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Pierre Fima

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KAZHDAN’S PROPERTY T FOR DISCRETE QUANTUM GROUPS

Pierre Fima∗†

Abstract
We give a simple definition of property T for discrete quantum groups. We prove the basic expected properties: discrete quantum groups with property T are finitely generated and unimodular. Moreover we show that, for “I.C.C.” discrete quantum groups, property T is equivalent to Connes’ property T for the dual von Neumann algebra. This allows us to give the first example of a property T discrete quantum group which is not a group using the twisting construction.

1 Introduction

In the 1980’s, Woronowicz [19], [20], [21] introduced the notion of a compact quantum group and generalized the classical Peter-Weyl representation theory. Many interesting examples of compact quantum groups are available by now: Drinfel’d and Jimbo [5], [9] introduced q-deformations of compact semi-simple Lie groups, and Rosso [13] showed that they fit into the theory of Woronowicz. Free orthogonal and unitary quantum groups were introduced by Van Daele and Wang [18] and studied in detail by Banica [1], [2].

Some discrete group-like properties and proofs have been generalized to (the dual of) compact quantum groups. See, for example, the work of Tomatsu [14] on amenability, the work of Banica and Vergnioux [6] on growth and the work of Vergnioux and Vaes [15] on boundary.

The aim of this paper is to define property T for discrete quantum groups. We give a definition analogous to the group case using almost invariant vectors. We show that a discrete quantum group with property T is finitely generated, i.e. the dual is a compact quantum group of matrices. Recall that a locally compact group with property T is unimodular. We show that the same result holds for discrete quantum groups, i.e. every discrete quantum group with property T is a Kac algebra. In [7] Connes and Jones defined property T for arbitrary von Neumann algebras and showed that an I.C.C. group has property T if and only if its group von Neumann algebra (which is a II₁ factor) has property T. We show that if the group von Neumann algebra of a discrete quantum group $\hat{G}$ is

∗Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, United States. Email: pfima@illinois.edu
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an infinite dimensional factor (i.e. \( \hat{G} \) is “I.C.C.”), then \( \hat{G} \) has property \( T \) if and only if its group von Neumann algebra is a II\(_1\) factor with property \( T \). This allows us to construct an example of a discrete quantum group with property \( T \) which is not a group by twisting an I.C.C. property \( T \) group. In addition we show that free quantum groups do not have property \( T \).

This paper is organized as follows: in Section 2 we recall the notions of compact and discrete quantum groups and the main results of this theory. We introduce the notion of discrete quantum sub-groups and prove some basic properties of the quasi-regular representation. We also recall the definition of property \( T \) for von Neumann algebras. In Section 3 we introduce property \( T \) for discrete quantum groups, we give some basic properties and we show our main result.

## 2 Preliminaries

### 2.1 Notations

The scalar product of a Hilbert space \( H \), which is denoted by \( \langle ., . \rangle \), is supposed to be linear in the first variable. The von Neumann algebra of bounded operators on \( H \) will be denoted by \( B(H) \) and the \( C^* \) algebra of compact operators by \( B_0(H) \). We will use the same symbol \( \otimes \) to denote the tensor product of Hilbert spaces, the minimal tensor product of \( C^* \) algebras and the spatial tensor product of von Neumann algebras. We will use freely the leg numbering notation.

### 2.2 Compact quantum groups

We briefly overview the theory of compact quantum groups developed by Woronowicz in [21]. We refer to the survey paper [12] for a smooth approach to these results.

**Definition 1.** A compact quantum group is a pair \( G = (A, \Delta) \), where \( A \) is a unital \( C^* \) algebra; \( \Delta \) is unital \(*\)-homomorphism from \( A \) to \( A \otimes A \) satisfying \((\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta \) and \( \Delta(A)(A \otimes 1) \) and \( \Delta(A)(1 \otimes A) \) are dense in \( A \otimes A \).

**Notation 1.** The Haar state need not be faithful. We denote by \( G_{\text{red}} \) the reduced quantum group obtained by taking \( C(G_{\text{red}}) = C(G)/I \) where \( I = \{ x \in A | \varphi(x^* x) = 0 \} \). The Haar measure is faithful on \( G_{\text{red}} \). We denote by \( L^\infty(G) \) the von Neumann algebra generated by the G.N.S. representation of the Haar state of \( G \). Note that \( L^\infty(G_{\text{red}}) = L^\infty(G) \).
Definition 2. A unitary representation $u$ of a compact quantum group $G$ on a Hilbert space $H$ is a unitary element $u \in \mathcal{M}(\mathcal{B}_0(H) \otimes C(G))$ satisfying

$$(\text{id} \otimes \Delta)(u) = u_{12}u_{13}.$$ 

Let $u^1$ and $u^2$ be two unitary representations of $G$ on the respective Hilbert spaces $H_1$ and $H_2$. We define the set of intertwiners

$$\text{Mor}(u^1, u^2) = \{ T \in \mathcal{B}(H_1, H_2) \mid (T \otimes 1)u^1 = u^2(T \otimes 1) \}.$$ 

A unitary representation $u$ is said to be irreducible if $\text{Mor}(u, u) = \mathbb{C}1$. Two unitary representations $u^1$ and $u^2$ are said to be unitarily equivalent if there is a unitary element in $\text{Mor}(u^1, u^2)$.

Theorem 2. Every irreducible representation is finite-dimensional. Every unitary representation is unitarily equivalent to a direct sum of irreducibles.

Definition 3. Let $u^1$ and $u^2$ be unitary representations of $G$ on the respective Hilbert spaces $H_1$ and $H_2$. We define the tensor product

$$u^1 \otimes u^2 = u^1_{13}u^2_{23} \in \mathcal{M}(\mathcal{B}_0(H_1 \otimes H_2) \otimes C(G)).$$

Notation 3. We denote by $\text{Irred}(G)$ the set of (equivalence classes) of irreducible unitary representations of a compact quantum group $G$. For every $x \in \text{Irred}(G)$, we choose representatives $u^x$ on the Hilbert space $H_x$. Whenever $x, y \in \text{Irred}(G)$, we use $x \otimes y$ to denote the (class of the) unitary representation $u^x \otimes u^y$. The class of the trivial representation is denoted by 1.

