z) \mapsto F(x,y,\phi_\varepsilon(z)) = F_\varepsilon(x,y,z)$ then Problem (P) is changed into the family of regularized problems

$$(P_\varepsilon) \begin{cases} \dfrac{\partial^2 u_\varepsilon}{\partial x \partial y} = F_\varepsilon(\cdot,\cdot,u_\varepsilon) \\ u_{\varepsilon|(Ox)} = \varphi \\ u_{\varepsilon|\gamma} = \psi. \end{cases}$$

Example 19 *If* $F(x,y,z) = G(z) = z^p$*, we have*

$$F_\varepsilon(x,y,u_\varepsilon(x,y)) = G_\varepsilon(u_\varepsilon(x,y)) = (\phi_\varepsilon(u_\varepsilon(x,y)))^p\,.$$

Verification of assumption $(H_{\varepsilon,\rho})$. φ*,* ψ *and* g *are some smooth one-variable functions. We fix* ρ *and set* $\varphi_\rho = \varphi$*,* $\psi_\rho = \psi$*. We have* $\dfrac{\partial G_\varepsilon}{\partial z}(z) = p\phi_\varepsilon^{p-1}(z)\phi'_\varepsilon(z)$*. Thus*

$$\left|\frac{\partial G_\varepsilon}{\partial z}(z)\right| \leq pr_\varepsilon^{p-1}\left|f_\varepsilon(z) + zf'_\varepsilon(z)\right| \leq pr_\varepsilon^{p-1}\left|1 + r_\varepsilon M_1\right| \leq \mu_1 r_\varepsilon^p$$

and $\mu_1 = 2p\max(M_1,1)$ *is independent of* ε*. Then assumption* $(H_\varepsilon) = (H_{\varepsilon,\rho})$ *is verified and Problem* (P_ε) *has a unique solution in* $C^\infty(\mathbb{R}^2)$*. When* $K_\lambda = [g(-\lambda), g(\lambda)] \times [-\lambda,\lambda]$*,* $m_{K_\lambda,\varepsilon} = m_{\lambda,\varepsilon}$*, we have the estimate*

$$\|u_\varepsilon\|_{\infty,K} \leq \|u_\varepsilon\|_{\infty,K_\lambda} \leq \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + \frac{\Phi_{\lambda,\varepsilon}}{m_{\lambda,\varepsilon}}\exp[m_{\lambda,\varepsilon}\left(g(\lambda) - g(-\lambda)\right)\lambda]. \quad (4)$$

4.2 Construction of $\mathcal{A}(\mathbb{R}^2)$

Let ε a parameter belonging to the interval $(0,1]$, let $(r_\varepsilon)_\varepsilon$ be in $(\mathbb{R}_*^+)^{(0,1]}$ such that $\lim_{\varepsilon\to 0} r_\varepsilon = +\infty$. We take $\mathcal{C} = A/I_A$ overgenerated by $(\varepsilon)_\varepsilon$, $(r_\varepsilon)_\varepsilon$ and $(\exp(r_\varepsilon))_\varepsilon$ (elements of $(\mathbb{R}_*^+)^{(0,1]}$). Then $\mathcal{A}(\mathbb{R}^2) = \mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2)$ is built on $\mathcal{C}$ with $(\mathcal{E},\mathcal{P}) = \left(\mathrm{C}^\infty(\mathbb{R}^2), (P_{K,l})_{K\Subset\mathbb{R}^2, l\in\mathbb{N}}\right)$.

We look for u, solution to problem (P), in the algebra $\mathcal{A}(\mathbb{R}^2)$.

4.2.1 Stability of $\mathcal{A}(\mathbb{R}^2)$

Proposition 20 *Set* $S_n = \left\{\alpha \in \mathbb{N}^3 : |\alpha| = n\right\}$ *when* $n \in \mathbb{N}^*$*. Let* $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$*,* F_ε *defined by* $F_\varepsilon(x,y,z) = F(x,y,\phi_\varepsilon(z))$*. Assume that*

$$\forall\varepsilon \in (0,1], \forall (x,y) \in \mathbb{R}^2, F_\varepsilon(x,y,0) = 0\,, \quad (A0)$$

$$\exists p > 0, \forall n \in \mathbb{N}, \exists\mu_n > 0, \forall\varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \sup_{(x,y)\in K;\ z\in\mathbb{R};\alpha\in Sn} \left|D^\alpha F_\varepsilon(x,y,z)\right| \leq \mu_n r_\varepsilon^p, \quad (A1)$$

then $\mathcal{A}(\mathbb{R}^2)$ *is stable under the family* $(F_\varepsilon)_\varepsilon$*.*

We refer the reader to [10] for a detailed proof.

Corollary 21 *Set* $F(x,y,z) = G(z) = z^p$*,* $G_\varepsilon(z) = F_\varepsilon(x,y,z)$*, then* $\mathcal{A}(\mathbb{R}^2)$ *is stable under* $(G_\varepsilon)_\varepsilon$*.*

We have $|G_\varepsilon(z)| = |z^p g_\varepsilon^p(z)| \le r_\varepsilon^p$, so $\sup\limits_{(x,y)\in\mathbb{R};z\in\mathbb{R}} |G_\varepsilon(z)| \le r_\varepsilon^p$. As $\phi_\varepsilon(z) = z g_\varepsilon(z)$, we obtain

$$\frac{\partial^n \phi_\varepsilon}{\partial z^n}(z) = z\frac{\partial^n g_\varepsilon}{\partial z^n}(z) + n\frac{\partial^{n-1} g_\varepsilon}{\partial z^{n-1}}(z).$$

Thus $\left|\frac{\partial^n \phi_\varepsilon}{\partial z^n}(z)\right| \le r_\varepsilon M_n + nM_{n-1} \le \alpha_n r_\varepsilon$, where $\alpha_n = 2\max(M_n; nM_{n-1})$. Set $w(z) = z^p$, then $\frac{\partial^m w}{\partial z^m}(z) = \left(\prod_{i=0}^{i=m-1}(p-i)\right) z^{p-m}$ for $1 \le m \le p$. According Francesco Faà di Bruno's formula, the nth order derivative of $G_\varepsilon = w \circ \phi_\varepsilon$ can be written

$$\frac{\partial^n G_\varepsilon}{\partial z^n} = \sum_{m=1}^{n} \sum_{\substack{i_1 \ge \ldots \ge i_m \\ i_1+\ldots+i_m=n}} t_{i_1,\ldots,i_m} w^{(m)} \circ \phi_\varepsilon \prod_{k=1}^{m} \phi_\varepsilon^{(i_k)},$$

where the coefficients $t_{i_1,\ldots,i_m}$ are integers. Then we get

$$\left|\frac{\partial^n G_\varepsilon}{\partial z^n}(z)\right| \le \sum_{m=1}^{p} \sum_{\substack{i_1 \ge \ldots \ge i_m \\ i_1+\ldots+i_m=n}} t_{i_1,\ldots,i_m} (\textstyle\prod_{i=0}^{i=m-1}(p-i)) r_\varepsilon^{p-m} \prod_{k=1}^{m} \alpha_{i_k} r_\varepsilon \le \mu_n r_\varepsilon^p,$$

where μ_n is independent of ε. So assumptions *(A0), (A1)* are verified.

4.3 Solution to (P_{gen})

Theorem 22 *With the previous Assumptions (H), (A0), (A1), if u_ε is the solution to Problem (P_ε) then Problem (P_{gen}) admits $u = [u_\varepsilon]_{\mathcal{A}(\mathbb{R}^2)}$ as solution.*

According to [8], $u = [u_\varepsilon]$ is solution to (P_{gen}) if $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R}^2)$. Then we shall prove that

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, (P_{K,l}(u_\varepsilon))_\varepsilon \in A.$$

We proceed by induction. We have: $\forall K \Subset \mathbb{R}^2_, \exists\, K_\lambda \Subset \mathbb{R}^2_,\ K \subset K_\lambda$,

$$\begin{aligned}\|u_\varepsilon\|_{\infty,K} \le \|u_\varepsilon\|_{\infty,K_\lambda} &\le \|u_{0,\varepsilon}\|_{\infty,K_\lambda} + \frac{\Phi_{\lambda,\varepsilon}}{m_{\lambda,\varepsilon}} \exp[(g(\lambda) - g(-\lambda))\,\lambda]. \\ &\le \|u_{0,\varepsilon}\|_{\infty,K_\lambda} \left(1 + \exp[\lambda \mu_1 r_\varepsilon^p \left(g(\lambda) - g(-\lambda)\right)]\right).\end{aligned}$$

As $\left(\|u_{0,\varepsilon}\|_{\infty,K_\lambda}\right)_\varepsilon \in A$ we have

$$\left(\|u_{0,\varepsilon}\|_{\infty,K_\lambda} \left(1 + \exp -\lambda \mu_1 r_\varepsilon^p \left(g(\lambda) - g(-\lambda)\right)]\right)\right)_\varepsilon \in A.$$

As A is stable, we deduce $(P_{K,0}(u_\varepsilon))_\varepsilon \in A$, then the 0th order estimate is verified.

