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# Energy-aware and entropy coding for Networked Controlled Linear Systems

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## SUMMARY

This paper addresses issues on coding design in the context of control of systems equipped with low-energy sensors networks. We particularly focus on issues concerning minimum bit and energy-aware coding. To this aim, we devise a coding strategy with the ability to quantify and to differentiate stand-still signal events from changes in the source (level crossing detector). Coding is effectuated by defining at least 3-valued alphabet for the minimum bit case, and  $(2L + 1)$ -valued alphabet for a general case with a precision depending on  $L$ . Energy saves are studied in two different scenarios; (1) in the word-by-word transmission case, the stand-still signal event is modulated with a low power transmission mechanisms, whereas the changes of levels will be modulated with high-power, (2) in the package-based transmission case, an entropy variable length coding is added to the previous encoding process. Entropy coding assigns some probability distribution to the events, so that the mean transmission energy can be substantially improved for systems where the stand-still events have higher probability to arise (i.e. stable systems). The paper studies the stability properties needed for this type of coding

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to operate properly, and quantify the energy saves for each of the considered scenarios. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS:

Differential coding, Entropy coding, Networked controlled systems, NCS, Quantized systems.

## 1. Introduction

Wireless low-cost sensor networks are an expanded technology in many new and varied areas such as: traffic monitoring and control (urban, highways), undersea monitoring/exploration, environment sensing (forest, farms, etc.), building services, large instruments with distributed sensing and actuators (Tokamak, telescopes), etc.

In this context, future generation of this type of sensors are expected to be packaged together with communication protocols, RF electronics, and energy management systems. Therefore, the development of such integrated sensors will be driven by constraints like: low cost, ease of replacement, low energy consumption, and energy-efficient communication links. In turn, these constraints bring new problems to be considered in the exploitation of this information. For instance, low cost will induce sensors with low resolution (binary sensors, at the extreme) advocating for minimum bit coding strategies, low consumption will impose issues on efficient sensor energy management (sleep and wake-up modes, differentiation of stand-still event), ease of replacement will imply the system ability to keep safe operation in a failure of one or several sensors, and finally communication links and protocols should be designed to account for energy savings, information loss, and varying fading characteristics. Some of the coding strategies proposed in the literature [8], [4], [11], [12], [10], [6], [1], often disregards issues related with energy use, and discrimination of type of events. The objective of this paper is specifically to treat aspect related to the energy management, in relation with the particular code to be used. To this aim, we propose to use a coding strategy based in the following 3 main ingredients:

- **Differential coding** (see [3], [7], [5]) encodes the differences (error prediction) between successive samples rather than the samples themselves. Since differences between samples are expected to be smaller than the actual samples amplitudes, fewer bits are required to represent the differences. However, in its standard form no specific distinction is made for the stand-still events.
- **Event-based coding** has the ability to quantify and to differentiate stand-still signal events from changes in the source (level crossing detector). Coding is effectuated by defining at least 3-valued alphabet for the minimum bit case, and  $(2L+1)$ -valued alphabet for a general case with a precision inversely proportional to  $L \in \mathbb{Z}^+$ . Hence, the stand-still signal event is modulated with a low-power signal, whereas the changes of levels will be modulated with a high-power one.
- **Entropy coding** can be added to improve the energy use by assigning a probability distribution to the events. In that way, the mean transmission energy can be substantially improved for systems where the stand-still events may have high probability to occur

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(i.e. stable systems).

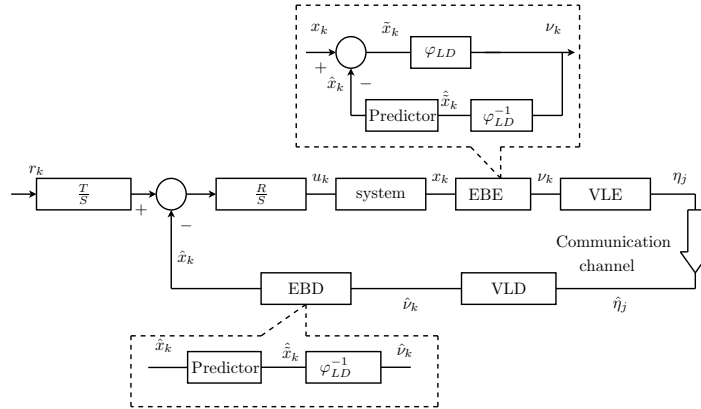


Figure 1. Block diagram of the non uniform entropy coding in the feedback loop.

A pre-requisite for the entropy coding strategy is to design a mechanism with the ability to quantify and to differentiate stand-still signal events, from changes in the source (level crossing detector, denoted here as  $\varphi_{LD}$ ). For instance, this can be done by defining an alphabet where the source signal information is contained in the time interval between level crossing and in the direction of the level crossing. As suggested in [9], by assigning strings of the 2-tuple 00 to represent the time between signal level crossing, and 01 and 10 to denote the direction of level crossing, the output of the level crossing detector contains a high probability of the 0 symbol with makes it suitable for an entropy encoder to attain a “good” overall compression ratio, and hence (as it will be shown here), a substantial improvements of energy saves. A fundamental difference with the classical Differential algorithm (i.e. delta modulation, see [3]) is that the error is coded on the basis of a 3-valued alphabet rather than a 2-valued one. If the levels are uniformly spaced and constant then the signal prediction precision, and hence the resulting closed-loop behavior, will be limited by the size of the level spacing. One alternative to improve the prediction precision is to make this level adaptive in the same spirit as [5], but in this context the stability analysis will be much more involved. Instead, we propose here to increase the number of levels to  $2L + 1$ , where  $L \in \mathbb{Z}^+$ , so as to match the required precision. This leads to a  $2L + 1$ -word alphabet, which can still be combined with a energy-efficient variable length (VLE) entropy coding to improve the use of energy for cases and systems where the stand-still events have a substantial probability to arise.

The overall coding strategy studied here is shown in Figure 1, and it is composed of two main blocks: an event-based (EB) coding, and a variable length-block entropy (VLE) coding scheme. The paper aims at studding the closed-loop properties of such arrangement, and investigate and evaluate the possible improvement in term of energy saves that the scheme may provide.

### 1.1. Definitions

Signals are sampled on the basis of the time interval  $T_s$ , that is at  $T_s, 2T_s, \dots, kT_s$ .