The set $\text{Irred}(G)$ is equipped with a natural involution $x \mapsto \bar{x}$ such that $u^x$ is the unique (up to unitary equivalence) irreducible representation such that

$$\text{Mor}(1, x \otimes \bar{x}) \neq 0 \neq \text{Mor}(1, \bar{x} \otimes x).$$

This means that $x \otimes \bar{x}$ and $\bar{x} \otimes x$ contain a non-zero invariant vector. Let $E_x \in H_x \otimes H_x$ be a non-zero invariant vector and $J_x$ the invertible antilinear map from $H_x$ to $H_x$ defined by

$$\langle J_x \xi, \eta \rangle = \langle E_x, \xi \otimes \eta \rangle, \quad \text{for all } \xi \in H_x, \eta \in H_x.$$ 

Let $Q_x = J_x^* J_x$. We will always choose $E_x$ and $E_x$ normalized such that $||E_x|| = ||E_x||$ and $J_x = J_x^{-1}$. Then $Q_x$ is uniquely determined, $\text{Tr}(Q_x) = ||E_x||^2 = \text{Tr}(Q_x^{-1})$ and $Q_x = (J_x J_x^*)^{-1}$. $\text{Tr}(Q_x)$ is called the quantum dimension of $x$ and is denoted by $\dim_q(x)$ The unitary representation $u^x$ is called the contragredient of $u^x$.

The G.N.S. representation of the Haar state is given by $(L^2(G), \Omega)$ where $L^2(G) = \bigoplus_{x \in \text{Irred}(G)} H_x \otimes H_x$, $\Omega \in H_1 \otimes H_1$ is the unique norm one vector, and

$$(\omega \xi, \eta \otimes \text{id})(u^x)\Omega = \frac{1}{||E_x||}\xi \otimes J_x(\eta), \quad \text{for all } \xi, \eta \in H_x.$$ 

It is easy to see that $\varphi$ is a trace if and only if $Q_x = \text{id}$ for all $x \in \text{Irred}(G)$. In this case $||E_x|| = \sqrt{n_x}$ where $n_x$ is the dimension of $H_x$ and $J_x$ is an anti-unitary operator.
Notation 4. Let $C(G)_a$ be the vector space spanned by the coefficients of all irreducible representations of $G$. Then $C(G)_a$ is a dense unital $*$-subalgebra of $C(G)$. Let $C(G_{\text{max}})$ be the maximal $C^*$ completion of the unital $*$-algebra $C(G)_a$. $C(G_{\text{max}})$ has a canonical structure of a compact quantum group. This quantum group is denoted by $G_{\text{max}}$ and it is called the maximal quantum group.

A morphism of compact quantum groups $\pi : G \to \mathbb{H}$ is a unital $*$-homomorphism from $C(G_{\text{max}})$ to $C(\mathbb{H}_{\text{max}})$ such that $\Delta_{\mathbb{H}} \circ \pi = (\pi \otimes \pi) \circ \Delta_G$, where $\Delta_G$ and $\Delta_{\mathbb{H}}$ denote the comultiplications for $G_{\text{max}}$ and $\mathbb{H}_{\text{max}}$ respectively. We will need the following easy Lemma.

Lemma 1. Let $\pi$ be a surjective morphism of compact quantum group from $G$ to $\mathbb{H}$ and $\tilde{\pi}$ be the surjective $*$-homomorphism from $C(G_{\text{max}})$ to $C(\mathbb{H})$ obtained by composition of $\pi$ with the canonical surjection $C(\mathbb{H}_{\text{max}}) \to C(\mathbb{H})$. Then for every irreducible unitary representation $v$ of $\mathbb{H}$ there exists an irreducible unitary representation $u$ of $G$ such that $v$ is contained in the unitary representation $(id \otimes \tilde{\pi})(u)$.

Proof. Let $\varphi$ be the Haar state of $\mathbb{H}$ and $v$ be an irreducible unitary representation of $\mathbb{H}$ on the Hilbert space $H_v$. Because $v$ is irreducible it is sufficient to show that there exists a unitary irreducible representation $u$ of $G$ such that $\text{Mor}(w, v) \neq \{0\}$, where $w = (id \otimes \tilde{\pi})(u)$. Suppose that the statement is false. Then for all irreducible unitary representations $u$ of $G$ on $H_u$, we have $\text{Mor}(w, v) = \{0\}$. By Lemma 6.3, for every operator $a : H_v \to H_u$ the operator $(id \otimes \varphi)(v^*(a \otimes 1)w)$ is in $\text{Mor}(w, v)$. It follows that for every irreducible unitary representation $u$ of $G$ and every operator $a : H_v \to H_u$ we have $(id \otimes \varphi)(v^*(a \otimes 1)w) = 0$. Using the same techniques as in Theorem 6.7, (because, by the surjectivity of $\pi$, $\tilde{\pi}(C(G)_a)$ is dense in $C(\mathbb{H})$) we find $(id \otimes \varphi)(v^*v) = 0$. But this is a contradiction as $v^*v = 1$.

The collection of all finite-dimensional unitary representations (given with the concrete Hilbert spaces) of a compact quantum group $G$ is a complete concrete monoidal $W^*$-category. We denote this category by $\mathcal{R}(G)$. We say that $\mathcal{R}(G)$ is finitely generated if there exists a finite subset $E \subset \text{Irred}(G)$ such that for all finite-dimensional unitary representations $r$ there exists a finite family of morphisms $b_k \in \text{Mor}(r_k, r)$, where $r_k$ is a product of elements of $E$, and $\sum_k b_k b_k^* = I_r$. It is not difficult to show that $\mathcal{R}(G)$ is finitely generated if and only if $G$ is a compact quantum group of matrices (see [20]).

2.3 Discrete quantum groups

A discrete quantum group is defined as the dual of a compact quantum group.

Definition 4. Let $G$ be a compact quantum group. We define the dual discrete quantum group $\hat{G}$ as follows:

$$c_0(\hat{G}) = \bigoplus_{x \in \text{Irred}(G)} B(H_x), \quad l^\infty(\hat{G}) = \bigoplus_{x \in \text{Irred}(G)} B(H_x).$$
We denote the minimal central projection of $l^\infty(\hat{G})$ by $p_x$, $x \in \text{Irred}(G)$. We have a natural unitary $V \in \text{M}(c^*_0(\hat{G}) \otimes C(G))$ given by

$$V = \bigoplus_{x \in \text{Irred}(G)} u^x.$$ 

We have a natural comultiplication

$$\hat{\Delta} : l^\infty(\hat{G}) \to l^\infty(\hat{G}) \otimes l^\infty(\hat{G}) : (\hat{\Delta} \otimes \text{id})(V) = V_{13}V_{23}.$$ 

The comultiplication is given by the following formula

$$\hat{\Delta}(a)S = Sa, \quad \text{for all } a \in \mathcal{B}(H_x), S \in \text{Mor}(x, yz), x, y, z \in \text{Irred}(G).$$ 

**Remark 1.** The maximal and reduced versions of a compact quantum group are different versions of the same underlying compact quantum group. This different versions give the same dual discrete quantum group, i.e. $\hat{G} = \hat{G}_{\text{red}} = \hat{G}_{\text{max}}$. This means that $\hat{G}$, $\hat{G}_{\text{red}}$ and $\hat{G}_{\text{max}}$ have the same $C^*$ algebra, the same von Neumann algebra and the same comultiplication.