We have

$$\frac{\partial u_\varepsilon}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) + \int_0^y F_\varepsilon(x,\zeta,u_\varepsilon(x,\zeta))d\zeta,$$

hence

$$P_{K,(1,0)}(u_\varepsilon) \leq \sup_K \left| \frac{\partial u_{0,\varepsilon}}{\partial x}(x,y) \right| + \lambda \left(\sup_{K_\lambda} |F_\varepsilon(x,\zeta,u_\varepsilon(x,\zeta))| \right).$$

Moreover

$$P_{K_\lambda,(0,0)}(F_\varepsilon(\cdot,\cdot,u_\varepsilon)) \leq P_{K_\lambda,0}(F_\varepsilon(\cdot,\cdot,u_\varepsilon)) \leq \mu_0 r_\varepsilon^p,$$

so

$$P_{K,(1,0)}(u_\varepsilon) \leq \left\| \frac{\partial u_{0,\varepsilon}}{\partial x} \right\|_{\infty,K} + \mu_0 r_\varepsilon^p \lambda.$$

As A is stable, we get $\left(P_{K,(1,0)}(u_\varepsilon)\right)_\varepsilon \in A$. We have

$$\frac{\partial u}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) + \int_{g(y)}^{x} F_\varepsilon(\xi,y,u_\varepsilon(\xi,y))d\xi - g'(y)\int_0^y F_\varepsilon(g(y),\zeta,u_\varepsilon(g(y),\zeta))d\zeta,$$

so

$$P_{K,(0,1)}(u_\varepsilon) \leq \sup_K \left| \frac{\partial u_{0,\varepsilon}}{\partial y}(x,y) \right| + ((g(\lambda)-g(-\lambda)) + \lambda g'(y)) \sup_{K_\lambda} |F_\varepsilon(x,\zeta,u_\varepsilon(x,\zeta))|.$$

Hence

$$P_{K,(0,1)}(u_\varepsilon) \leq \left\| \frac{\partial u_{0,\varepsilon}}{\partial y} \right\|_{\infty,K} + \mu_0 r_\varepsilon^p \left(g(\lambda) - g(-\lambda) + \lambda g'(y) \right)$$

and, as previously $\left(P_{K,(0,1)}(u_\varepsilon)\right)_\varepsilon \in A$. Finally

$$\left(P_{K,1}(u_\varepsilon)\right)_\varepsilon \in A.$$

Assume that $\left(P_{K,l}(u_\varepsilon)\right)_\varepsilon \in A$ for any $l \leq n$. In fact we have

$$P_{K,n+1} = \max\left(P_{K,n}, P_{1,n}, P_{2,n}, P_{3,n}, P_{4,n}\right)$$

with

$$P_{1,n} = P_{K,(n+1,0)} \ ; \ P_{2,n} = P_{K,(0,n+1)K,(0,n+1)}$$
$$P_{3,n} = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)} \ ; \ P_{4,n} = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}.$$

For $n \geq 1$, we have by successive derivations

$$\frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) = \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}}(x,y) + \int_0^y \frac{\partial^n}{\partial x^n} F_\varepsilon(x,\zeta,u_\varepsilon(x,\zeta))d\zeta.$$

As $K \subset K_\lambda$, we can write

$$\sup_{(x,y)\in K} \left| \frac{\partial^{n+1} u_\varepsilon}{\partial x^{n+1}}(x,y) \right| \leq \left\| \frac{\partial^{n+1} u_{0,\varepsilon}}{\partial x^{n+1}} \right\|_{\infty,K} + \lambda \left(\sup_{(x,y)\in K} \left| \frac{\partial^n}{\partial x^n} F_\varepsilon(x,y,u_\varepsilon(x,y)) \right| \right).$$

We have

$$\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial x^n}F_\varepsilon(x,y,u_\varepsilon(x,y))\right| \leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\varepsilon)).$$

Moreover $\left(\left\|\frac{\partial^{n+1}u_{0,\varepsilon}}{\partial x^{n+1}}\right\|_{\infty,K}\right)_\varepsilon \in A$. According to the stability hypothesis, a simple calculation shows that for any $K \Subset \mathbb{R}^2$, $\left(P_{K,(n+1,0)}\left(u_\varepsilon\right)\right)_\varepsilon \in A$, then $\left(P_{1,n}(u_\varepsilon)\right)_\varepsilon \in A$. Let us show that $\left(P_{2,n}(u_\varepsilon)\right)_\varepsilon \in A$ for every $n \in \mathbb{N}$. We have by successive derivations, for $n \geq 1$,

$$\begin{aligned}
&\frac{\partial^{n+1}u_\varepsilon}{\partial y^{n+1}}(x,y) = \frac{\partial^{n+1}u_{0,\varepsilon}}{\partial y^{n+1}}(x,y) \\
&\quad -\sum\nolimits_{j=0}^{n-1} C_n^j g^{(n-j)}(y)\frac{\partial^j}{\partial y^j}F_\varepsilon(g(y),y,\psi_\varepsilon(y)) - \int_x^{g(y)}\frac{\partial^n}{\partial y^n}F_\varepsilon(\xi,y,u_\varepsilon(\xi,y))d\xi \\
&-\sum\nolimits_{j=0}^{n-1} C_n^{j+1} g^{(n-j)}(y)\frac{\partial^j}{\partial y^j}F_\varepsilon(g(y),y,\psi_\varepsilon(y)) - g^{(n+1)}(y)\int_0^y F_\varepsilon(g(y),\zeta,u_\varepsilon(g(y),\zeta))d\zeta\ .
\end{aligned}$$

As $K \subset K_\lambda$, we can write

$$\begin{aligned}
\sup_{(x,y)\in K}\left|\frac{\partial^{n+1}u_\varepsilon}{\partial y^{n+1}}(x,y)\right| &\leq \sup_{y\in[-\lambda,\lambda]}\sum\nolimits_{j=0}^{n-1} C_{n+1}^{j+1}\left|g^{(n-j)}(y)\right|\left|\frac{\partial^j}{\partial y^j}F_\varepsilon(g(y),y,\psi_\varepsilon(y))\right| \\
&\quad + (g(\lambda)-g(\lambda))\sup_{(x,y)\in K}\left|\frac{\partial^n}{\partial y^n}F_\varepsilon(x,y,u_\varepsilon(x,y))\right| \\
&\qquad +\lambda g^{(n+1)}(y)\sup_{(x,y)\in K}\left|F_\varepsilon(x,y,u_\varepsilon(x,y))\right| + P_{K,(0,n+1)}\left(u_{0,\varepsilon}\right)\ .
\end{aligned}$$

For any $K \Subset \mathbb{R}^2$, we have

$$\sup_{(x,y)\in K}\left|\frac{\partial^i}{\partial y^i}F(x,y,u_\varepsilon(x,y))\right| \leq P_{K,n}(F(\cdot,\cdot,u_\varepsilon)$$

Then, for any $n \in \mathbb{N}$, $\left(P_{K,(0,n+1)}\left(u_\varepsilon\right)\right)_\varepsilon \in A$. So $\left(P_{2,n}(u_\varepsilon)\right)_\varepsilon \in A$.

For $\alpha+\beta = n$ and $\beta \geq 1$, we have now

$$\begin{aligned}
P_{K,(\alpha+1,\beta)}(u_\varepsilon) &= \sup_{(x,y)\in K}\left|D^{(\alpha,\beta-1)}D^{(1,1)}u_\varepsilon\left(x,y\right)\right| = \sup_{(x,y)\in K}\left|D^{(\alpha,\beta-1)}F_\varepsilon(x,y,u_\varepsilon\left(x,y\right))\right| \\
&\leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\varepsilon)).
\end{aligned}$$

So we finally have

$$P_{3,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\beta\geq 1} P_{K,(\alpha+1,\beta)}(u_\varepsilon) \leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis implies $\left(P_{3,n}(u_\varepsilon)\right)_\varepsilon \in A$. In the same way, for $\alpha+\beta = n$ and $\alpha \geq 1$, we have

$$P_{K,(\alpha,\beta+1)}(u_\varepsilon) = \sup_{(x,y)\in K}\left|D^{(\alpha-1,\beta)}F_\varepsilon(x,y,u_\varepsilon\left(x,y\right))\right| \leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\varepsilon)).$$

So we have

$$P_{4,n}(u_\varepsilon) = \sup_{\alpha+\beta=n;\alpha\geq 1} P_{K,(\alpha,\beta+1)}(u_\varepsilon) \leq P_{K,n}(F_\varepsilon(\cdot,\cdot,u_\varepsilon))$$

and the stability hypothesis implies $(P_{4,n}(u_\varepsilon))_\varepsilon \in A$. Finally, we have $(P_{K,n+1}(u_\varepsilon))_\varepsilon \in A$.