- $r_k$ : reference signal,
- $x_k$ : system output,
- $\hat{x}_k$ : estimated(reconstructed) output,
- $\tilde{x}_k$ : true estimated error,  $\tilde{x}_k = x_k - \hat{x}_k$ ,
- $\hat{\tilde{x}}_k$ : approximated estimated error, obtained after reconstruction, i.e.  $\hat{\tilde{x}}_k = \{\varphi_{LD}^{-1} \circ \varphi_{LD}\}(\tilde{x}_k)$ , with  $\varphi_{LD}^{-1} \circ \varphi_{LD} \neq 1$ .
- $\Delta$ : step interval used for level detection and to reconstruct  $\hat{\tilde{x}}_k$ ,
- $\delta_k$ :  $(2L + 1)$ -level valued integer signal, belonging to the interval  $[-L, L]$ , with  $L > 1, \in \mathcal{Z}^+$
- $\nu_k$ :  $\lceil \log_2(2L + 1) \rceil$ -bits binary signal
- $\eta_j$ : binary signal to be send by the channel (output of the VLE block) in asynchronous manner, at the time instants multiples of  $T_s$ . The index  $j$  captures this asynchronism.
- $u_k$ : control input

### 1.2. Assumptions

The hypothesis used in the results presented in this paper, are the following:

- The transmitted information is binary
- Only encoder-to-decoder information transmission is allowed (feedback between decoder to encoder is not allowed here),
- Reliable noiseless channel transmission is considered (no data lost, or information distortion, no transmission delays). See [2] for the treatment of transmission delay in this context.

## 2. Problem set up

We consider the following SISO discrete-time linear system (possible unstable), of the form,

$$x_k = \frac{B(q^{-1})}{A(q^{-1})} u_k \quad (1)$$

together with a RST controller,

$$u_k = \frac{R(q^{-1})}{S(q^{-1})} \left\{ \frac{\gamma}{T(q^{-1})} r_k - \hat{x}_k \right\} \quad (2)$$

where  $r_k$  is the reference,  $\hat{x}_k$  is the estimated of the system output  $x_k$ , and  $R(q^{-1}), S(q^{-1}), T(q^{-1})$  are the control polynomials in the delay operator  $q^{-1}$ . They also satisfy:

$$T = RB, \quad SA + RB = A_{cl}, \quad \gamma \triangleq A_{cl}(1)$$

with  $A_{cl}$  being the stable closed-loop polynomial, and  $\gamma$  the static gain needed to reach unitary zero-frequency gain. For simplicity, we will omit the use of the argument ( $q^{-1}$ ) when not explicitly needed.

The coding process consists of: 1) encoding the system output  $x_k$ , 2) transmitting the coding sequence through the communication channel, and 3) decoding the received information to produce the estimated  $\hat{x}_k$ . The complete sequence can be seen as estimation process.

**Forced synchronization under asynchronous transmission** The time basis for the system and the controller description is defined with respect to  $T_s$ . However, due to the variable length characteristic inherent to the entropy coding, the transmission of the coded signal  $\nu_j$  is then done asynchronously at multiples of  $T_s$ . For instance, for a choice of a VLE coding of length  $M = 3$ , the coded signal can be sent either at;  $T_s$ , or  $2T_s$ , or  $3T_s$  as shown in Table II. The index  $j$  captures this asynchronism (more details on the VLE code is given in latter sections).

It is assumed here that when the receiver does not receive information (this may happen if the run sequence includes a stand-still event for some  $k$ ) the receiver hold the last-received value  $\hat{\eta}_{k-1}$  until a new change of level is detected. By this mechanism, the signals at the receiver can be re-synchronized to the time basis  $T_s$ . This is the reason why the control formulation is stated in a discrete-time synchronous representation with the sole index  $k$ , as described next.

**Nominal closed-loop transfer function** Assume a perfect transmission process (i.e.  $\hat{x}_k \equiv x_k$ ), then the control law (2) gives the following nominal closed-loop relation,

$$x_k = \frac{\gamma}{A_{cl}(q^{-1})} r_k$$

**Perturbed closed-loop transfer function** Consider the case of interest where information is transmitted by the channel and quantized, i.e.  $\hat{x}_k \neq x_k$ . Then, the error transfer function is

$$x_k = \frac{\gamma}{A_{cl}(q^{-1})} r_k + W(q^{-1}) \tilde{x}_k$$

where  $\tilde{x}_k = x_k - \hat{x}_k$  is the estimation error, and  $W = BR/A_{cl}$ . As  $A_{cl}$  defines a stable polynomial, the output  $x_k$  is kept bounded as long as  $\tilde{x}_k$  is bounded as well.

The problem is then to design the coding process that defines the output  $\hat{x}_k$  preserving closed-loop properties.

### 2.1. Coding Process

The coding (encoding/decoding) process is composed of several steps, described by the following operations:

$$x_k \xrightarrow{\text{EBE}} \nu_k \xrightarrow{\text{VLE}} \eta_k \xrightarrow{\text{channel}} \hat{\eta}_k \xrightarrow{\text{VLD}} \hat{\nu}_k \xrightarrow{\text{EBD}} \hat{x}_k$$

As shown in Figure 1, the encoder (respectively the inverse decoder) operation is composed of two separate blocks:

- The event-based encoder EBE (respectively, event-based decoder EBD). This block maps  $x_k \mapsto \nu_k$  (respectively, the decoder maps  $\hat{\nu}_k \mapsto \hat{x}_k$ ). This block includes a level detector  $\varphi_{LD}$ , and a model-based predictor, MBP, similar to the one proposed in [3], and

- A variable length entropy encoder VLE (respectively, variable length decoder VLD) mapping the binary signal  $\nu_k$  (run sequence, see TableII) to the  $M$ -bits  $\eta_j$  (respectively, the decoder maps  $\hat{\eta}_j \mapsto \hat{\nu}_k$ ). This block includes the synchronization process described before.

We first address the description of the EBC and the resulting stability results, then in latter section we discuss the VLE algorithm in connection with the Energy saves.

### 3. Description of the Event-based Coding EBC

Elements composing the Event-based Coding EBC are the level detector  $\varphi_{LD}$ , and the model-based predictor.

#### 3.1. The Level Detector $\varphi_{LD}$

The operation principle of the level detector is shown in Figure 2. The signal detection levels are uniformly spaced by the quantum  $\Delta$ . The level detector device produces a signal (identified by '01' or '10') whenever a level crossing takes place, and a '00' if the signal remains within the level. While two symbols are used to characterize the level changes, one more symbol can be used to quantize time intervals. Then 01 indicates upward crossing, 10 downward crossing, and 00 is used to code the time-interval between crossing.

To illustrate this, consider the example of Figure 2 ( $L=1$ ), see [9] for more details. We assume that uniform samples are taken every time  $T_s$ , then  $m$  samples are taken in the time interval  $T_i = t_i - t_{i-1}$  before a cross level takes place. As two levels (upward) crossing happen within this interval, the binary representation of this situation by the level crossing detector produce the following signal,

$$01, \underbrace{00, 00, \dots, 00}_{m\text{-pairs}}, 01$$

This sequence has then high probability of 0's, and thus suited for entropy coding.