A morphism of discrete quantum groups $\pi : \hat{G} \to \hat{H}$ is a non-degenerate $^*$-homomorphism from $c_0(\hat{G})$ to $M(c_0(\hat{H}))$ such that $\Delta_H \circ \pi = (\pi \otimes \pi) \circ \Delta_G$, where $\Delta_G$ and $\Delta_H$ denote the comultiplication for $G$ and $H$ respectively. Every morphism of compact quantum groups $\pi : G \to H$ admits a canonical dual morphism of discrete quantum groups $\hat{\pi} : \hat{G} \to \hat{H}$. Conversely, every morphism of discrete quantum groups $\hat{\pi} : \hat{G} \to \hat{H}$ admits a canonical dual morphism of compact quantum groups $\pi : G \to H$. Moreover, $\pi$ is surjective (resp. injective) if and only if $\hat{\pi}$ is injective (resp. surjective).

We say that a discrete quantum group $\hat{G}$ is finitely generated if the category $\mathcal{R}(\hat{G})$ is finitely generated.

We will work with representations in the von Neumann algebra setting.

**Definition 5.** Let $\hat{G}$ be a discrete quantum group. A unitary representation $U$ of $\hat{G}$ on a Hilbert space $H$ is a unitary $U \in l^\infty(\hat{G}) \otimes \mathcal{B}(H)$ such that :

$$(\hat{\Delta} \otimes \text{id})(U) = U_{13}U_{23}.$$ 

Consider the following maximal version of the unitary $V$:

$$V = \bigoplus_{x \in \text{Irred}(G)} u^x \in \text{M}(c^*_0(\hat{G}) \otimes C(G_{\text{max}})).$$

For every unitary representation $U$ of $\hat{G}$ on a Hilbert space $H$ there exists a unique $^*$-homomorphism $\rho : C(G_{\text{max}}) \to \mathcal{B}(H)$ such that $(\text{id} \otimes \rho)(V) = U$.

**Notation 5.** Whenever $U$ is a unitary representation of $\hat{G}$ on a Hilbert space $H$ we write $U = \sum_{x \in \text{Irred}(G)} U^x$ where $U^x = Up_x$ is a unitary in $\mathcal{B}(H_x) \otimes \mathcal{B}(H)$. 

5
The discrete quantum group $l^\infty(\hat{G})$ comes equipped with a natural modular structure. Let us define the following canonical states on $B(H_x)$:

$$\phi_x(A) = \frac{\text{Tr}(Q_xA)}{\text{Tr}(Q_x)}, \quad \text{and} \quad \psi_x(A) = \frac{\text{Tr}(Q_x^{-1}A)}{\text{Tr}(Q_x^{-1})},$$

for all $A \in B(H_x)$.

The states $\phi_x$ and $\psi_x$ provide a formula for the invariant normal semi-finite faithful (n.s.f.) weights on $l^\infty(\hat{G})$.

**Proposition 1.** The left invariant weight $\hat{\phi}$ and the right invariant weight $\hat{\psi}$ on $\hat{G}$ are given by

$$\hat{\phi}(a) = \sum_{x \in \text{Irred}(G)} \text{dim}_q(x)^2 \phi_x(ap_x) \quad \text{and} \quad \hat{\psi}(a) = \sum_{x \in \text{Irred}(G)} \text{dim}_q(x)^2 \psi_x(ap_x),$$

for all $a \in l^\infty(\hat{G})$ whenever this formula makes sense.

A discrete quantum is unimodular (i.e. the left and right invariant weights are equal) if and only if the Haar state $\phi$ on the dual is a trace. In general, a discrete quantum group is not unimodular, and it is easy to check that the Radon-Nikodym derivative is given by

$$[D\hat{\psi} : D\hat{\phi}]_t = \hat{\delta}^t \quad \text{where} \quad \hat{\delta} = \sum_{x \in \text{Irred}(G)} Q_x^{-2}p_x.$$

The positive self-adjoint operator $\hat{\delta}$ is called the modular element: it is affiliated with $c_0(\hat{G})$ and satisfies $\hat{\Delta}(\hat{\delta}) = \hat{\delta} \otimes \hat{\delta}$.

The following Proposition is very easy to prove.

**Proposition 2.** Let $\Gamma$ be the subset of $\mathbb{R}_+^*$ consisting of all the eigenvalues of the operators $Q_x^{-2}$ for $x \in \text{Irred}(G)$. Then $\Gamma$ is a subgroup of $\mathbb{R}_+^*$ and $\text{Sp}(\hat{\delta}) = \Gamma \cup \{0\}$.

**Proof.** Note that, because $J_x = J_x^{-1}$, the eigenvalues of $Q_x$ are the inverse of the eigenvalues of $Q_x$. Using the formula $SQ_z = Q_z \otimes Q_y S$, when $z \subset x \otimes y$ and $S \in \text{Mor}(z,x \otimes y)$ is an isometry, the Proposition follows immediately.

### 2.4 Discrete quantum subgroups

Let $G$ be a compact quantum group with representation category $\mathcal{C}$. Let $\mathcal{D}$ be a full subcategory such that $1_\mathcal{C} \in \mathcal{D}$, $\mathcal{D} \otimes \mathcal{D} \subset \mathcal{D}$ and $\overline{\mathcal{D}} = \mathcal{D}$. By the Tannaka-Krein Reconstruction Theorem of Woronowicz, we know that there exists a compact quantum group $\mathbb{H}$ such that the representation category of $\mathbb{H}$ is $\mathcal{D}$. We say that $\mathbb{H}$ is a discrete quantum subgroup of $\hat{G}$. We have $\text{Irred}(\mathbb{H}) \subset \text{Irred}(G)$. We collect some easy observations in the next proposition. We denote by a subscript $\mathbb{H}$ the objects associated to $\mathbb{H}$.
Proposition 3. Let $p = \sum_{x \in \text{Irred}(\mathbb{H})} p_x$. We have:

1. $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$;
2. $l^\infty(\mathbb{H}) = p(l^\infty(\mathbb{G}))$;
3. $\hat{\Delta}_\mathbb{H}(a) = \hat{\Delta}(a)(p \otimes p)$ for all $a \in l^\infty(\mathbb{H})$;
4. $\hat{\phi}(p) = \hat{\phi}_\mathbb{H}$ and $\hat{\delta}_\mathbb{H} = p \hat{\delta}$.