4.4 Generalized solution only depend on the class of cut off functions

Consider $\mathcal{T}(\mathbb{R})$ the set of families of smooth one-variable functions $(h_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{R})$, verfying the following assumptions

$$(f_\varepsilon - h_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}), \tag{E0}$$

$$\exists\, (s_\varepsilon)_\varepsilon \in \mathbb{R}_*^{(0,1]} : \sup_{z\in[-s_\varepsilon,s_\varepsilon]} |h_\varepsilon(z)| = 1,\ h_\varepsilon(z) = \begin{cases} 0, \text{ if } \ |z| \geq s_\varepsilon \\ 1, \text{ if } \ -s_\varepsilon + 1 \leq z \leq s_\varepsilon - 1 \end{cases}, \tag{E1}$$

$$\exists q \in \mathbb{N}^*, \forall\, (h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R}), \forall \varepsilon, s_\varepsilon \leq r_\varepsilon^q, \tag{E2}$$

moreover $\dfrac{\partial^n h_\varepsilon}{\partial z^n}$ is bounded on $[-s_\varepsilon, s_\varepsilon]$ for any integer n, $n > 0$.

Recall that $\phi_\varepsilon(z) = zf_\varepsilon(z)$ for $z \in \mathbb{R}$, $F_\varepsilon(x,y,z) = F(x,y,\phi_\varepsilon(z))$ for $(x,y,z) \in \mathbb{R}^3$ and

$$\sup_{z\in[-r_\varepsilon,r_\varepsilon]} \left| \frac{\partial^n f_\varepsilon}{\partial z^n}(z) \right| = M_n.$$

Take $(h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R})$. Set $\sigma_\varepsilon(z) = zh_\varepsilon(z)$ for $z \in \mathbb{R}$, $H_\varepsilon(x,y,z) = F(x,y,\sigma_\varepsilon(z))$ for $(x,y,z) \in \mathbb{R}^3$ and

$$\sup_{z\in[-s_\varepsilon,s_\varepsilon]} \left| \frac{\partial^n h_\varepsilon}{\partial z}(z) \right| = M'_n.$$

Set $I_\varepsilon = [-r_\varepsilon, r_\varepsilon]$.

Lemma 23 *Set $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$, $\phi \in \mathrm{C}^\infty(\mathbb{R},\mathbb{R})$, $\mathcal{F}(x,y,z) = F(x,y,\phi(z))$. For any $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$ with $|\alpha| = \alpha_1+\alpha_2+\alpha_3 = n \neq 0$, we have*

$$\frac{\partial^n \mathcal{F}}{\partial x^{\alpha_1}\partial y^{\alpha_2}\partial z^{\alpha_3}}(x,y) = \sum_{1\leq|\beta|\leq n} \left(D^\beta F\right)(x,y,\phi(z)) \sum_{i=1}^{n} \sum_{p_i(\alpha,\beta)} d_{i,\alpha,\beta} \prod_{j=1}^{i} \left(\frac{\partial^{l_j}}{\partial z^{l_j}} \phi(z) \right)^{k_j}$$

where $\beta \in \mathbb{N}^3$. The set $p_i(\alpha,\beta)$ mentioned in the inner sum consists of all nonzero multi-indices $(k_1,...,k_i,l_1,...,l_i) \in (\mathbb{N})^{2i}$, such that

$$0 < l_1 < ... < l_i, \quad \sum_{j=1}^{i} k_j = \beta_3, \quad \sum_{j=1}^{i} k_j l_j = \alpha_3.$$

The proof uses the Multivariate Faà di Bruno's formula (see [2]).

Corollary 24 *Set* $F \in \mathrm{C}^{\infty}(\mathbb{R}^3, \mathbb{R})$, $\sigma_\varepsilon(z) = z h_\varepsilon(z)$ *with* $(h_\varepsilon)_\varepsilon \in \mathcal{T}(\mathbb{R})$, $H_\varepsilon(x,y,z) = F(x,y,\sigma_\varepsilon(z))$,$\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$ *with* $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = n \neq 0$. *Then, for* $\beta \in \mathbb{N}^3$, $1 \leq |\beta| \leq n$, *there exist constants* $C_{|\beta|}$ *which no depend of* F *and* ϕ_ε, *such that* $\forall K \Subset \mathbb{R}^2$, $\forall (x,y) \in K$, $\forall z \in [-s_\varepsilon, s_\varepsilon]$,

$$\left| \frac{\partial^n H_\varepsilon}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}(x,y,z) \right| \leq \sum_{1 \leq |\beta| \leq n} P_{K,|\beta|}(F)\, C_{|\beta|} s_\varepsilon^{\alpha_3}$$

We have

$$\frac{\partial^n \sigma_\varepsilon}{\partial z^n}(z) = z \frac{\partial^n h_\varepsilon}{\partial z^n}(z) + n \frac{\partial^{n-1} h_\varepsilon}{\partial z^{n-1}}(z).$$

Thus $\left| \frac{\partial^n \sigma_\varepsilon}{\partial z^n}(z) \right| \leq s_\varepsilon M'_n + n M'_{n-1} \leq \alpha_n s_\varepsilon \leq \alpha_n r_\varepsilon^q$, where $\alpha_n = 2\max(M'_n; nM'_{n-1})$. So we deduce the formula. Moreover, according (E2), we have $s_\varepsilon \leq r_\varepsilon^q$, so

$$\left| \frac{\partial^n H_\varepsilon}{\partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}}(x,y,z) \right| \leq \sum_{1 \leq |\beta| \leq n} P_{K,|\beta|}(F)\, C_{|\beta|} r_\varepsilon^{q\alpha_3}$$

Corollary 25 *Set* $S_n = \left\{ \alpha \in \mathbb{N}^3 : |\alpha| = n \right\}$ *when* $n \in \mathbb{N}^*$. *Let* $F \in \mathrm{C}^{\infty}(\mathbb{R}^3, \mathbb{R})$, H_ε *defined by* $H_\varepsilon(x,y,z) = F(x,y,\sigma_\varepsilon(z))$. *Assume that*

$$\forall (x,y) \in \mathbb{R}^2, F(x,y,0) = 0\ ,$$

$$\exists p_0 > 0, \forall \alpha \in \mathbb{N}^3, |\alpha| = n > p_0, D^\alpha F(x,y,z) = 0,$$

$$\forall n \in \mathbb{N}, n \leq p_0, \exists d_n > 0, \forall \varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \sup_{(x,y) \in K;\ z \in [-r_\varepsilon, r_\varepsilon]; \alpha \in Sn} |D^\alpha F(x,y,z)| \leq d_n r_\varepsilon^{p_0}, \tag{1}$$

then $\mathcal{A}\left(\mathbb{R}^2\right)$ *is stable under the family* $(H_\varepsilon)_\varepsilon$.

Indeed, we have $\forall K \Subset \mathbb{R}^2$, $\forall (x,y) \in K$, $\forall z \in [-s_\varepsilon, s_\varepsilon]$, $\forall \alpha \in \mathbb{N}^3$,

$$\begin{aligned} \left| \frac{\partial^n H_\varepsilon}{\partial x^{\alpha_1} \partial^{\alpha_2} y \partial z^{\alpha_3}}(x,y,z) \right| &\leq \sum_{1 \leq |\beta| \leq n} P_{K,|\beta|}(F)\, C_{|\beta|} r_\varepsilon^{q\alpha_3} \leq \sum_{1 \leq |\beta| \leq p} d_{|\beta|} r_\varepsilon^{p_0} C_{|\beta|} r_\varepsilon^{qp_0} \\ &\leq \mu_n r_\varepsilon^{p_0(1+q)} \end{aligned}$$

where μ_n no depend to ε and r_ε. So, as $\sigma_\varepsilon(z) = 0$ if $z \notin [-s_\varepsilon, s_\varepsilon]$,

$$\sup_{(x,y) \in K;\ z \in \mathbb{R}; \alpha \in Sn} |D^\alpha H_\varepsilon(x,y,z)| \leq \mu_n r_\varepsilon^{p_0(1+q)},$$

and, according to Proposition 20, $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under the family $(H_\varepsilon)_\varepsilon$.

Set $p = p_0(1+q)$. Then, considering the family $(H_\varepsilon)_\varepsilon$ associated to F via the family $(h_\varepsilon)_\varepsilon$, we can build a solution to (P_{gen}) in the same algebra $\mathcal{A}\left(\mathbb{R}^2\right)$.

Theorem 26 *Under the same hypotheses as Corollary 25, the solution* $u = [u_\varepsilon]$ *does not depend of the choice of the representative* $(f_\varepsilon)_\varepsilon$ *of the class* $f \in \mathcal{T}(\mathbb{R})/\mathcal{N}(\mathbb{R})$.