To make this process operational, we introduce the operator  $\varphi_{LD} : \tilde{x}_k \mapsto \nu_k$ , which takes the error signal,  $\tilde{x}_k$ , and codes the output  $\nu_k$  into a  $2L+1$ -valued one  $\delta_k \in \{-L, -1, 0, 1, L\}$ . That is:

$$l_k = \left\lfloor \frac{\tilde{x}_k}{\Delta} + \frac{1}{2} \right\rfloor$$

$$\delta_k = \begin{cases} L & \text{if } l_k > l_{k-1} & \text{and } L \text{ levels are crossed upwards} \\ 1 & \text{if } l_k > l_{k-1} & \text{and } 1 \text{ level is crossed upwards} \\ 0 & \text{if } l_k = l_{k-1}; & \text{and signal stay at the actual level} \\ -1 & \text{if } l_k < l_{k-1} & \text{and } 1 \text{ level is crossed downwards} \\ -L & \text{if } l_k < l_{k-1} & \text{and } L \text{ levels are crossed downwards} \end{cases}$$

with  $\Delta$  the level threshold and  $\lfloor \cdot \rfloor$  the floor operator which rounds to the smaller integer.

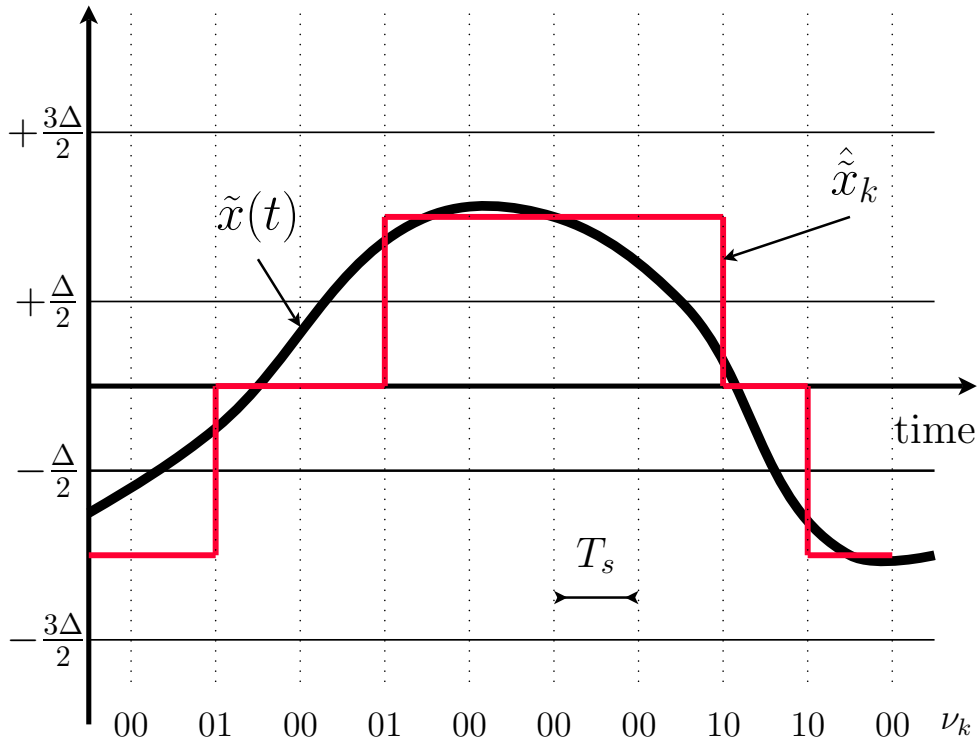


Figure 2. Illustration of the level detector working operation principle for  $L = 1$ .

For instance with  $L = 1$ , the 3-valued signal  $\delta_k$  is transformed into a 2-bits binary number  $\nu_k \in \{00, 01, 10\}$ , by the following operation.

$$\nu_k = \begin{cases} 00 & \text{if } \delta_k = 0 \\ 01 & \text{if } \delta_k = -1 \\ 10 & \text{if } \delta_k = 1 \end{cases}$$

The combination '11' is not used in this process.

### 3.2. The model-based predictor

Has the role to estimate (reconstruct) the encoded signal  $x_k$ , namely  $\hat{x}_k$ , from the 2-bits binary signal  $\nu_k$ . It is composed of:

- The inverse of the level detector:  $\varphi_{LD}^{-1} : \nu_k \mapsto \delta_k \mapsto \hat{x}_k$ , which equations are:

$$\delta_k = \begin{cases} 0 & \text{if } \nu_k = 00 \\ -1 & \text{if } \nu_k = 01 \\ 1 & \text{if } \nu_k = 10 \end{cases}$$



and,

$$\hat{x}_k = \hat{x}_{k-1} + \Delta \cdot \delta_k$$

Due to the quantization, this map does not describe an “exact inverse operator” as it will be explained latter.

- *The predictor.* The model-based predictor, as its name indicated, uses the target closed-loop model as a basis for its design. The predictor is a dynamic operator mapping the “reconstructed” error  $\hat{\tilde{x}}_k$  to the “reconstructed” state  $\hat{x}_k$ . Its structure is inspired by our previous works in [3], [5], and also in [6]. The predictor is a dynamic linear discrete-time operator that maps the output of the inverse level detector, to the signal prediction  $\hat{x}_k$ . Its structure depends upon the particular control used (state feedback or output feedback). For instance, for the RST-control discussed here, it has the following form:

$$\hat{x}_k = W \left[ \frac{\gamma}{T} r_k + \hat{\tilde{x}}_k \right], \quad W \triangleq \frac{BR}{A_{cl}} \quad (3)$$

Which results in the following error equation:

$$\tilde{x}_k = W \left[ \tilde{x}_k - \hat{\tilde{x}}_k \right] \quad (4)$$

Due to the fact that the VLE block is a distortion-less coding, and for simplicity reasons and without loss of generality, the stability properties of the first coding block, are basically the same as those with the VLE block, which is mainly used for energy efficiency transmission.

#### 4. Error system and stability conditions

Following the assumptions made in this paper (lossless channel transmission), we then have that  $\nu_k = \hat{\nu}_k$ , and that  $\delta_k = \hat{\delta}_k$ . In this case binary variables are not needed, and hence error equation can be described by real variables only.

##### 4.1. Error equations

Introducing the following error definitions:

- $e_k = x_k - \frac{\gamma}{A_{cl}} r_k$ : the tracking error,
- $\tilde{x}_k = x_k - \hat{x}_k$ : the prediction error, and
- $\varepsilon_k = \tilde{x}_k - \hat{\tilde{x}}_k$ : quantized error due to the non exact inverse mapping of the level detector, i.e. due to the map  $\varphi_{LD} \circ \varphi_{LD}^{-1}$ .

we have the closed-loop error system:

$$e_k = W(q^{-1})\tilde{x}_k \quad (5)$$

$$\tilde{x}_k = W(q^{-1})\varepsilon_k \quad (6)$$

with  $W = BR/A_{cl}$  being the stable operator defined previously. Note that the  $\varepsilon_k = \varepsilon_k(\tilde{x}_k)$ , and thereby the above error equation can be seen as two systems in cascade, i.e. the output

of the autonomous system (6) is the input of the stable system (5). For stability purposes it is thus sufficient to demonstrate the stability properties of the sub-system (6).