Proof. For $x, y, z \in \text{Irred}(\mathbb{G})$ such that $y \subset z \otimes x$, we denote by $p^y \otimes x \in \text{End}(x \otimes y)$ the projection on the sum of all sub-representations equivalent to $y$. Note that

$$\hat{\Delta}(p)(p \otimes p) = \left\{ \begin{array}{ll} p^y \otimes x & \text{if } y \subset z \otimes x, \\ 0 & \text{otherwise}. \end{array} \right.$$  \hspace{1cm} (1)

Thus:

$$\hat{\Delta}(p)(p \otimes p) = \sum_{y \in \text{Irred}(\mathbb{H}), y \subset z \otimes x} p^y \otimes x.$$

Note that if $y \subset z \otimes x$ and $y, z \in \text{Irred}(\mathbb{H})$ then $x \in \text{Irred}(\mathbb{H})$. It follows that:

$$\hat{\Delta}(p)(p \otimes p) = \left\{ \begin{array}{ll} p \otimes p_x & \text{if } x \in \text{Irred}(\mathbb{H}), \\ 0 & \text{otherwise}. \end{array} \right.$$

Thus, $\hat{\Delta}(p)(p \otimes 1) = p \otimes p$. The other assertions are obvious.

We introduce the following equivalence relation on $\text{Irred}(\mathbb{G})$ (see [17]): if $x, y \in \text{Irred}(\mathbb{G})$ then $x \sim y$ if and only if there exists $t \in \text{Irred}(\mathbb{H})$ such that $x \subset y \otimes t$. We define the right action of $\hat{\mathbb{H}}$ on $l^\infty(\mathbb{G})$ by translation:

$$\alpha : l^\infty(\mathbb{G}) \rightarrow l^\infty(\mathbb{G}) \otimes l^\infty(\hat{\mathbb{G}}), \quad \alpha(a) = \hat{\Delta}(a)(1 \otimes p).$$

Using $\hat{\Delta}(p)(p \otimes p) = p \otimes p$ and $\hat{\Delta}_\mathbb{H} = \hat{\Delta}(p \otimes p)$ it is easy to see that $\alpha$ satisfies the following equations:

$$(\alpha \otimes 1)\alpha = (\text{id} \otimes \hat{\Delta}_\mathbb{H})\alpha \quad \text{and} \quad (\hat{\Delta} \otimes \text{id})\alpha = (\text{id} \otimes \alpha)\hat{\Delta}.$$

The first equality means that $\alpha$ is a right action of $\hat{\mathbb{H}}$ on $l^\infty(\mathbb{G})$. Let $l^\infty(\mathbb{G}/\hat{\mathbb{H}})$ be the set of fixed points of the action $\alpha$:

$$l^\infty(\mathbb{G}/\hat{\mathbb{H}}) := \{ a \in l^\infty(\mathbb{G}), | \alpha(a) = a \otimes 1 \}.$$

Using the second equality for $\alpha$ it is easy to see that:

$$\hat{\Delta}(l^\infty(\mathbb{G}/\hat{\mathbb{H}})) \subset l^\infty(\mathbb{G}) \otimes l^\infty(\mathbb{G}/\hat{\mathbb{H}}).$$

Thus the restriction of $\hat{\Delta}$ to $l^\infty(\mathbb{G}/\hat{\mathbb{H}})$ gives an action of $\hat{\mathbb{G}}$ on $l^\infty(\mathbb{G}/\hat{\mathbb{H}})$. We denote this action by $\beta$. 7
**Proposition 4.** Let $T_\alpha = (\otimes \circ \hat{\varphi})\alpha$ be the normal faithful operator valued weight from $l^\infty(\hat{G})$ to $l^\infty(\hat{G}/\hat{H})$ associated to $\alpha$. $T_\alpha$ is semi-finite and there exists a unique n.s.f. weight $\theta$ on $l^\infty(\hat{G}/\hat{H})$ such that $\hat{\varphi} = \theta \circ T_\alpha$.

**Proof.** It follows from Eq. 8 that $T_\alpha(p_y)p_z = 0$ if $z \prec y$. Take $z \sim y$, we have:

$$T_\alpha(p_y)p_z = \sum_{x \in \text{Irred}(\hat{H})} \dim_q(x)^2 (\otimes \varphi_x)(p^{x \otimes y})$$

$$\leq \sum_{x \in \text{Irred}(\hat{G})} \dim_q(x)^2 (\otimes \varphi_x)(p^{x \otimes y})$$

$$= (\otimes \varphi_x)(\hat{\Delta}(p_y))p_z = \hat{\varphi}(p_y)p_z$$

$$= \dim_q(y)^2 p_z.$$ 

It follows that $T_\alpha(p_y) < \infty$ for all $y$. This implies that $T_\alpha$ is semi-finite. Note that $\alpha(\delta^{-it}) = \delta^{-it} \otimes \delta^{-it}$. It follows from [8], Proposition 8.7, that there exists a unique n.s.f. weight $\theta$ on $l^\infty(\hat{G}/\hat{H})$ such that $\hat{\varphi} = \theta \circ T_\alpha$. \[\square\]

Denote by $l^1(\hat{G}/\hat{H})$ the G.N.S. space of $\theta$ and suppose that $l^\infty(\hat{G}/\hat{H}) \subset B(l^1(\hat{G}/\hat{H}))$. Let $U^* \in l^\infty(\hat{G}) \otimes B(l^1(\hat{G}/\hat{H}))$ be the unitary implementation of $\beta$ associated to $\theta$ in the sense of [10]. Then $U$ is a unitary representation of $\hat{G}$ on $l^2(\hat{G}/\hat{H})$ and $\beta(x) = U^*(1 \otimes x)U$. We call $U$ the quasi-regular representation of $\hat{G}$ modulo $\hat{H}$.

**Lemma 2.** We have $p \in l^\infty(\hat{G}/\hat{H}) \cap N_\theta$. Put $\xi = \Lambda_\theta(p)$. If $\hat{G}$ is unimodular then $U^* \eta \otimes \xi = \eta \otimes \xi$ for all $x \in \text{Irred}(\hat{H})$ and all $\eta \in H_x$.