We have $\forall n \in \mathbb{N}, \exists \mu_n > 0, \forall \varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \forall \alpha \in \mathbb{N}^3$ with $|\alpha| = n$,

$$\sup_{(x,y)\in K;\ z\in\mathbb{R}} |D^\alpha F_\varepsilon(x,y,z)| \le \mu_n r_\varepsilon^p \text{ and } \sup_{(x,y)\in K;\ z\in\mathbb{R}} |D^\alpha H_\varepsilon(x,y,z)| \le \mu_n r_\varepsilon^p.$$

According to the results of Theorem 22, let $u = [u_\varepsilon]$ the solution to (P_{gen}) defined with the family $(F_\varepsilon)_\varepsilon$ and $v = [v_\varepsilon]$ another solution to (P_{gen}) defined with the family $(H_\varepsilon)_\varepsilon$. As $v_{0,\varepsilon}(x,y) = u_{0,\varepsilon}(x,y)$ we have

$$v_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \iint_{D(x,y,f)} F(\xi,\varsigma,\sigma_\varepsilon(v_\varepsilon(\xi,\varsigma)))d\xi d\varsigma.$$

We will prove that $(w_\varepsilon)_\varepsilon = (v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$. Let

$$\Delta_\varepsilon(x,y) = \sigma_\varepsilon(v_\varepsilon(x,y)) - \phi_\varepsilon(u_\varepsilon(x,y)).$$

Set $J_\varepsilon = [-s_\varepsilon, s_\varepsilon]$. We have $\forall K \Subset \mathbb{R}^2$, $\forall (x,y) \in K$,

$$\Delta_\varepsilon(x,y) = v_\varepsilon(x,y) h_\varepsilon(v_\varepsilon(x,y)) - u_\varepsilon(x,y) f_\varepsilon(u_\varepsilon(x,y)),$$

so

$$\Delta_\varepsilon(x,y) = w_\varepsilon(x,y) h_\varepsilon(v_\varepsilon(x,y)) + u_\varepsilon(x,y)\left(h_\varepsilon(v_\varepsilon(x,y)) - f_\varepsilon(u_\varepsilon(x,y))\right) \quad (2)$$

and

$$h_\varepsilon \circ v_\varepsilon - f_\varepsilon \circ u_\varepsilon = (h_\varepsilon \circ v_\varepsilon - h_\varepsilon \circ u_\varepsilon) + (h_\varepsilon \circ u_\varepsilon - f_\varepsilon \circ u_\varepsilon)$$

As

$$h_\varepsilon(v_\varepsilon(x,y)) - h_\varepsilon(u_\varepsilon(x,y)) = (v_\varepsilon(x,y) - u_\varepsilon(x,y)) \int_0^1 \frac{\partial h_\varepsilon}{\partial z}(u_\varepsilon(x,y) + \mu(v_\varepsilon(x,y) - u_\varepsilon(x,y)))d\mu, \quad (3)$$

so

$$h_\varepsilon(v_\varepsilon(x,y)) - f_\varepsilon(u_\varepsilon(x,y)) = w_\varepsilon(x,y) \int_0^1 \frac{\partial h_\varepsilon}{\partial z}(u_\varepsilon(x,y) + \mu w_\varepsilon(x,y))d\mu + (h_\varepsilon - f_\varepsilon)(u_\varepsilon(x,y)). \quad (4)$$

We deduce that $\forall (x,y) \in K$,

$$\begin{aligned} |h_\varepsilon(v_\varepsilon(x,y)) - f_\varepsilon(u_\varepsilon(x,y))| &\le |w_\varepsilon(x,y)| \int_0^1 M_1' d\mu + |(h_\varepsilon - f_\varepsilon)(u_\varepsilon(x,y))| \\ &\le |w_\varepsilon(x,y)| M_1' + p_{J_\varepsilon,1}(h_\varepsilon - f_\varepsilon). \end{aligned}$$

Then

$$\begin{aligned} |\Delta_\varepsilon(x,y)| &\le |w_\varepsilon(x,y)| + |u_\varepsilon(x,y)| \left(|w_\varepsilon(x,y)| M_1' + p_{J_\varepsilon,1}(h_\varepsilon - f_\varepsilon)\right) \\ &\le |w_\varepsilon(x,y)| (1 + |u_\varepsilon(x,y)| M_1') + |u_\varepsilon(x,y)| p_{J_\varepsilon,1}(h_\varepsilon - f_\varepsilon). \end{aligned}$$

We have $\forall K \Subset \mathbb{R}^2_, \exists K_\lambda \Subset \mathbb{R}^2_, K \subset K_\lambda$,

$$\|u_\varepsilon\|_{\infty,K} \leq \|u_{0,\varepsilon}\|_{\infty,K_\lambda} (1 + \exp[2\lambda'\lambda\mu_1 r_\varepsilon^p])$$

with $2\lambda' = (g(\lambda) - g(-\lambda))$. Let $c_{\lambda,\varepsilon} = \left(1 + \|u_{0,\varepsilon}\|_{\infty,K_\lambda} (1 + \exp[2\lambda'\lambda\mu_1 r_\varepsilon^p])\, M_1'\right)$ and

$$a_{\lambda,\varepsilon} = \|u_{0,\varepsilon}\|_{\infty,K_\lambda} (1 + \exp[2\lambda'\lambda\mu_1 r_\varepsilon^p])\, p_{J_\varepsilon,1}(h_\varepsilon - f_\varepsilon).$$

As $(p_{J_\varepsilon,1}(h_\varepsilon - f_\varepsilon))_\varepsilon \in I_A$, so $(a_{\lambda,\varepsilon})_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$.

$$|\Delta_\varepsilon(x,y)| \leq |w_\varepsilon(x,y)|\, c_{\lambda,\varepsilon} + a_{\lambda,\varepsilon}. \qquad (P1)$$

As

$$\begin{aligned} &F(\xi,\varsigma,\sigma_\varepsilon(v_\varepsilon(\xi,\varsigma))) - F(\xi,\varsigma,\phi_\varepsilon(u_\varepsilon(\xi,\varsigma))) \qquad (P2)\\ &= \Delta_\varepsilon(\xi,\varsigma)\left(\int_0^1 \frac{\partial F}{\partial z}(\xi,\varsigma,\phi_\varepsilon(u_\varepsilon(\xi,\varsigma))) + \mu(\sigma_\varepsilon(v_\varepsilon(\xi,\varsigma)) - \phi_\varepsilon(u_\varepsilon(\xi,\varsigma)))d\mu\right), \end{aligned}$$

we have

$$w_\varepsilon(x,y) = \iint_{D(x,y,f)} \Delta_\varepsilon(\xi,\varsigma)\left(\int_0^1 \frac{\partial F}{\partial z}(\xi,\varsigma,\phi_\varepsilon(u(\xi,\varsigma))) + \mu\Delta_\varepsilon(\xi,\varsigma))d\mu\right) d\xi d\varsigma.$$

Let $(x,y) \in K$, since $D(x,y,g) \subset K_\lambda$, if $g(y) \leq x$, we deduce

$$\begin{aligned} |w_\varepsilon(x,y)| &\leq m_{\lambda,\varepsilon} \int_{g(y)}^{x} \int_0^y |\Delta_\varepsilon(\xi,\zeta)|\, d\xi d\zeta \\ &\leq m_{\lambda,\varepsilon} \int_{-g(\lambda)}^{+g(\lambda)} \int_0^y (|w_\varepsilon(\xi,\varsigma)|\, c_{\lambda,\varepsilon} + a_{\lambda,\varepsilon})\, d\xi d\zeta \\ &\leq m_{\lambda,\varepsilon} c_{\lambda,\varepsilon} \int_{-g(\lambda)}^{+g(\lambda)} \int_0^y |w_\varepsilon(\xi,\varsigma)|\, d\xi d\varsigma + 2\lambda'\lambda m_{\lambda,\varepsilon} a_{\lambda,\varepsilon}. \end{aligned}$$

Set $b_{\lambda,\varepsilon} = 2\lambda'\lambda m_{\lambda,\varepsilon} a_{\lambda,\varepsilon}$ and $e_\varepsilon(y) = \sup_{\xi \in [g(-\lambda);g(\lambda)]} |w_\varepsilon(\xi,y)|$, then

$$|w_\varepsilon(x,y)| \leq m_{\lambda,\varepsilon} c_{\lambda,\varepsilon} 2\lambda' \int_0^y e_\varepsilon(\zeta) d\zeta + b_{\lambda,\varepsilon},$$

we deduce that,

$$e_\varepsilon(y) \leq m_{\lambda,\varepsilon} c_{\lambda,\varepsilon} 2\lambda' \int_0^y e_\varepsilon(\zeta) d\zeta + b_{\lambda,\varepsilon},$$

for every $y \in [0,\lambda]$. Thus according to the Gronwall's lemma

$$e_\varepsilon(y) \leq b_{\lambda,\varepsilon} \exp(\int_0^y m_{\lambda,\varepsilon} c_{\lambda,\varepsilon} 2\lambda' d\zeta).$$

We obtain the same result in the other cases, hence

$$\forall y \in [-\lambda,\lambda]\,,\ e_\varepsilon(y) \leq b_{\lambda,\varepsilon} \exp(m_{\lambda,\varepsilon} c_{\lambda,\varepsilon} 2\lambda'\lambda),$$

consequently

$$\|w_\epsilon\|_{\infty,K_\lambda} \leq b_{\lambda,\varepsilon} \exp(m_{\lambda,\varepsilon} c_{\lambda,\varepsilon} 2\lambda' \lambda).$$

As $(b_{\lambda,\varepsilon})_\varepsilon \in I_A$ and $\exp(m_{\lambda,\varepsilon} c_{\lambda,\varepsilon} 2\lambda'\lambda)$ is a constant, consequently $\left(\|w_\varepsilon\|_{\infty,K_\lambda}\right)_\varepsilon \in I_A$. According $(P1)$, we have $\left(\|\Delta_\varepsilon\|_{\infty,K_\lambda}\right)_\varepsilon \in I_A$ and according to $(P2)$, we have

$$\left(P_{K,0}\left(F(\cdot,\cdot,\sigma_\varepsilon\left(v_\varepsilon\right)) - F(\cdot,\cdot,\phi_\varepsilon\left(u_\varepsilon\right))\right)\right)_\varepsilon \in I_A.$$

So, according to Proposition 9, we deduce $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$; consequently u depends solely on the class $[f_\varepsilon]$ as a generalized function, not on the particular representative. Moreover, according to Proposition 9, $(\Delta_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ and $\left(F(\cdot,\cdot,\sigma_\varepsilon\left(v_\varepsilon\right)) - F(\cdot,\cdot,\phi_\varepsilon\left(u_\varepsilon\right))\right)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$.