Note that  $\varepsilon_k$  writes as:

$$\begin{aligned}\varepsilon_k &= \tilde{x}_k - \hat{\tilde{x}}_k \\ &= \tilde{x}_k - \varphi_{LD} \circ \varphi_{LD}^{-1} \{\tilde{x}_k\} \\ &= \tilde{x}_k - \tilde{\varphi}_{LD} \{\tilde{x}_k\}\end{aligned}$$

where  $\tilde{\varphi}_{LD} \triangleq \varphi_{LD} \circ \varphi_{LD}^{-1} : \tilde{x}_k \mapsto \hat{\tilde{x}}_k$ . Note that this map is dynamic, defined by the following relation:

$$\hat{\tilde{x}}_k = \hat{\tilde{x}}_{k-1} + \Delta \cdot \delta_k \quad (7)$$

with  $\delta_k = f(\tilde{x}_k)$  as defined before. The sub-system (6)-(7) can be then seen as a feedback system, i.e.

$$\begin{aligned}\tilde{x}_k &= W(q^{-1})\varepsilon_k \\ &= W(q^{-1})\left(\tilde{x}_k - \hat{\tilde{x}}_k\right) \\ &= W(q^{-1})\left(\tilde{x}_k - \frac{\Delta}{1 - q^{-1}}\delta_k(\tilde{x}_k)\right)\end{aligned}$$

Ideally we would like that the map  $\tilde{\varphi}_{LD}$  be a linear map with unitary gain. This ideal goal is hampered by several factors:

- unknown initial conditions of  $\tilde{x}_0$ ,
- badly chosen  $T_s$ , and  $\Delta$ , and
- chattering in the neighborhood of the quantum  $\Delta$ .

In particular, large sampling times  $T_s$ , and too small quantum  $\Delta$  may result in signal variation of more than one level, which may lead to unrecovered bias in the estimated, leading to potential instabilities for unstable open-loop systems. This stability issues are analyzed next for a system of one dimension.

#### 4.2. Stability properties

Consider the stabilization problem ( $r = 0$ ) of the following simple unstable system  $\frac{B(q^{-1})}{A(q^{-1})} = \frac{bq^{-1}}{1 - aq^{-1}}$ , with  $2 > |a| > 1$ , and the control law  $u = -k\hat{x}_k$ . Let  $1 > a_c > 0$  be the desired closed loop poles, the required gain to reach such closed-loop specification is  $k = (a - a_c)/b$ . This particular choice leads to the error equations (5)-(6) with  $W(q^{-1}) = \frac{(a - a_c)q^{-1}}{1 - a_cq^{-1}}$ . Due to the cascade structure of such error equation arrangement, we mainly will concentrate in the equation (6) which captures most of the difficulties. To this aim we will concentrate on the following set of equations, which describes the error feedback interconnection.

$$\tilde{x}_{k+1} = a_c\tilde{x}_k + (a - a_c)\varepsilon_k, \quad \varepsilon_k = \tilde{x}_k - \hat{\tilde{x}}_k \quad (8)$$

$$\hat{\tilde{x}}_k = \hat{\tilde{x}}_{k-1} + \Delta\delta_k, \quad l_k = \left\lfloor \frac{\tilde{x}_k}{\Delta} + \frac{1}{2} \right\rfloor \quad (9)$$

The analysis is divided into two steps:

- Rate level condition. We first derive conditions on  $a$ , and a domain  $B_{\rho_1}$  for  $\tilde{x}_k$  that ensures that no more than one level change can be effectuated, i.e.  $|l_k - l_{k-1}| \leq 1$ ,
- Invariance condition. Then, by a Lyapunov-like analysis we show that this domain is indeed an invariant; solutions  $\tilde{x}_k$  starting in  $B_{\rho_1}$  do not leave this domain.

#### 4.3. Rate level condition

We seek here to establish condition on  $|\tilde{x}_k|$ ,  $\forall k \in \mathbb{Z}^+$  such that the rate change in the level detector be at most one. To be consistent with this aim, we need to assume in the sequel that the encoder/decoder internal states are suitable initialized. That is,  $\hat{x}_0$ , and  $l_0$  are such that:  $|\varepsilon_0| \leq \Delta/2$ , and  $\hat{x}_0 = \Delta l_0$  at  $k = 0$ .

**Lemma 1.** *Consider unstable systems limited by the relation  $a < 2 + a_c < 3$ , and let define the compact set,  $B_{\rho_1}$ , as:*

$$B_{\rho_1} = \{\tilde{x}_k : |\tilde{x}_k| < \rho_1\}, \quad \rho_1 = \frac{(1 - \frac{a-a_c}{2})}{1 - a_c} \Delta$$

with  $\rho_1 > 0$ . Then for all  $|\tilde{x}_k| \in B_{\rho_1}$  the following holds,  $\forall k \in \mathbb{Z}^+$ :

i)  $|\tilde{x}_k - \tilde{x}_{k-1}| \leq \Delta$ ,

furthermore, i) implies the following two equivalent inequalities:

ii)  $|l_k - l_{k-1}| \leq 1$

iii)  $|\varepsilon_k| \leq \Delta/2$

**Proof.** Let us start with the last part of this result, i.e.  $(i) \Rightarrow \{(ii) \Leftrightarrow (iii)\}$ . By inspection, it is easy to see that  $(i) \Rightarrow (ii)$ ; if the rate of change of  $\tilde{x}_k$  is strictly smaller than  $\Delta$  then the level change is limited, by definition, to a maximum one. In turn,  $(ii) \Rightarrow (iii)$ , results from the following arguments.

If the initialization condition  $\hat{x}_0 = \Delta l_0$  and  $(ii)$  holds, then we have that  $\hat{x}_k = \Delta l_k$ ,  $\forall k \in \mathbb{Z}^+$ . Then it follows that the error between the true estimation error, and the reconstructed one is bounded by the amount  $\Delta/2$ , i.e.

$$\begin{aligned} \varepsilon_k &= \tilde{x}_k - \hat{x}_k = \tilde{x}_k - \Delta l_k = \tilde{x}_k - \Delta \left[ \frac{\tilde{x}_k}{\Delta} + \frac{1}{2} \right] \\ &= \Delta \left( \frac{\tilde{x}_k}{\Delta} - \left[ \frac{\tilde{x}_k}{\Delta} + \frac{1}{2} \right] \right). \end{aligned}$$

Thus, we obtain

$$|\varepsilon_k| \leq \frac{\Delta}{2}$$

Inversely, if (iii) holds, it is easy by inspection to see that (ii) is true.

Now the first part of the lemma (i) is proven. From (8) we have

$$|\tilde{x}_{k+1} - \tilde{x}_k| < (1 - a_c)|\tilde{x}_k| + (a - a_c)|\varepsilon_k|$$

assume for the moment that (iii) holds, then condition for fulfilling (i) is that

$$|\tilde{x}_{k+1} - \tilde{x}_k| < (1 - a_c)|\tilde{x}_k| + (a - a_c)\Delta/2 < \Delta$$

or equivalently that

$$|\tilde{x}_k| < \frac{(1 - \frac{(a - a_c)}{2})}{1 - a_c} \Delta = \rho_1 \quad (10)$$

For this expression to be valid (i.e.  $\rho_1 > 0$ ), we require the condition  $a < 2 + a_c < 3$ . To conclude, assume that (10) holds independently to (i) – (iii) (as it will be shown latter in the next section). Now if (i) holds we just show that this implies (iii), which in turn and together with (10), implies (i).