**Proof.** Using $\hat{\Delta}(p_1)(1 \otimes p_z) = p^{z \otimes x}_{\otimes y}$ it is easy to see that $T_\alpha(p_1) = p$. It follows that $p \in l^\infty(\hat{G}/\hat{H})$ and $\theta(p) = \hat{\varphi}(p_1) = 1$. Thus $p \in N_\theta$. Let $x \in M^+$ such that $T_\alpha(x) < \infty$, $\omega \in l^\infty(\hat{G})_+$ and $\mu$ a n.s.f. weight on $l^\infty(\hat{G}/\hat{H})$. Using $(\hat{\Delta} \otimes \text{id})\alpha = (\otimes \alpha)\hat{\Delta}$ we find:

$$\begin{align*}
(\omega \otimes \mu)\beta(T_\alpha(x)) &= (\omega \otimes \mu)\hat{\Delta}(T_\alpha(x)) = (\omega \otimes \mu)(\hat{\Delta}(\hat{\Delta}(\otimes \text{id})\alpha(x))\\
&= (\omega \otimes \mu \otimes \hat{\varphi})(\hat{\Delta}(\otimes \text{id})\alpha(x))\\
&= (\omega \otimes \mu \otimes \hat{\varphi})(\hat{\Delta}(\otimes \alpha)\hat{\Delta}(x))\\
&= (\omega \otimes \mu \circ T_\alpha)\hat{\Delta}(x).
\end{align*}$$

(2)

It follows that, for all $\omega \in l^\infty(\hat{G})_+$ and all $y \in l^\infty(\hat{G})_+$ such that $T_\alpha(y) < \infty$, we have:

$$(\omega \otimes \theta)\beta(T_\alpha(y)) = (\omega \otimes \hat{\varphi})(\hat{\Delta}(y)) = \hat{\varphi}(y)\omega(1) = \theta(T_\alpha(y))\omega(1).$$

Let $x \in l^\infty(\hat{G}/\hat{H})^+$. Because $T_\alpha$ is a faithful and semi-finite, there exists an increasing net of positive elements $y_i$ in $l^\infty(\hat{G})^+$ such that $T_\alpha(y_i) < \infty$ for all $i$ and $\text{Sup}(T_\alpha(y_i)) = x$. It follows that:

$$(\omega \otimes \theta)\beta(x) = \text{Sup}(\omega \otimes \theta)\beta(T_\alpha(y_i))) = \text{Sup}(\theta(T_\alpha(y_i)))\omega(1) = \theta(x)\omega(1),$$
for all \( \omega \in l^\infty(\hat{G})^+ \). This means that \( \theta \) is \( \beta \)-invariant. Using this invariance we define the following isometry:

\[
V^*(\hat{\Lambda}(x) \otimes \Lambda_\theta(y)) = (\hat{\Lambda} \otimes \Lambda_\theta)(\beta(y)(x \otimes 1))
\]

Because \( \hat{G} \) is unimodular we know from [16], Proposition 4.3, that \( V^* \) is the unitary implementation of \( \beta \) associated to \( \theta \) i.e. \( V = U \). Using \( \hat{\Delta}(p)(p \otimes 1) = p \otimes p \), it follows that, for all \( x \in N_\beta \), we have:

\[
U^*(p\hat{\Lambda}(x) \otimes \Lambda_\theta(p)) = (\hat{\Lambda} \otimes \Lambda_\theta)(\hat{\Delta}(p)(px \otimes 1)) = p\hat{\Lambda}(x) \otimes \Lambda_\theta(p).
\]

This concludes the proof. \( \square \)

**Remark 2.** For general discrete quantum groups it can be proved, as in [16], Théorème 2.9, that \( V^* \) is a unitary implementing the action \( \beta \) and, as in [16], Proposition 4.3, that \( V^* \) is the unitary implementation of \( \beta \) associated to \( \theta \).

Thus the previous lemma is also true for general discrete quantum groups.

**Lemma 3.** Suppose that \( U \) has a non-zero invariant vector \( \xi \in l^2(\hat{G}/\hat{H}) \). Then \( \text{Irred}(G)/\text{Irred}(H) \) is a finite set.

**Proof.** Let \( \xi \in l^2(\hat{G}/\hat{H}) \) be a normalized \( U \)-invariant vector. Using \( \beta(x) = U^*(1 \otimes x)U \) it is easy to see that \( \omega_\xi \) is a \( \beta \)-invariant normal state on \( l^\infty(\hat{G}/\hat{H}) \), i.e. \( (\text{id} \otimes \omega_\xi)\beta(x) = \omega_\xi(x)1 \) for all \( x \in l^\infty(\hat{G}/\hat{H}) \). Let \( s \) be the support of \( \omega_\xi \) and \( e = 1 - s \). Let \( \omega \) be a faithful normal state on \( l^\infty(\hat{G}) \). Because the support of \( \omega \otimes \omega_\xi \) is \( 1 \otimes s \) and \( (\omega \otimes \omega_\xi)\beta(e) = \omega_\xi(e) = 0 \) we find \( \hat{\Delta}(e) = \beta(e) \leq 1 \otimes e \).

It follows from [16], Lemma 6.4, that \( e = 0 \) or \( e = 1 \). Because \( \xi \) is a non-zero vector we have \( e = 0 \). Thus \( \omega_\xi \) is faithful. Let \( x \in M^+ \) such that \( T_\alpha(x) < \infty \). By Eq. (2) we have:

\[
(\omega \otimes \omega_\xi \circ T_\alpha)(\hat{\Delta}(x)) = (\omega \otimes \omega_\xi)\beta(T_\alpha(x)) = \omega_\xi(T_\alpha(x))\omega(1),
\]

for all \( \omega \in l^\infty(\hat{G})^+ \). Because \( T_\alpha \) is n.s.f., it follows easily that \( \omega_\xi \circ T_\alpha \) is a left invariant n.s.f. weight on \( \hat{G} \). Thus, up to a positive constant, we have \( \omega_\xi \circ T_\alpha = \hat{\phi} \).

Suppose that \( \text{Irred}(G)/\text{Irred}(H) \) is infinite, and let \( x_i \in \text{Irred}(G), i \in \mathbb{N} \) be a complete set of representatives of \( \text{Irred}(G)/\text{Irred}(H) \). Let \( a \) be the positive element of \( l^\infty(\hat{G}) \) defined by \( a = \sum_{i \geq 0} \frac{1}{\text{dim}(x_i)} p_{x_i} \). Then we have \( \hat{\phi}(a) = +\infty \) and \( T_\alpha(a) = \sum_i \sum_{x \in x_i} p_x = 1 < \infty \), which is a contradiction. \( \square \)

### 2.5 Property \( T \) for von Neumann algebras

Here we recall several facts from [16]. If \( M \) and \( N \) are von Neumann algebras then a correspondence from \( M \) to \( N \) is a Hilbert space \( H \) which is both a left \( M \)-module and a right \( N \)-module, with commuting normal actions \( \pi_l \) and \( \pi_r \) respectively. The triple \( (\text{H}, \pi_l, \pi_r) \) is simply denoted by \( H \) and we shall write \( a \xi b \) instead of \( \pi_l(a) \pi_r(b) \xi \) for \( a \in M, b \in N \) and \( \xi \in H \). We shall denote by
$C(M)$ the set of unitary equivalence classes of correspondences from $M$ to $M$. The standard representation of $M$ defines an element $L^2(M)$ of $C(M)$, called the identity correspondence.