4.5 Comparison with classical solutions

Remark 27 *The generalized solution to Problem (P_{gen}) is defined from the integral representation (3). Thus, we are going to study the relationship between this generalized function and the classical solutions to (P_{form}) (when they exist) on a domain Ω such that $\forall\,(x,y) \in \Omega$, $D(x,y,g) \subset \Omega$. This justified to choose $\Omega = \left]g(\mu), g(\nu)\right[\times\left]\mu,\nu\right[$ when $(\mu,\nu) \in \mathbb{R}^2$ with $\mu < 0 < \nu$.*
If the non regularized problem (P_{form}) has a smooth solution v on Ω then, necessarily we have $\Omega \subset \mathbb{R}^2 \backslash \operatorname{singsupp}(u)$.

Recall that there exists a canonical sheaf embedding of $\mathrm{C}^\infty(\cdot)$ into $\mathcal{A}(\cdot)$, through the morphism of algebra

$$\sigma_\Upsilon : \mathrm{C}^\infty(O) \to \mathcal{A}(O)\,,\ \ f \mapsto [f_\varepsilon]\ \ \text{(where } O \text{ is any open subset of } \mathbb{R}^2 \text{and } f_\varepsilon = f).$$

The presheaf $\mathcal{A}$ allows to restriction and as usually we denote by $u|_O$ the restriction on O of $u \in \mathcal{A}\left(\mathbb{R}^2\right)$.

Theorem 28 *Let Ω be an open subset of $\mathbb{R}^2$ such that $\Omega \subset \mathbb{R}^2 \backslash \operatorname{singsupp}(u)$. Assume that $\Omega = \bigcup_\varepsilon \Omega_\varepsilon$ with $(\Omega_\varepsilon)_\varepsilon$ is an increasing family of open subsets of $\mathbb{R}^2$ such that $\Omega_\varepsilon = \left]g(\mu_\varepsilon), g(\nu_\varepsilon)\right[\times\left]\mu_\varepsilon,\nu_\varepsilon\right[$ when $(\mu_\varepsilon,\nu_\varepsilon) \in \mathbb{R}^2$ with $\mu_\varepsilon < 0 < \nu_\varepsilon$. Assume that the non regularized problem has a smooth solution v on Ω such that $\sup\limits_{(x,y)\in\Omega_\varepsilon} |v(x,y)| < r_\varepsilon - 1$ for any ε. Let $u = [u_\varepsilon]$ be the solution to Problem (P_{gen}) given in Theorem 22. Then $\sigma_\Omega(v) = u|_\Omega$.*

We can choose as representative of $\sigma_\Omega(v)$ the net $(v)_\varepsilon$. We clearly have $\forall\,(x,y) \in \Omega, \exists\varepsilon_0, \forall\varepsilon \leq \varepsilon_0, (x,y) \in \Omega_\varepsilon$. Then $D(x,y,g) \subset \Omega_\varepsilon \subset \Omega$ and

$$v(x,y) = v_0(x,y) + \iint_{D(x,y,g)} F(\xi,\zeta,v(\xi,\zeta))d\xi d\zeta.$$

We take has representative of u the net $(u_\varepsilon)_\varepsilon$ given by Theorem 22. This net satisfies

$$\forall (x,y) \in \Omega, \quad u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \iint_{D(x,y,g)} F_\varepsilon(\xi,\zeta,u_\varepsilon(\xi,\zeta))d\xi d\zeta,$$

moreover $v_0 = u_{0,\varepsilon}$. Set $(w_\varepsilon)_\varepsilon = \left(u_\varepsilon|_\Omega - v\right)_\varepsilon$ and take $K \Subset \Omega$. There exists ε_1 such that, for all $\varepsilon < \varepsilon_1$, $K \Subset \Omega_\varepsilon$. According the definition of Ω_ε, there exists λ, $0 < \lambda < (\nu_\varepsilon - \mu_\varepsilon)/2$, such that $K \subset Q_\lambda \subset \Omega$ with $Q_\lambda = [g(\mu_\varepsilon + \lambda), g(\nu_\varepsilon - \lambda)] \times [\mu_\varepsilon + \lambda, \nu_\varepsilon - \lambda]$. Take $(x,y) \in K$, then $D(x,y,g) \subset Q_\lambda$. Note that, for $(\xi,\varsigma,z) \in \Omega_\varepsilon \times \left]-r_\varepsilon + 1, r_\varepsilon - 1\right[$, we have $F(\xi,\varsigma,z) = F_\varepsilon(\xi,\varsigma,z)$ by construction of F_ε. Thus v, which values are in $\left]-r_\varepsilon + 1, r_\varepsilon - 1\right[$, is solution of the same integral equation as u_ε, which admits a unique solution since F_ε is a smooth function of its arguments. Thus, for all $\varepsilon \leq \varepsilon_1$, v and u_ε are equal on Ω_ε. It follows that, for all $\varepsilon \leq \varepsilon_1$, $\sup_{(x,y)\in Q_\lambda} |w_\varepsilon(x,y)| = 0$, hence $(P_{K,l}(w_\varepsilon))_\varepsilon \in I_A$ for any $l \in \mathbb{N}$ as w_ε vanishes on K. Thus $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$ and $\sigma_\Omega(V) = U|_\Omega$ as claimed.

5 Case of irregular data

In this section, we assume that φ and ψ are themselves irregular data, say $\varphi \in \mathcal{A}(\mathbb{R})$, $\psi \in \mathcal{A}(\mathbb{R})$, where $\mathcal{A}(\mathbb{R})$ is define below. We replace problem (P_{form}) with the family of problems

$$\left(P_{(\varepsilon,\rho)}\right) \begin{cases} \dfrac{\partial^2}{\partial x \partial y} u_{\varepsilon,\rho}(x,y) = F\left(x,y,\phi_\varepsilon(u_{\varepsilon,\rho}(x,y))\right) = F_\varepsilon\left(x,y,u_{\varepsilon,\rho}(x,y)\right), \\ u_{\varepsilon,\rho}(x,0) = \varphi_\rho(x), \\ u_{\varepsilon,\rho}(g(y),y) = \psi_\rho(y), \end{cases}$$

where ε, ρ are parameters belonging to the interval $(0,1]$, $(\varphi_\rho)_\rho$ and $(\psi_\rho)_\rho$ are representatives of φ and ψ, $f \in \mathrm{C}^\infty(\mathbb{R})$. The parameter ε permits to replace Problem (P_{form}) by Lipschitz problems $\left(P_{(\varepsilon,\rho)}\right)$, whereas the parameter ρ makes it regular. Keeping assumption (H), assume that

$$\begin{cases} \mathcal{C} = A/I_A \text{ is overgenerated by the following elements of } \mathbb{R}_*^{(0,1]\times(0,1]} \\ \qquad (\varepsilon)_{(\varepsilon,\rho)}, (\rho)_{(\varepsilon,\rho)}, \left(e^{r_\varepsilon^p}\right)_{(\varepsilon,\rho)}. \\ \mathcal{A}\left(\mathbb{R}^2\right) \text{ and } \mathcal{A}(\mathbb{R}) \text{ are built on the same ring } \mathcal{C} \text{ of generalized constants.} \end{cases} \tag{H_1}$$

Assumption (H), implies that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under the family $(F_\varepsilon)_\varepsilon$.