#### 4.4. Invariance condition

Question here is to find under which conditions we can ensure the invariance of the set  $B_{\rho_1}$ . This invariance condition is clearly needed to preserve the rate level condition mentioned previously. We assume initially that the observation error is inside that set, and we look for the condition such that this signal does not leave  $B_{\rho_1}$ .

To this aim, consider the Lyapunov function  $V_k = \tilde{x}_k^2$ , and its rate variation  $\nabla V_k \triangleq \tilde{x}_{k+1}^2 - \tilde{x}_k^2$ , i.e.

$$\begin{aligned} \nabla V_k &= (a_c^2 - 1)|\tilde{x}_k|^2 + 2(a - a_c)a_c\varepsilon_k\tilde{x}_k + (a - a_c)^2\varepsilon_k^2 \\ &\leq (a_c^2 - 1)|\tilde{x}_k|^2 + (a - a_c)a_c|\tilde{x}_k|\Delta + \frac{(a - a_c)^2}{4}\Delta^2 \\ &\triangleq \Phi(|\tilde{x}_k|) \end{aligned}$$

Where the last inequality results from the hypothesis that initially we assume  $\tilde{x}_k \in B_{\rho_1}$ , or equivalently (see Lemma 1) that  $|\varepsilon_k| \leq \Delta/2$ .

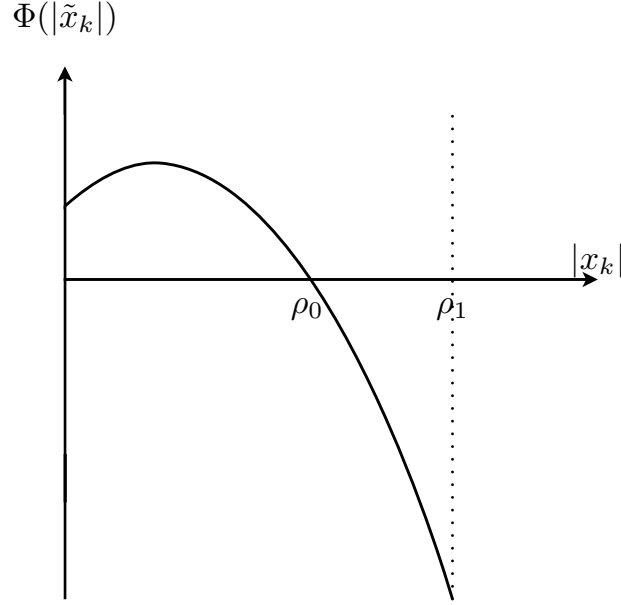
The shape of the polynomial  $\Phi(|\tilde{x}_k|)$  is shown in Figure 3, this function has two roots, one negative and other positive. The positive root, is  $\rho_0$ , and is given by

$$\rho_0 = \frac{(a - a_c)}{2(1 - a_c)}\Delta$$

It is necessary for stability that  $\rho_0 < \rho_1$ , else a local stability region may not exist. It is easy to show that this condition is valid as long as  $a < a_c + 1$ , which is the same condition already assumed by Lemma 1.

The value of  $\rho_0$  defines the limit (somewhat conservative) after which the function  $V_k$ , and hence the norm of  $x_k$  decreases. Below that limit the function may grow. The worst case growth in the interval  $|\tilde{x}_k| \in [0, \rho_0]$ , can be estimated from the relation

$$\tilde{x}_{k+1}^2 \leq \left( a_c|\tilde{x}_k| + \frac{(a - a_c)}{2}\Delta \right)^2 \triangleq \psi(|\tilde{x}_k|)$$

Figure 3.  $\Phi(|\tilde{x}_k|)$ .

As the function  $\psi(|\tilde{x}_k|)$  is convex in  $|\tilde{x}_k| \in [0, \rho_0]$  its maximum is located at the extremes of this interval. The worst case growth is then defined by the following relation:

$$x_{max} = \max \left\{ \sqrt{\psi(0)}, \sqrt{\psi(\rho_0)} \right\} = \Delta \frac{(a - a_c)}{2(1 - a_c)}$$

Finally, the set  $B_{\rho_1}$  is invariant if  $x_{max} < \rho_1$ , i.e.

$$\Delta \frac{(a - a_c)}{2(1 - a_c)} < \Delta \frac{(1 - \frac{(a - a_c)}{2})}{1 - a_c}$$

working out details of this inequality, it can be shown that this equality holds if  $a - a_c < 1$ , for all  $a_c \in (0, 1)$ . Note that this is a stronger condition than the one in Lemma 1 as it is derived from a more conservative analysis.

The following theorem summarizes the main result.

**Theorem 1.** *Assume that the coding algorithm is initialized such that  $\hat{x}_0$ , and  $l_0$  are such that:  $\varepsilon_0 < \Delta/2$ , and  $\hat{x}_0 = \Delta l_0$ . Consider system satisfying  $a - a_c < 1$ , with initial condition in the set  $\tilde{x}_0 \in B_{\rho_1}$ . Then:*

- $\tilde{x}_k \in B_{\rho_1}, \forall k \in \mathbb{Z}^+$ ,

- $\exists k_0 : |\tilde{x}_k| \leq \rho_0, \forall k \geq k_0$ , and
- $\lim_{k \rightarrow \infty} d(x_k, \mathcal{B}_\beta) = 0$ .

where  $d(x_k, \mathcal{B}_\beta)$  is the minimum Euclidean distance from  $x_k$  to any point within the ball

$$\mathcal{B}_\beta := \{x \in \mathbb{R} : \|x\| < \beta\},$$

and  $\beta$  is a constant that depends on  $\rho_0$ , and on the infinite norm of  $W(q^{-1})$ .

**Proof.** The first two statements follow from the previous analysis, the last statement result from equation (5), i.e;  $|x_k| \leq \|W\| \cdot |\tilde{x}_k|$ . Details for the derivation of this property are similar to the ones used in [3], and [5].

#### 4.5. Stability of the $2L+1$ level detector

We consider now the case of a  $2L + 1$ -level detector. Increase of level numbers allows for a better system precision, but also permits to the ability to handle systems with a higher level of instability. It is also important to have rules for the design of  $L$  and  $\Delta$  as a function of the desired specifications.

Consider the following problem. Given a system  $(a, b)$ , and a desired closed loop system precision and pole location  $(\rho_2, a_c)$ , being  $\rho_2$  the desired asymptotic maximum value of the regulation error  $e_k$ , we wish to find how many level values  $2L + 1$ , and the level width  $\Delta$  so that the system is stable (in the sense defined below) and reply to the given specifications. Because the nature of the problem treated here, the result can only be local. Then, the attraction domain also need to be computed as a function of the design parameters.