Given $H \in C(M)$, $\epsilon > 0$, $\xi_1, \ldots, \xi_n \in H$, $a_1, \ldots, a_p \in M$, let $V_H(\epsilon, \xi_i, a_i)$ be the set of $K \in C(M)$ for which there exist $\eta_1, \ldots, \eta_n \in K$ with

$$|(a_j \eta_k a_k, \eta_j') - (a_j \xi_i a_k, \xi_i')| < \epsilon, \quad \text{for all } i, i', j, k.$$  

Such sets form a basis of a topology on $C(M)$ and, following [4], $M$ is said to have property $T$ if there is a neighbourhood of the identity correspondence, each member of which contains $L^2(M)$ as a direct summand.

When $M$ is a $\text{II}_1$ factor the property $T$ is easier to understand. A $\text{II}_1$ factor $M$ has property $T$ if we can find $\epsilon > 0$ and $a_1, \ldots, a_p \in M$ satisfying the following condition: every $H \in C(M)$ such that there exists $\xi \in H$, $||\xi|| = 1$, with $||a_i \xi - \xi a_i|| < \epsilon$ for all $i$, contains a non-zero central vector $\eta$ (i.e. $a \eta = \eta a$ for all $a \in M$). We recall the following Proposition from [4].

**Proposition 5.** If $M$ is a $\text{II}_1$ factor with property $T$ then there exist $\epsilon > 0$, $b_1, \ldots, b_m \in M$ and $C > 0$ with the following property: for any $\delta \leq \epsilon$, if $H \in C(M)$ and $\xi \in H$ is a unit vector satisfying $||b_i \xi - \xi b_i|| < \delta$ for all $1 \leq i \leq m$, then there exists a unit central vector $\eta \in H$ such that $||\xi - \eta|| < C\delta$.

It is proved in [4] that a discrete I.C.C. group has property $T$ if and only if the group von Neumann algebra $L(G)$ has property $T$.

### 3 Property $T$ for Discrete Quantum Groups

**Definition 6.** Let $\hat{G}$ be a discrete quantum group.

- Let $E \subset \text{Irred}(\hat{G})$ be a finite subset, $\epsilon > 0$ and $U$ a unitary representation of $\hat{G}$ on a Hilbert space $K$. We say that $U$ has an $(E, \epsilon)$-invariant vector if there exists a unit vector $\xi \in K$ such that for all $x \in E$ and $\eta \in H_x$ we have:

$$||UX \xi \otimes \eta - \eta \otimes \xi|| < \epsilon ||\eta||.$$  

- We say that $U$ has almost invariant vectors if, for all finite subsets $E \subset \text{Irred}(\hat{G})$ and all $\epsilon > 0$, $U$ has an $(E, \epsilon)$-invariant vector.

- We say that $\hat{G}$ has property $T$ if every unitary representation of $\hat{G}$ having almost invariant vectors has a non-zero invariant vector.

**Remark 3.** Let $G = (C^*(\Gamma), \Delta)$, where $\Gamma$ is a discrete group and $\Delta(g) = g \otimes g$ for $g \in \Gamma$. It follows from the definition that $\hat{G}$ has property $T$ if and only if $\Gamma$ has property $T$.

The next proposition will be useful to show that the dual of a free quantum group does not have property $T$. 

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Proposition 6. Let $G$ and $H$ be compact quantum groups. Suppose that there is a surjective morphism of compact quantum groups from $G$ to $H$ (or an injective morphism of discrete quantum groups from $\hat{H}$ to $\hat{G}$). If $\hat{G}$ has property $T$ then $\hat{H}$ has property $T$.

Proof. We can suppose that $G = G_{\text{max}}$ and $H = H_{\text{max}}$. We will denote by a subscript $G$ (resp. $H$) the object associated to $G$ (resp. $H$). Let $\pi$ be the surjective morphism from $C(G)$ to $C(H)$ which intertwines the comultiplications.

Let $U$ be a unitary representation of $\hat{H}$ on a Hilbert space $K$ and suppose that $U$ has almost invariant vectors. Let $\rho$ be the unique morphism from $C(H)$ to $B(K)$ such that $(\text{id} \otimes \rho)(V_G) = U$. Consider the following unitary representation of $\hat{G}$ on $K$: $V = (\text{id} \otimes (\rho \circ \pi))(V_G)$. We will show that $V$ has almost invariant vectors. Let $E \subset \text{Irred}(G)$ be a finite subset and $\epsilon > 0$. For $x \in \text{Irred}(G)$ and $y \in \text{Irred}(H)$ denote by $w^x \in B(H_x) \otimes C(G)$ and $v^y \in B(H_y) \otimes C(H)$ a representative of $x$ and $y$ respectively. Note that $w^x = (\text{id} \otimes \pi)(u^x)$ is a finite dimensional unitary representation of $G$, thus we can suppose that $w^x = \oplus n_{x,y}v^y$. Let $L = \{ y \in \text{Irred}(H) | \exists x \in E, n_{x,y} \neq 0 \}$. Because $U$ has almost invariant vectors, there exists a norm one vector $\xi \in K$ such that $|\|U^y\xi \otimes \eta \otimes \xi\| - \epsilon |\eta||$ for all $y \in L$ and all $\eta \in H_y$. Using the isomorphism

$$H_x = \bigoplus_{y \in \text{Irred}(G), n_{x,y} \neq 0} (H_y \oplus \ldots \oplus H_y) \bigoplus_{n_{x,y}}$$

we can identify $V^x$ with $\oplus n_{x,y}U^y$ in $\bigoplus_y B(H_y) \oplus B(H_y) \oplus \ldots \oplus B(H_y) \oplus B(K)$. With this identification it is easy to see that, for all $x \in E$ and all $\eta$ in $H_x$, we have $|\|V^x\eta \otimes \xi - \eta \otimes \xi\| - \epsilon |\eta||$. It follows that $V$ has almost invariant vectors and thus there is a non-zero $V$-invariant vector, say $l$, in $K$. To show that $l$ is also $U$-invariant it is sufficient to show that for every $y \in \text{Irred}(H)$ there exists $x \in \text{Irred}(G)$ such that $n_{x,y} \neq 0$. This follows from Lemma [1] \(\Box\)

Corollary 1. The discrete quantum groups $\widetilde{A}_\omega(n)$, $\widetilde{A}_\nu(n)$ and $\widetilde{A}_s(n)$ do not have property $T$ for $n \geq 2$.