Theorem 29 *If $u_{\varepsilon,\rho}$ is the solution to Problem $\left(P_{(\varepsilon,\rho)}\right)$ then Problem (P_{gen}) admits $u = [u_{\varepsilon,\rho}]_{\mathcal{A}(\mathbb{R}^2)}$ as solution.*

According to [8], [9], $u = [u_{\varepsilon,\rho}]$ is a solution to (P_{gen}) if $(u_{\varepsilon,\rho})_{(\varepsilon,\rho)} \in \mathcal{X}(\mathbb{R}^2)$. Then we will prove that

$$\forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \left(P_{K,l}(u_{\varepsilon,\rho})\right)_{(\varepsilon,\rho)} \in A.$$

We proceed by induction. We have: $\forall K \Subset \mathbb{R}^2, \exists K_\lambda \Subset \mathbb{R}^2, K \subset K_\lambda$,

$$\|u_{\varepsilon,\rho}\|_{\infty,K} \leq \|u_{\varepsilon,\rho}\|_{\infty,K_\lambda} \leq \|u_{0,\varepsilon,\rho}\|_{\infty,K_\lambda} \left(1+\exp[\lambda\mu_1 r_\varepsilon^p \left(g(\lambda)-g(-\lambda)\right)]\right).$$

Like previously, we can prove that $\left(\|u_{\varepsilon,\rho}\|_{\infty,K}\right)_{(\varepsilon,\rho)} \in A$ that is $\left(P_{K,0}\left(u_{\varepsilon,\rho}\right)\right)_{(\varepsilon,\rho)} \in A$, then the 0th order estimate is verified. We have

$$\frac{\partial u_{\varepsilon,\rho}}{\partial x}(x,y) = \frac{\partial u_{0,\varepsilon,\rho}}{\partial x}(x,y) + \int_0^y F_\varepsilon(x,\zeta,u_{\varepsilon,\rho}(x,\zeta))d\zeta,$$

and

$$\frac{\partial u_{\varepsilon,\rho}}{\partial y}(x,y) = \frac{\partial u_{0,\varepsilon,\rho}}{\partial y}(x,y)+\int_{g(y)}^x F_\varepsilon(\xi,y,u_{\varepsilon,\rho}(\xi,y))d\xi - g'(y)\int_0^y F_\varepsilon(g(y),\zeta,u_{\varepsilon,\rho}(g(y),\zeta))d\zeta,$$

hence $\left(P_{K,(1,0)}(u_{\varepsilon,\rho})\right)_{(\varepsilon,\rho)} \in A$ and $\left(P_{K,1}(u_{\varepsilon,\rho})\right)_{(\varepsilon,\rho)} \in A$.

Suppose that we have $\left(P_{K,l}(u_{\varepsilon,\rho})\right)_{(\varepsilon,\rho)} \in A$ for any $l \leq n$. We have $u_{0,\varepsilon,\rho}(x,y) = \psi_\rho(y)+\varphi_\rho(x)-\varphi_\rho(g(y))$, then $\left(\left\|\frac{\partial^{n+1}u_{0,\varepsilon,\rho}}{\partial x^{n+1}}\right\|_{\infty,K}\right)_{(\varepsilon,\rho)} \in A$ because $[\varphi_\rho]$ and $[\psi_\rho]$ are elements of $\mathcal{A}(\mathbb{R})$. For $n \geq 1$, replacing u_ε by $u_{\varepsilon,\rho}$ and computing the successive derivatives $\dfrac{\partial^{n+1}u_{\varepsilon,\rho}}{\partial y^{n+1}}$ and $\dfrac{\partial^{n+1}u_{\varepsilon,\rho}}{\partial y^{n+1}}$, we get similar estimates as those ones of Theorem 22. Like previously we can prove that $\left(P_{K,(n+1,0)}\left(u_{\varepsilon,\rho}\right)\right)_{(\varepsilon,\rho)} \in A$ and $\left(P_{K,(0,n+1)}\left(u_{\varepsilon,\rho}\right)\right)_{(\varepsilon,\rho)} \in A$ for any $K \Subset \mathbb{R}^2$ and $n \in \mathbb{N}$. In the same way $\left(P_{3,n}(u_{\varepsilon,\rho})\right)_{(\varepsilon,\rho)} \in A$ and $\left(P_{4,n}(u_{\varepsilon,\rho})\right)_{(\varepsilon,\rho)} \in A$. Finally, we have $\left(P_{K,n+1}(u_{\varepsilon,\rho})\right)_{(\varepsilon,\rho)} \in A$.

Theorem 30 *Under the same hypotheses as subsection 4.4, the solution* $u = [u_{\varepsilon,\rho}]_{\mathcal{A}(\mathbb{R}^2)}$ *does not depend of the choice of the representative* $(f_\varepsilon)_\varepsilon$ *of the class* $f \in \mathcal{T}(\mathbb{R})/\mathcal{N}(\mathbb{R})$.

We have $\forall n \in \mathbb{N}, \exists \mu_n > 0, \forall \varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \forall \alpha \in \mathbb{N}^3$ with $|\alpha| = n$,

$$\sup_{(x,y)\in K;\ z\in\mathbb{R}} |D^\alpha F_\varepsilon(x,y,z)| \leq \mu_n r_\varepsilon^p \text{ and } \sup_{(x,y)\in K;\ z\in\mathbb{R}} |D^\alpha H_\varepsilon(x,y,z)| \leq \mu_n r_\varepsilon^p.$$

According to the results of Theorem 29, let $u = [u_{\varepsilon,\rho}]$ the solution to (P_{gen}) defined by the family $(F_\varepsilon)_\varepsilon$ associated to F *via* the family $(f_\varepsilon)_\varepsilon$and $v = [v_{\varepsilon,\rho}]$ another solution to (P_{gen}) defined by the family $(H_\varepsilon)_\varepsilon$ associated to F *via* the family $(h_\varepsilon)_\varepsilon$. We have

$$v_{\varepsilon,\rho}(x,y) = u_{0,\varepsilon,\rho}(x,y) + \iint_{D(x,y,g)} F(\xi,\zeta,\sigma_{\varepsilon,\rho}\left(v_{\varepsilon,\rho}(\xi,\zeta)\right))d\xi d\zeta + j_{\varepsilon,\rho}(x,y),$$

with $(j_{\varepsilon,\rho})_{\varepsilon,\rho} \in \mathcal{N}(\mathbb{R}^2)$ .We will prove that $(w_{\varepsilon,\rho})_{(\varepsilon,\rho)} = (v_{\varepsilon,\rho} - u_{\varepsilon,\rho})_{(\varepsilon,\rho)} \in \mathcal{N}(\mathbb{R}^2)$. Let,

$$\Delta_{\varepsilon,\rho}(x,y) = \sigma_\varepsilon\left(v_{\varepsilon,\rho}(x,y)\right) - \phi_\varepsilon\left(u_{\varepsilon,\rho}(x,y)\right),$$

with $\sigma_\varepsilon(z) = zh_\varepsilon(z)$.

We have $\forall K \Subset \mathbb{R}^2_, \ \exists\ K_\lambda \Subset \mathbb{R}^2_, \ K \subset K_\lambda$ and replacing u_ε (resp.v_ε) by $u_{\varepsilon,\rho}$ (resp. $v_{\varepsilon,\rho}$) we get estimates like those ones of Theorem 26, so

$$\left|\Delta_{\varepsilon,\rho}(x,y)\right| \le \left|w_{\varepsilon,\rho}(x,y)\right| c_{\lambda,\varepsilon} + a_{\lambda,\varepsilon}.$$

where $(a_{\lambda,\varepsilon})_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$ and $c_{\lambda,\varepsilon} = \left(1 + \|u_{0,\varepsilon,\rho}\|_{\infty,K_\lambda}(1+\exp[2\lambda'\lambda\mu_1 r_\varepsilon^p])\, M_1'\right)$ with $2\lambda' = (g(\lambda) - g(-\lambda))$. As

$$\begin{aligned}
&F(\xi,\zeta,\sigma_\varepsilon(v_{\varepsilon,\rho}(\xi,\zeta))) - F(\xi,\zeta,\phi_\varepsilon(u_{\varepsilon,\rho}(\xi,\zeta))) \qquad (P2)\\
&= \Delta_{\varepsilon,\rho}(\xi,\varsigma)\left(\int_0^1 \frac{\partial F}{\partial z}(\xi,\varsigma,\phi_\varepsilon(u_{\varepsilon,\rho}(\xi,\varsigma))) + \mu\Delta_{\varepsilon,\rho}(\xi,\varsigma))d\mu\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
w_{\varepsilon,\rho}(x,y) &= j_{\varepsilon,\rho}(x,y)\\
&+ \iint_{D(x,y,g)} w_{\varepsilon,\rho}(\xi,\zeta)\left(\int_0^1 \frac{\partial F}{\partial z}(\xi,\varsigma,\phi_\varepsilon(u_{\varepsilon,\rho}(\xi,\varsigma))) + \mu\Delta_{\varepsilon,\rho}(\xi,\varsigma))d\mu\right) d\xi d\zeta.
\end{aligned}$$

Let $(x,y) \in K_\lambda$, since $D(x,y,g) \subset K_\lambda$, if $g(y) \le x$, we have

$$\begin{aligned}
\left|w_{\varepsilon,\rho}(x,y)\right| &\le m_{\lambda,\varepsilon} \int_{g(y)}^{x}\int_0^y \left|w_{\varepsilon,\rho}(\xi,\zeta)\right| d\xi d\zeta + \|j_{\varepsilon,\rho}\|_{\infty,K_\lambda}\\
&\le m_{\lambda,\varepsilon} \int_{-g(\lambda)}^{+g(\lambda)}\int_0^y \left|w_{\varepsilon,\rho}(\xi,\zeta)\right| d\xi d\zeta + \|j_{\varepsilon,\rho}\|_{\infty,K_\lambda}.
\end{aligned}$$