**Theorem 2.** *Given  $a, b, a_c$ , and  $\rho_2$ , and assuming that the coding algorithm is initialized such that:  $\varepsilon_0 < \Delta/2$ , and  $\hat{x}_0 = \Delta l_0$ . Then, if the estimation error is initially within the set*

$$B_{\rho_1} = \{|\tilde{x}_k| < \frac{L - \frac{|a-a_c|}{2}}{|1-a_c|} \Delta\}$$

*the tracking error will converge to a domain bounded by  $\rho_2$ , provided that  $(L, \Delta)$  are chosen to satisfy the following inequalities:*

$$\Delta \leq 2 \frac{(1-a_c)^2}{(a-a_c)^2} \rho_2 \quad (11)$$

$$L \geq \lceil |a-a_c| \rceil \quad (12)$$

**Proof.** The proof is realized in two steps:

**1. Number of levels  $L$  design** Assuming first that  $\Delta$  is given, we seek to establish a condition on  $L$  to ensure that the rate of change of the encoded error prediction does not over exceed  $L$  levels at any time, that means  $|\tilde{x}_{k+1} - \tilde{x}_k| < L\Delta$ , for all  $k$ . As before, this condition implies that the error  $\varepsilon_k = \tilde{x}_k - \hat{x}_k$  is always bounded as  $|\varepsilon_k| \leq \Delta/2$ . Assuming this holds, we have

$$\begin{aligned} |\tilde{x}_{k+1} - \tilde{x}_k| &\leq |1 - a_c||\tilde{x}_k| + |a - a_c||\varepsilon_k| \\ &\leq |1 - a_c||\tilde{x}_k| + |a - a_c|\Delta/2 \\ &\leq L\Delta \end{aligned}$$

where the last inequality reflects the desired objective. As in Lemma 1, from here we get that

$$|\tilde{x}_k| < \frac{(L - \frac{|a-a_c|}{2})}{|1 - a_c|} \Delta = \rho_1 \quad (13)$$

Now following the same steps as in the invariant condition of Lemma 1, with  $\rho_0 = \frac{|a-a_c|}{2|1-a_c|} \Delta$ , the condition on  $L$  follows from the necessity of having  $\rho_1 > \rho_0$ . This results in

$$L \geq |a - a_c|$$

which gives the choice of  $L = \lceil |a - a_c| \rceil$  with  $\lceil \cdot \rceil$  being the ceil function.

**2. Level width  $\Delta$  design** Consider the function  $v_k = x_k^2$ , and its rate variation  $\nabla v_k \triangleq x_{k+1}^2 - x_k^2$ , we have previously seen that  $\tilde{x}_k$  is bounded by  $\rho_1 \quad \forall k$  and there exists a  $k_0 : |\tilde{x}_k| < \rho_0 \quad \forall k \geq k_0$ . Moreover the closed loop system is stable, so we assure that  $x_{k_0}$  is bounded. Now, all the analysis is realized for  $k > k_0$

$$\begin{aligned} \nabla v_k &= (a_c^2 - 1)|x_k|^2 + 2(a - a_c)a_c x_k \tilde{x}_k + (a - a_c)^2 \tilde{x}_k^2 \\ &\leq (a_c^2 - 1)|x_k|^2 + 2(a - a_c)a_c |x_k| \rho_0 + (a - a_c)^2 \rho_0^2 \\ &\triangleq \phi_2(|x_k|) \end{aligned}$$

The function  $\phi_2$  is negative for  $|x_k| > \frac{a-a_c}{1-a_c} \rho_0$ , so  $x_k$  will decrease until  $\frac{a-a_c}{1-a_c} \rho_0$  and after with the same analysis of convexity with  $\phi_2$  as  $\Phi$ , we assure that for any  $x_{k_0}$ ,  $\exists k_1 : \quad \forall k > k_1 \quad |x_k| < \frac{a-a_c}{1-a_c} \rho_0$ . Hence, we obtain that the regulation error converges in finite time to  $\rho_2 = \frac{a-a_c}{1-a_c} \rho_0$ . This analysis leads to

$$\Delta = 2 \frac{(1 - a_c)^2}{(a - a_c)^2} \rho_2$$

## 5. Energy saves

In this section, we discuss energy saves resulting from the previous scheme in two possible contexts:

- Word-by-word transmissions, and
- Packet-based transmissions

In the latter case, we introduce an additional entropy encoding algorithm that is suited the energy efficiency of the transmission.

### 5.1. Word-by-word transmission

In this scenario we assume that we dispose of a transmission mechanisms that transmit a single word-code per unit of time  $T_s$ , where  $T_s$  in our context is the sampling time used in our problem formulation. We assume also that we have to our disposal, two transmission power levels;  $P_H$  is the high power level, whereas  $P_L$  describes the low power one. This set up allows for "sleep" transmission modes, where the stand-till events are transmitted with a low-power, whereas changes in the signal uses high-power instead.

Consider the use of the 3-word code described previously, and assume that the coding sequence is independent and identically distributed, and that the upward crossing frequency equals the downwards crossing frequency, the probability of each event to happen is

$$p = \mathcal{P}(00), \quad \mathcal{P}(01) = \mathcal{P}(10) = \frac{1}{2}(1 - p)$$

where  $p \in [0, 1]$  is the probability to have an stand-still event. Then we can associate the power  $P_L$  to the code 00, and  $P_H$  to the events 01, and 10. This is summarized in Table I.

Codeword	Power distribution	Associated Probability
00-stand-still	$P_L$ -low power	$p$
01-source change	$P_H$ -high power	$\frac{1}{2}(1 - p)$
10-source change	$P_H$ -high power	$\frac{1}{2}(1 - p)$

Table I. Power distribution and associated probability as a function of the codeword.

We are interested here in quantifying the mean energy per unit of time  $T_s$  as a function of  $p$ , and  $P_H$ , and  $P_L$ . We can then define this mean energy as:

$$E_T = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N E_k = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N T_s P_k$$

where  $E_k = T_s P_k$  correspond to the delivered energy per period with  $P_k \in \{P_H, P_L\}$ . Under the assumptions taking here, the mean power use can be substituted by its probability value, leading to the following expression for  $E_T$

$$\begin{aligned} E_T &= T_s \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N P_k \\ &= T_s (\mathcal{P}(00)P_L + \mathcal{P}(01)P_H + \mathcal{P}(10)P_H) \\ &= \underbrace{T_s P_H}_{\text{Total energy}} - \underbrace{T_s p (P_H - P_L)}_{\text{Energy saves}} \end{aligned} \quad (14)$$

The expression above suggests as expected that energy saves, we can maximized by taking  $P_L$  as small as possible, and it also quantifies the impact of those saves as a function of the probability to the stand-still events. Energy-efficiency of this proposed coding scheme can then



substantial improved for systems where the prediction error is within the stand-still zone (i.e. low perturbed systems including stable ones). Note also that the size of  $\Delta$  will also have an important impact as this will affect the probability  $p$ .