Proof. It follows directly from the preceding proposition and the following surjective morphisms:

$$A_\phi(n) \to C^*(\ast_{i=1}^n \mathbb{Z}_2), \quad A_\nu(n) \to C^*(\mathbb{F}_n), \quad A_s(n) \to C^*(\ast_{i=1}^n \mathbb{Z}_{n_i}),$$

where $\sum n_i = n$. \(\Box\)

In the next Proposition we show that discrete quantum groups with property $T$ are unimodular.

Proposition 7. Let $\widehat{G}$ be a discrete quantum group. If $\widehat{G}$ has property $T$ then it is a Kac algebra, i.e. the Haar state $\varphi$ on $G$ is a trace.
Proof. Suppose \( \hat{G} \) has property \( T \) and let \( \Gamma \) be the discrete group introduced in Proposition 3. Because \( \text{Sp}(\hat{\delta}) = \Gamma \cup \{0\} \) and \( \hat{\Delta} = \delta \otimes \hat{\delta} \), we have an injective \(*\)-homomorphism
\[
\alpha : c_0(\Gamma) \to c_0(\hat{G}), \quad \alpha(f) = f(\hat{\delta})
\]
satisfying \( \Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_\Gamma \). By Proposition 3, \( \Gamma \) has property \( T \). It follows that \( \Gamma = \{1\} \) and \( \hat{\delta} = 1 \). Thus \( Q_x = 1 \) for all \( x \in \text{Irred}(\hat{G}) \). This means that \( \varphi \) is a trace.

\[\text{Proposition 8.}\] Let \( \hat{G} \) be a discrete quantum group. If \( \hat{G} \) has property \( T \) then it is finitely generated.

**Proof.** Let \( \text{Irred}(\hat{G}) = \{x_n \mid n \in \mathbb{N}\} \) and \( \mathcal{C} \) be the category of finite dimensional unitary representations of \( \hat{G} \). For \( i \in \mathbb{N} \) let \( \mathcal{D}_i \) be the full subcategory of \( \mathcal{C} \) generated by \( \{x_0, \ldots, x_i\} \). This means that the irreducibles of \( \mathcal{D}_i \) are the irreducible representations \( \pi \) of \( \hat{G} \) such that \( \pi \) is equivalent to a sub-representation of \( x_{k_1}^i \otimes \cdots \otimes x_{k_n}^i \) for \( l \geq 1 \), \( 0 \leq k_j \leq n \), and \( \epsilon_j \) is nothing or the contragredient. The Hilbert spaces and the morphisms are the same in \( \mathcal{D}_i \) or in \( \mathcal{D} \). Thus we have \( 1_\mathcal{C} \in \mathcal{D}_i \), \( \mathcal{D}_i \cong \mathcal{D}_i \) and \( \mathcal{D}_i = \mathcal{D}_i \). Let \( \mathcal{H}_i \) be the compact quantum group such that \( \mathcal{D}_i \) is the category of representation of \( \mathcal{H}_i \). Let \( U_i \in l^\infty(\mathcal{G}/\mathcal{H}_i) \otimes B(l^2(\mathcal{G}/\mathcal{H}_i)) \) be the quasi-regular representation of \( \hat{G} \) modulo \( \mathcal{H}_i \). Let \( U \) be the direct sum of the \( U_i \); this a unitary representation on \( K = \bigoplus l^2(\mathcal{G}/\mathcal{H}_i) \). Let us show that \( U \) has almost invariant vectors. Let \( E \subset \text{Irred}(\hat{G}) \) be a finite subset. There exists \( i_0 \) such that \( E \subset \text{Irred}(\mathcal{H}_i) \) for all \( i \geq i_0 \). By Lemma 3, we have a unit vector \( \xi \) in \( l^2(\mathcal{G}/\mathcal{H}_i) \) such that \( U_i \xi \otimes \xi = \eta \otimes \xi \) for all \( x \in E \) and all \( \eta \in H_x \). Let \( \xi = (\xi_i) \in K \) where \( \xi_i = 0 \) if \( i \neq i_0 \) and \( \xi_{i_0} = \xi \). Then \( \xi \) is a unit vector in \( K \) such that \( U^\xi \eta \otimes \xi = \eta \otimes \xi \) for all \( x \in E \). It follows that \( U \) has an almost invariant vector. By property \( T \) there exists a non-zero invariant vector \( l = (l_i) \in K \). There exists \( m \) such that \( l_m \neq 0 \). Then \( l_m \) is an invariant vector for \( U_m \). By Lemma 3, \( \text{Irred}(\hat{G})/\text{Irred}(\mathcal{H}_m) \) is a finite set. Let \( y_1, \ldots, y_l \) be a complete set of representatives of \( \text{Irred}(\hat{G})/\text{Irred}(\mathcal{H}_m) \). Then \( \mathcal{C} \) is generated by \( \{y_1, \ldots, y_l, x_0, \ldots, x_m, x_0, \ldots, x_m\} \).}

As in the classical case, we can show that property \( T \) is equivalent to the existence of a Kazhdan pair.

\[\text{Proposition 9.}\] Let \( \hat{G} \) be a finitely generated discrete quantum group. Let \( E \subset \text{Irred}(\hat{G}) \) be a finite subset with \( 1 \in E \) such that \( \mathcal{R}(\hat{G}) \) is generated by \( E \). The following assertions are equivalent:

1. \( \hat{G} \) has property \( T \).

2. There exists \( \epsilon > 0 \) such that every unitary representation of \( \hat{G} \) having an \((E, \epsilon)\)-invariant vector has a non-zero invariant vector.

**Proof.** It is sufficient to show that 1 implies 2. Let \( n \in \mathbb{N}^* \) and \( E_n = \{y \in \text{Irred}(\hat{G}) \mid y \subset x_1 \ldots x_n, x_i \in E\} \). Because \( 1 \in E \), the sequence \( (E_n)_{n \in \mathbb{N}} \) is increasing. Let us show that \( \text{Irred}(\hat{G}) = \bigcup E_n \). Let \( r \in \text{Irred}(\hat{G}) \). Because \( \mathcal{R}(\hat{G}) \)
is generated by $E$, there exists a finite family of morphisms $b_k \in \text{Mor}(r_k, r)$, where $r_k$ is a product of elements of $E$ and $\sum b_k b_k^* = I_r$. Let $L$ be the maximum of the length of the elements $r_k$. Because $1 \in E$, we can suppose that all the $r_k$ are of the form $x_1 \ldots x_L$ with $x_i \in E$. Put $t_k = b_k^*$. Note that $t_k^* t_k \in \text{Mor}(r, r)$. Because $r$ is irreducible and $\sum t_k^* t_k = I_r$, there exists a unique $k$ such that $t_k^* t_k = I_r$ and $t_k^* t_i = 0$ if $i \neq k$. Thus $t_k \in \text{Mor}(r, r_k)$ is an isometry. This means that $r \subseteq r_k = x_1 \ldots x_L$, i.e. $r \in E_L$.