Put $e_{\varepsilon,\rho}(y) = \sup\limits_{\xi\in[g(-\lambda);g(\lambda)]} \left|w_{\varepsilon,\rho}(\xi,y)\right|$, then

$$\left|w_{\varepsilon,\rho}(x,y)\right| \le m_{\lambda,\varepsilon} 2\lambda' \int_0^y e_{\varepsilon,\rho}(\zeta)d\zeta + \|j_{\varepsilon,\rho}\|_{\infty,k_\lambda},$$

we deduce that, for every $y \in [0,\lambda]$, if $g(y) \le x$,

$$e_{\varepsilon,\rho}(y) \le m_{\lambda,\varepsilon} 2\lambda' \int_0^y e_{\varepsilon,\rho}(\eta)d\eta + \|j_{\varepsilon,\rho}\|_{\infty,K_\lambda}.$$

Thus according to the Gronwall's lemma, for every $y \in [0,\lambda]$, if $g(y) \le x$,

$$e_{\varepsilon,\rho}(y) \le \left(\exp(\int_0^y m_{\lambda,\varepsilon} 2\lambda d\zeta)\right) \|j_\varepsilon\|_{\infty,K_\lambda}.$$

For every $y \in [0,\lambda]$, if $g(y) \le x$,

$$e_{\varepsilon,\rho}(y) \le (\exp(m_{\lambda,\varepsilon} 2\lambda'\lambda))\, \|j_{\varepsilon,\rho}\|_{\infty,K_\lambda}.$$

We obtain the same result in the other cases, hence

$$\forall y \in [-\lambda,\lambda],\ e_{\varepsilon,\rho}(y) \le \|j_{\varepsilon,\rho}\|_{\infty,K_\lambda} \exp(m_{\lambda,\varepsilon} 2\lambda'\lambda),$$

consequently
$$\|w_{\varepsilon,\rho}\|_{\infty,K_\lambda} \leq \|j_{\varepsilon,\rho}\|_{\infty,K_\lambda} (\exp(m_{\lambda,\varepsilon} 2\lambda'\lambda) .$$
As $(j_{\varepsilon,\rho})_{\varepsilon,\rho} \in \mathcal{N}(\mathbb{R}^2)$ so $\left(\|j_{\varepsilon,\rho}\|_{\infty,K_\lambda}\right)_{\varepsilon,\rho} \in I_A$. Moreover $(\exp(m_{\lambda,\varepsilon} 2\lambda'\lambda))$ is a constant, consequently $\left(\|w_{\varepsilon,\rho}\|_{\infty,K_\lambda}\right)_\varepsilon \in I_A$. Which implies the 0th order estimate. According to Proposition 9, we deduce $(w_{\varepsilon,\rho})_{\varepsilon,\rho} \in \mathcal{N}(\mathbb{R}^2)$; consequently u depends solely on the class $[f_\varepsilon]$ as a generalized function, not on the particular representative.

Consider the family of problems
$$(P_\rho) \begin{cases} \dfrac{\partial^2}{\partial x \partial y} u_\rho(x,y) = F(x,y,u_\rho(x,y)), \\ u_\rho(x,0) = \varphi_\rho(x), \\ u_\rho(g(y),y) = \psi_\rho(y), \end{cases}$$
where $(\varphi_\rho)_\rho$ and $(\psi_\rho)_\rho$ are representatives of φ and ψ in $\mathcal{A}(\mathbb{R})$ defined in assumption (H_1).

Theorem 31 *Let Ω be an open subset of $\mathbb{R}^2$ such that $\Omega \subset \mathbb{R}^2 \backslash \mathrm{singsupp}(u)$. Assume that $\Omega = \bigcup_\varepsilon \Omega_\varepsilon$ with $(\Omega_\varepsilon)_\varepsilon$ is an increasing family of open subsets of $\mathbb{R}^2$ such that $\Omega_\varepsilon =]g(\mu_\varepsilon), g(\nu_\varepsilon)[\times]\mu_\varepsilon, \nu_\varepsilon[$ when $(\mu_\varepsilon, \nu_\varepsilon) \in \mathbb{R}^2$ with $\mu_\varepsilon < 0 < \nu_\varepsilon$. Assume that, for any ρ, problem (P_ρ) has a smooth solution v_ρ on Ω such that $\sup_{(x,y)\in\Omega_\varepsilon} |v_\rho(x,y)| < r_\varepsilon - 1$ for any ε. Let $u = [u_{\varepsilon,\rho}]$ be the solution to Problem (P_{gen}) given in Theorem 29. Then the family $(v_\rho)_{(\varepsilon\rho)}$ is a representative of a generalized function v which belongs to the algebra $\mathcal{A}(\Omega)$ and $v = u|_\Omega$.*

We clearly have $\forall (x,y) \in \Omega$, $D(x,y,g) \subset \Omega_\varepsilon \subset \Omega$ and following [8], [9]
$$v_\rho(x,y) = v_{0,\rho}(x,y) + \iint_{D(x,y,g)} F(\xi,\zeta,v_\rho(\xi,\zeta)) d\xi d\zeta.$$
Replacing $u_{\varepsilon,\rho}$ by v_ρ we can prove, like in Theorem 29, that $(P_{K,n}(v_\rho))_{(\varepsilon,\rho)} \in A$ for any $K \Subset \Omega$ and $n \in \mathbb{N}$. Then $v \in \mathcal{A}(\Omega)$.

We take has representative of u the net $(u_\varepsilon)_\varepsilon$ given by Theorem 29. This net satisfies
$$\forall (x,y) \in \Omega, \quad u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) + \iint_{D(x,y,g)} F_\varepsilon(\xi,\zeta,u_\varepsilon(\xi,\zeta)) d\xi d\zeta$$
and $v_{0,\rho}(x,y) = u_{0,\varepsilon,\rho}(x,y)$.

Take $K \Subset \Omega$. There exists ε_1, such that, for all $\varepsilon < \varepsilon_1$, $K \Subset \Omega_\varepsilon$. According the definition of Ω_ε, there exists λ, $0 < \lambda < (\nu_\varepsilon - \mu_\varepsilon)/2$, such that $K \subset Q_\lambda \subset \Omega$ with $Q_\lambda = [g(\mu_\varepsilon + \lambda), g(\nu_\varepsilon - \lambda)] \times [\mu_\varepsilon + \lambda, \nu_\varepsilon - \lambda]$. Note that, for $(\xi,\varsigma,z) \in \Omega_\varepsilon \times]-r_\varepsilon + 1, r_\varepsilon - 1[$, we have $F(\xi,\varsigma,z) = F_\varepsilon(\xi,\varsigma,z)$ by construction of F_ε. Set $(w_{\varepsilon,\rho})_{(\varepsilon,\rho)} = \left(u_{\varepsilon,\rho}|_\Omega - v_\rho\right)_{(\varepsilon,\rho)}$. Take $(x,y) \in K$, then

$D(x,y,g)\subset Q_\lambda$. As previously, Theorem 28, we can prove that, for all $\varepsilon\leq\varepsilon_1$, $\sup_{(x,y)\in Q_\lambda}|w_{\varepsilon,\rho}(x,y)|=0$, hence $(P_{K,l}(w_{\varepsilon,\rho}))_{(\varepsilon,\rho)}\in I_A$ for any $l\in\mathbb{N}$ as $w_{\varepsilon,\rho}$ vanishes on K. Thus $(w_{\varepsilon,\rho})_{(\varepsilon,\rho)}\in\mathcal{N}(\Omega)$ and $v=u|_\Omega$ as claimed, that is, there exists $(\sigma_{\varepsilon,\rho})_{(\varepsilon,\rho)}\in\mathcal{N}(\Omega)$ such that

$$\forall\,(x,y)\in\Omega,\forall\varepsilon,\quad u_{\varepsilon,\rho}(x,y)=v_\rho(x,y)+\sigma_{\varepsilon,\rho}(x,y).$$

5.1 A degenerate Goursat problem in $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras

We study the Goursat problem in the case where φ and ψ are one-variable generalized functions, $\gamma=(Oy)$. (We take $g=0$).

We replace Problem (P_{form}) with the family of problems

$$\left(P_{(\varepsilon,\rho)}\right)\begin{cases}\dfrac{\partial^2}{\partial x\partial y}u_{\varepsilon,\rho}(x,y)=F_\varepsilon(x,y,u_{\varepsilon,\rho}(x,y)),\\ u_{\varepsilon,\rho}(x,0)=\varphi_\rho(x),\\ u_{\varepsilon,\rho}(0,y)=\psi_\rho(y),\end{cases}$$

where $(\varphi_\rho)_\rho$ and $(\psi_\rho)_\rho$ are representatives of φ and ψ in $\mathcal{A}(\mathbb{R}^2)$.