### 5.2. Packet-based transmission

We consider a different scenario, where bits are sent in a packet fashion. We assume that we dispose for each package of a fixed amount of energy,  $E_0 = (P_0/m)(mT_s)$  but that we allow to distribute this energy along the time-interval  $T_I = mT_s$ , by changing the power associated to each sent package  $P_0/m$ . Note that both the time-interval and the associated power per package will vary in time as a function of the positive integer  $1 \leq m \leq (2^{(M-1)} - 1)$ , where  $M$  is the block length of the variable length entropy coding described next. The idea is shown by Fig. 4.

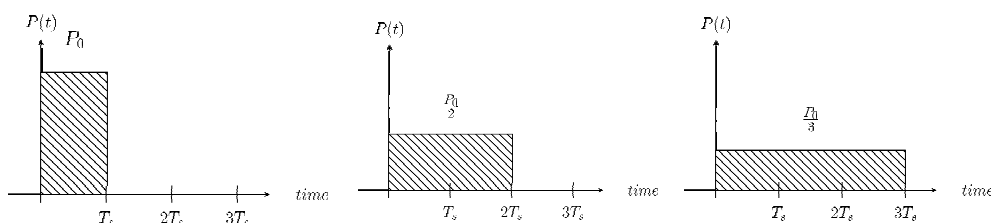


Figure 4. Transmission power distribution in the variable length entropy coding

**Variable length entropy coding VLE** Entropy codes are typically used to reach high compression rates. In here we show the interplay of these benefits with respect to energy saves in the context described above. The main idea is to assign some probability distribution to the events, so that the mean energy per period can be optimized. Run-length codes\*, are a class of variable-length codes that are sub-optimal in term of compaction ratio (when compared to the Huffman code), but have the advantage of avoiding buffering at the decoder side, and therefore reducing data transmission latency.

The VLE of length  $M$  can be described as a memory map from  $\nu_k$  to  $\eta_j$ . The VLE block contains a buffer of dimension  $l$  that stores information of past values of  $\nu_k$ . The buffer dimension  $l$  depends on  $M$ . The buffer information is used to build a run sequence resulting from the composition of  $\nu_{k-l}, \dots, \nu_k$ . This sequence, which is a variable-length binary signal, is named the “run sequence” used to produce the output  $\eta_j$ . The variable-length nature of this sequence introduces a variable latency (asynchronous output) which is multiple of  $T_s$ . An example of a coding scheme of block length  $M = 3$ , inspired from [9], is described in Table II.

As a metric for energy-improvements, let consider again the mean energy for unit of time  $E_T$ , written as the product of the mean power  $P_m$  times the time interval under consideration

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\*Class of coding strategy that can decode information instantaneously.

Run sequence $\nu_{k-2}, \nu_{k-1}, \nu_k$	Transmission period (sec)	Output $\eta_j$ ( $M = 3$ )
01	$T_s$	000
10	$T_s$	001
00 01	$2T_s$	010
00 10	$2T_s$	011
00 00 01	$3T_s$	100
00 00 10	$3T_s$	101
00 00 00	$3T_s$	110
unused	-	111

Table II. Run-length encoding.

$T_s$ . It follows then that

$$E_T = P_m T_s = \frac{E_0}{T_m} T_s$$

where  $T_m$  is the mean transmission period that need to be calculated. For the case shown in Table II,  $T_m$  can be calculated using the associated probability values, i.e.

$$\begin{aligned} T_m &= T_s \cdot 2 \left( \frac{1-p}{2} + 2T_s \cdot 2 \cdot \frac{1-p}{2} + 3T_s \cdot (2 \cdot p^2 \frac{1-p}{2} + p^3) \right) \\ &= T_s ((1-p)(1+2p+3p^2) + 3p^3) \\ &= T_s \frac{1-p^3}{1-p} \end{aligned}$$

Note that, for the case of of a VLE of block length  $M$  bits, the result generalized to

$$T_m = T_s \frac{1-p^{(2^{(M-1)}-1)}}{1-p}$$

And hence the  $E_T$  is given for the general case as:

$$E_T = E_0 \frac{1-p}{1-p^{(2^{(M-1)}-1)}} = E_0 f(p, M)$$

with extreme values being:

$$E_{max} = \lim_{p \rightarrow 0} E_T = E_0, \quad E_{min} = \lim_{p \rightarrow 1} E_T = \frac{E_0}{(2^{(M-1)} - 1)}$$

For comparison purposes, it is suited to compute the ratio between the mean energy cost using VLE, and the energy cost not using the VLE. This last cost can be computed by first noticing that the Energy cost per bit is assumed to be constant and equal to  $E_0/M$ , hence as two bits are sent in the case of not using VLE, the energy cost for that case will be  $2E_0/M$ . Then we have

$$\alpha(p, M) = \frac{E_T}{2E_0/M} = \frac{M}{2} f(p, M)$$

Ratios  $\alpha < 1$ , provide energy saves, whereas ratios  $\alpha > 1$ , have a penalty energy cost. Table III gives the efficiency ratio  $\alpha$ , for different values of  $M$  and  $p$ .

	$\alpha$			
	$p = 0; p = 0.4; p = 0.8; p = 1$			
$M$	$a = 1.6$	$a = 1.1$		
3	1.50	0.96	0.61	0.50
4	2.00	1.20	0.51	0.29
5	2.50	1.50	0.52	0.17
6	3.00	1.80	0.60	0.09

Table III. Efficiency-energy ratio  $\alpha(p, M)$  for  $p = 0, p = 0.4, p = 0.8$  and  $p = 1$

From this table we can see that for a given probability  $p$  it becomes critical to select the code block length  $M$ .

## 6. Simulation evaluation

The purpose of this section is to evaluate in simulation the proposed algorithm and to discuss several issues concerning the algorithm implementation, and energy saves.

We consider the following simple system

$$\frac{B(q^{-1})}{A(q^{-1})} = \frac{bq^{-1}}{1 - aq^{-1}} \quad (15)$$

The controller is:  $u_k = -k\hat{x}_k + \gamma r_k$  obtained from the closed-loop specification given by  $A_{cl} = (1 - a_{cl}q^{-1})$ ,  $k = \frac{a - a_{cl}}{b}$  and  $\gamma = 1 - a_c$ . The system in the simulation is given by:  $a = 1.1$ ,  $b = 1$ ,  $a_c = 0.9$ ,  $T_s = 0.05$  (sec),  $x_0 = 0$  and  $\hat{x}_0 = -0.01$  so  $\tilde{x}_0 = 0.01$ , and the asymptotic bound on the regulation error is  $\rho_2 = 0.04$ , with the analysis of Theorem 2 we obtain  $L = 1$  and  $\Delta = 0.02$ .

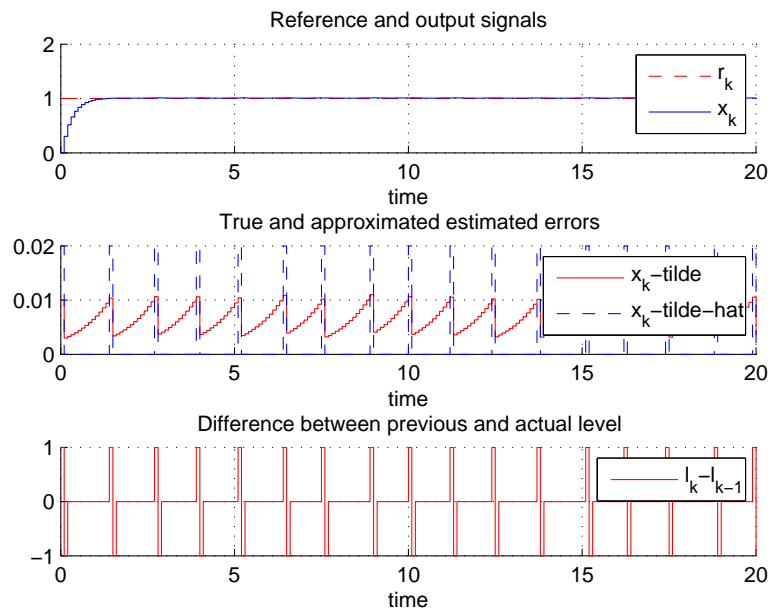


Figure 5. Simulation results with  $\Delta = 0.02$  ;  $T_s = 0.05$  yielding  $|l_k - l_{k-1}| \leq 1$ . Output and reference (upper),  $\tilde{x}$  vs.  $\hat{\tilde{x}}$  (middle), and  $l_k - l_{k-1}$  (lower)

Initial conditions of the predictor at the encoder side  $\hat{x}_0$  need to be synchronized with initial condition predictor at the decoder side. This requires a specific initialization procedure that send this initial information before the coding process is triggered. The Figure 5, shows a simulation where parameters are selected to satisfy the stability conditions. As a consequence, the change of level is limited to one.

### 6.1. Event distribution and its impact in the energy saves

Note that practically all parameters of the control scheme affect the spectrum of the evolution of  $\delta_k$ , and in particular the  $\Delta$ , and  $T_s$ , but also the magnitude of open-loop unstable poles. Figure 6 shows the resulting histogram of  $\delta_k$  for two simulations with two different values of  $a$ .

The results show that for small values of  $a$  the frequency spectrum of  $\delta_k$  is reduced, and hence the event  $\delta_k = 0$  has higher probability to occur. We recall that distributions with high roll-off will be benefic for data compression, as illustrated by the VLE algorithm. We can then conclude that open-loop instable systems with a high degree of instability are less adapted for entropy coding. The resulting compression ratio are reported in Table ???. Best energy-efficiency rates are thus obtained for the cases where  $p$  is higher, i.e. the bottom figures in Fig.6.

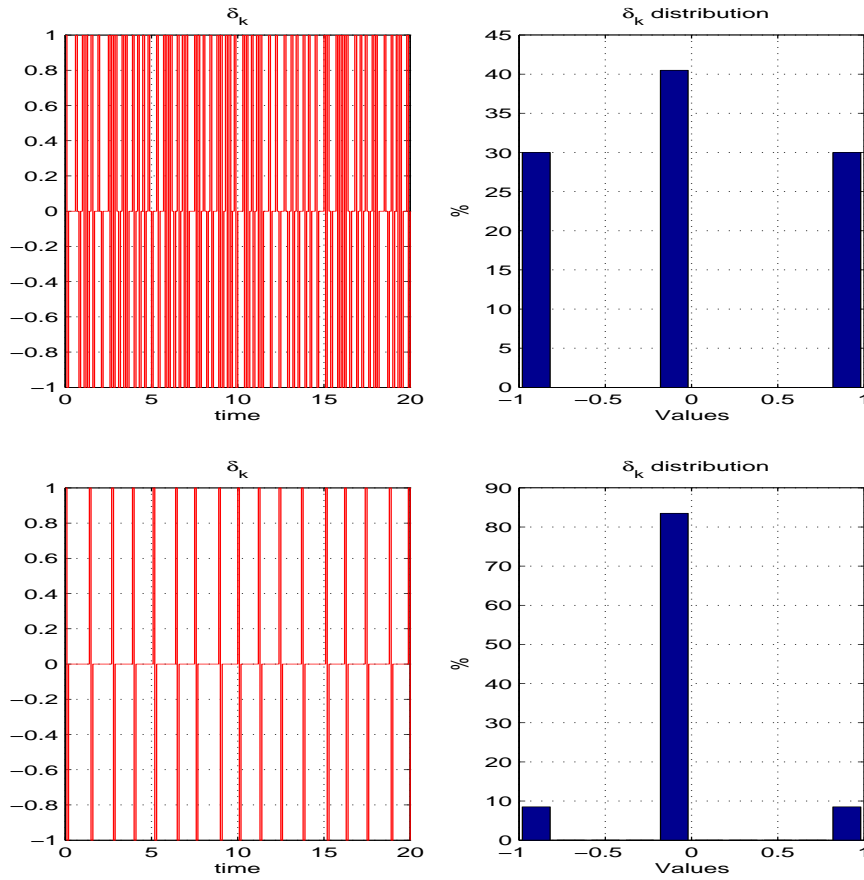


Figure 6. Simulations with  $\Delta = 0.02$ ;  $T_s = 0.05$ ;  $a = 1.6$  (upper),  $\Delta = 0.02$ ;  $T_s = 0.05$ ;  $a = 1.1$  (bottom), Time evolution of  $\delta_k$  (left), histogram of events (right)

From the Table III we can see that for  $a = 1.1$ , the best choice is a VLE coding of length  $M = 4$ , or  $M = 5$ , whereas for the system with  $a = 1.6$  only the VLE with  $M = 3$  improves over the one without entropy coding.

## 7. Conclusions

In this paper we have investigated new event-based coding algorithms that are suited to improve the energy transmission efficiency in the context of networked controlled systems. We have also study the energy impact of using entropy coding and the event-based algorithms. The main motivation has been to explore the benefits in terms of mean energy per period.

In particular we have analyzed the case of variable-length encoder VLE, which in spite of its sub-optimality do not require buffering at the decoder side, and hence reduce latency.

It has been shown that the scheme results in a local stable system with an attraction domain and a final precision depending on the value of the level granularity  $\Delta$ . It was also possible to devise a design algorithm that determines the number of levels (or bits) and the size of  $\Delta$  needed to reach a given precision.

Finally, we have evaluated the energy saves that can be reached in two possible scenarios; word-by-word, and packaged-based transmissions. The first case, which better adapted to scenarios where sleep modes can be implemented with low-power, the energy saves are directly proportional to the probability,  $p$  to these events to occur. In the second case, we have analyzed the energy impact of using VLE. We have derived a energy-efficiency ratio,  $\alpha$  that quantifies the potential energy saves as a function of  $p$ , and the VLE code length,  $M$ . Here the optimal choices are more involved as this efficiency ratio depends nonlinearly on  $p$ , and  $M$ . However, for a given  $p$  it is still possible to find the optimal value of  $M$ .

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