Proposition 32 *If $u_{\varepsilon,\rho}$ is the solution to Problem $\left(P_{(\varepsilon,\rho)}\right)$ then Problem (P_{gen}) admits $u=[u_{\varepsilon,\rho}]_{\mathcal{A}(\mathbb{R}^2)}$ as solution.*

Moreover

$$\begin{aligned}u_{\varepsilon,\rho}(x,y)&=u_{0,\varepsilon,\rho}(x,y)+\iint_{D(x,y,0)}F_\varepsilon(\xi,\eta,u_{\varepsilon,\rho}(\xi,\eta))d\xi d\eta\\&=u_{0,\varepsilon,\rho}(x,y)+\int_0^x\left(\int_0^y F_\varepsilon(\xi,\eta,u_{\varepsilon,\rho}(\xi,\eta))d\eta\right)d\xi,\end{aligned}$$

with $u_{0,\varepsilon,\rho}(x,y)=\psi_\rho(y)+\varphi_\rho(x)-\varphi_\rho(0)$.

6 Examples

Example 33 *For α real, consider the functions*

$$h_{+,\alpha}(x)=\begin{cases}x^{-\alpha}\ \text{if } x>0\\ 0\ \text{if } x<0\end{cases}\quad\text{and } h_{-,\alpha}(x)=\begin{cases}0\ \text{if } x>0\\ |x|^{-\alpha}\ \text{if } x<0\end{cases}.$$

Set

$$\begin{aligned}h_\alpha(x)&=h_{+,\alpha}(x)+h_{-,\alpha}(x)\\ g_\alpha(x)&=h_{+,\alpha}(x)-h_{-,\alpha}(x).\end{aligned}$$

Then

$$\begin{aligned}h'_\alpha(x)&=(-\alpha)\left(h_{+,\alpha+1}(x)-h_{-,\alpha+1}(x)\right)=(-\alpha)\,g_{\alpha+1}(x)\\ g'_\alpha(x)&=(-\alpha)\left(h_{+,\alpha}(x)+h_{-,\alpha}(x)\right)=(-\alpha)\,h_{\alpha+1}(x).\end{aligned}$$

Consider $Pf.h_\alpha$ (resp. $Pf.g_\alpha$) the Hadamard's finite-part of h_α (resp g_α). Consider the problem

$$(P_{form})\begin{cases} \dfrac{\partial^2 u}{\partial x\partial y} = |u|^P, \\ u_{|(Ox)} = \varphi, \\ u|_\gamma = \psi \end{cases}$$

where $\varphi \in \mathcal{A}(\mathbb{R})$, $\psi \in \mathcal{A}(\mathbb{R})$, γ is the curve of equation $x = \eta y$ where $\eta \in (0,1]$, η fixed, p is an integer, $p > 2$. Set $\alpha = (p-1)^{-1}$, $\beta = \alpha^{2\alpha}$. Keeping assumption (H), suppose that

$$\begin{cases} \mathcal{C} = A/I_A \text{ is overgenerated by the following elements of } \mathbb{R}_*^{]0,1]\times]0,1]} \\ (\rho)_{(\varepsilon,\rho)}, (\varepsilon)_{(\varepsilon,\rho)}, \left(e^{r_\varepsilon^p}\right)_{(\varepsilon,\rho)}. \\ \mathcal{A}\left(\mathbb{R}^2\right) \text{ and } \mathcal{A}(\mathbb{R}) \text{ are built on the same ring } \mathcal{C} \text{ of generalized constants.} \end{cases} \qquad (H_2)$$

Let

$$(P_\rho)\begin{cases} \dfrac{\partial^2 u}{\partial x\partial y} = |u|^p, \\ u_{|(Ox)} = \varphi_\rho, \\ u|_\gamma = \psi_\rho \end{cases}$$

with

$$\varphi_\rho(x) = \beta\left[(l_\rho * Pf.h_\alpha)(x)\right]\left[(l_\rho * Pf.h_\alpha)(0)\right],$$
$$\psi_\rho(y) = \beta\left[(l_\rho * Pf.h_\alpha)(\eta y)\right]\left[(l_\rho * Pf.h_\alpha)(y)\right],$$

where $(l_\rho)_\rho$ is a family of mollifiers ($l \in \mathcal{D}(\mathbb{R})$ $\int l(x)\,dx = 1$ and $l_\rho(x) = \frac{1}{\rho} l\left(\frac{x}{\rho}\right)$). Assume that $(\varphi_\rho)_\rho$ is a representative of φ and $(\psi_\rho)_\rho$ is a representative of ψ. We replace Problem (P_{form}) with the family of problems

$$\left(P_{(\varepsilon,\rho)}\right)\begin{cases} \dfrac{\partial^2}{\partial x\partial y} u_{\varepsilon,\rho}(x,y) = |\phi_\varepsilon(u_{\varepsilon,\rho}(x,y))|^p, \\ u_{,\varepsilon,\rho}(x,0) = \varphi_\rho(x), \\ u_{\varepsilon,\rho}(\eta y, y) = \psi_\rho(y). \end{cases}$$

If $u_{\varepsilon,\rho}$ is solution to $\left(P_{(\varepsilon,\rho)}\right)$ then $u = [u_{\varepsilon,\rho}]$ is solution to (P_{gen}).
Set $\Omega =]\eta\mu, \eta\nu[\times]\mu,\nu[$ with $\mu < 0 < \nu$. The solution v_ρ to the non regularized problem (P_ρ) on Ω is defined by

$$v_\rho(x,y) = \beta\left[(l_\rho * Pf.h_\alpha)(x)\right]\left[(l_\rho * Pf.h_\alpha)(y)\right].$$

According the previous results the family $(v_\rho)_{(\varepsilon\rho)}$ is a representative of a generalized function v and $v = u|_\Omega$.

Example 34 *We consider the problem*

$$(P'_{form})\begin{cases} \dfrac{\partial^2 u}{\partial x\partial y} = |u|^{p-1} u, \\ u_{|(Ox)} = \varphi, \\ u|_\gamma = \psi \end{cases}$$

where $\varphi \in \mathcal{A}(\mathbb{R})$, $\psi \in \mathcal{A}(\mathbb{R})$, γ *is the curve of equation* $x = \eta y$ *with* $\eta \in (0,1]$, η *fixed,* p *is an integer,* $p > 2$. *Set* $\alpha = (p-1)^{-1}$, $\beta = \alpha^{2\alpha}$. *Keep hypotheses (H) and* (H_2). *Let*

$$\left(P'_\rho\right)\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = |u|^{p-1}\, u, \\ u_{|(Ox)} = \varphi_\rho, \\ u|_\gamma = \psi_\rho \end{cases}$$

with

$$\varphi_\rho(x) = \beta\left[\left(l_\rho * Pf.g_\alpha\right)(x)\right]\left[\left(l_\rho * Pf.g_\alpha\right)(0)\right],$$
$$\psi_\rho(y) = \beta\left[\left(l_\rho * Pf.g_\alpha\right)(\eta y)\right]\left[\left(l_\rho * Pf.g_\alpha\right)(y)\right],$$

where $\left(l_\rho\right)_\rho$ *is a family of mollifiers (*$l \in \mathcal{D}(\mathbb{R})$ $\int l(x)\, dx = 1$ *and* $l_\rho(x) = \frac{1}{\rho} l\left(\frac{x}{\rho}\right)$*). Assume that* $\left(\varphi_\rho\right)_\rho$ *is a representative of* φ *and* $\left(\psi_\rho\right)_\rho$ *is a representative of* ψ. *We replace Problem* $\left(P'_{form}\right)$ *with the family of problems*

$$\left(P'_{(\varepsilon,\rho)}\right)\begin{cases} \dfrac{\partial^2}{\partial x \partial y} u_{\varepsilon,\rho}(x,y) = \left|\phi_\varepsilon(u_{\varepsilon,\rho}(x,y))\right|^{p-1} \phi_\varepsilon(u_{\varepsilon,\rho}(x,y)), \\ u_{,\varepsilon,\rho}(x,0) = \varphi_\rho(x), \\ u_{\varepsilon,\rho}(\eta y, y) = \psi_\rho(y). \end{cases}$$

If $u_{\varepsilon,\rho}$ *is solution to* $\left(P'_{(\varepsilon,\rho)}\right)$ *then* $[u_{\varepsilon,\rho}]$ *is solution to* $\left(P'_{gen}\right)$.
Set $\Omega =]\eta\mu, \eta\nu[\, \times\,]\mu, \nu[$, $\mu < 0 < \nu$, *the solution* v_ρ *to the non regularized problem* $\left(P'_\rho\right)$ *on* Ω *is defined by*

$$v_\rho(x,y) = \beta\left[\left(l_\rho * Pf.g_\alpha\right)(x)\right]\left[\left(l_\rho * Pf.g_\alpha\right)(y)\right].$$

According the previous results the family $\left(v_\rho\right)_{(\varepsilon\rho)}$ *is a representative of a generalized function* v *and* $v = u|_\Omega$.

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