Suppose that $\mathcal{G}$ has property $T$ and 2 is false. Let $N = \max \{ n_x | x \in E \}$ and $\epsilon_n = \frac{1}{\sqrt{n} \sqrt{N}}$. For all $n \in \mathbb{N}^*$ there exists a unitary representation $U_n$ of $\mathcal{G}$ on a Hilbert space $K_n$ with an $(E, \epsilon_n)$-invariant vector but without a non-zero invariant vector. Let $\xi_n$ be a unit vector in $\text{Mor}(r_k, \epsilon_n)$ with an $(E, \epsilon_n)$-invariant vector. Write $U_n = \sum_{y \in \text{Irred}(\mathcal{G})} U_{n,y}$ where $U_{n,y}$ is a unitary element in $\mathcal{B}(H_y) \otimes \mathcal{B}(K_n)$. Let us show the following:

\[ \| U_{n,y} \xi_n - \xi_n \|_{H_y \otimes K_n} < \frac{1}{n} \| \eta \|_{H_y}, \quad \forall n \in \mathbb{N}^*, \forall y \in E_n, \forall \eta \in H_y. \tag{3} \]

Let $y \in E_n$ and $t_y \in \text{Mor}(y, x_1 \ldots x_n)$ such that $t_y^* t_y = I_y$. Note that, by the definition of a representation and using the description of the coproduct on $\mathcal{G}$, we have $(t_y \otimes 1) U_{n,y} = U_{n,x_1} U_{n,x_2} U_{n,x_n} \ldots U_{n,x_1}^n (t_y \otimes 1)$ where the subscripts are used for the leg numbering notation. It follows that, for all $\eta \in H_y$, we have:

\[
\| U_{n,y} \eta \xi_n - \eta \xi_n \| = \| (t_y \otimes 1) U_{n,y} \eta \xi_n - (t_y \otimes 1) \eta \xi_n \|
= \| U_{n,x_1} U_{n,x_2} U_{n,x_n} \ldots U_{n,x_1}^n t_y \eta \xi_n - t_y \eta \xi_n \|
\leq \sum_{k=1}^n \| U_{n,x_k} t_y \eta \xi_n - t_y \eta \xi_n \|.
\]

Let $(e_j^{x_i})_{1 \leq j \leq n}$ be an orthonormal basis of $H_{x_i}$ and put

\[ t_y \eta = \sum \lambda_{i_1 \ldots i_n} e_{i_1}^{x_1} \otimes \ldots \otimes e_{i_n}^{x_n}. \]

Then we have, for all $y \in E_n$ and $\eta \in H_y$,

\[
\| U_{n,y} \eta \xi_n - \eta \xi_n \| \leq \sum_k \| \sum \lambda_{i_1 \ldots i_n} (U_{n,x_k} e_{i_1}^{x_1} \otimes \ldots \otimes e_{i_n}^{x_n} \xi_n - e_{i_1}^{x_1} \otimes \ldots \otimes e_{i_n}^{x_n} \xi_n) \|
\leq \sum_k \sum_{i_1 \ldots i_n} | \lambda_{i_1 \ldots i_n} | \| U_{n,x_k} e_{i_k} \xi_n - e_{i_k} \xi_n \|
\leq n \epsilon_n \| t_y \eta \|_1,
\]

where $\| t_y \eta \|_1 = \sum | \lambda_{i_1 \ldots i_n} |$. Note that $\| t_y \eta \|_1 \leq \sqrt{N} \| \eta \|$, thus we have

\[
\| U_{n,y} \eta \xi_n - \eta \xi_n \| \leq n \epsilon_n \sqrt{N} \| \eta \|
\leq \frac{1}{n} \| \eta \|.
\]

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This proves Eq. (1). It is now easy to finish the proof. Let $U$ be the direct sum of the $U_n$. It is a unitary representation of $\hat{G}$ on $K = \bigoplus K_n$. Let $\delta > 0$ and $L \subset \text{Irred}(G)$ a finite subset. Because $\text{Irred}(G) = \bigcup E_n$ there exists $n_1$ such that $L \subset E_n$ for all $n \geq n_1$. Choose $n \geq n_1$ such that $\frac{1}{n} < \delta$. Put $\xi = (0, \ldots, 0, \xi_n, 0, \ldots)$ where $\xi_n$ appears in the $n$-th place. Let $x \in L$ and $\eta \in H_x$. We have:

$$||U^x \eta \otimes \xi - \eta \otimes \xi|| = ||U^{n,x} \eta \otimes \xi_n - \eta \otimes \xi_n|| \leq \frac{1}{n}||\eta|| < \delta||\eta||.$$ 

Thus $U$ has almost invariant vectors. It follows from property $T$ that $U$ has a non-zero invariant vector, say $l = (l_n)$. There is a $n$ such that $l_n \neq 0$ and from the $U$-invariance of $l$ we conclude that $l_n$ is $U_n$-invariant. This is a contradiction.

Such a pair $(E, \epsilon)$ as defined Proposition 3 is called a Kazhdan pair for $\hat{G}$. Let us give an obvious example of a Kazhdan pair.

**Proposition 10.** Let $\hat{G}$ be a finite-dimensional discrete quantum group. Then $(\text{Irred}(G), \sqrt{2})$ is a Kazhdan pair for $\hat{G}$.

**Proof.** If $\hat{G}$ is finite-dimensional then it is compact, $\varphi$ is a trace and $\hat{\varphi}$ is a normal functional. For $x \in \text{Irred}(G)$ let $(e_i^x)$ be an orthonormal basis of $H_x$ and $e_{ij}^x$ the associated matrix units. As $Q_x = 1$, we have $\hat{\varphi}(e_{ij}^x) = \frac{\dim_x(x)}{n_x} \delta_{ij}$. Let $U \in l^\infty(\hat{G}) \otimes B(K)$ be a unitary representation of $\hat{G}$ with a unit vector $\xi \in K$ such that:

$$\sup_{x \in \text{Irred}(G), 1 \leq j \leq n_x} ||U^x e_j^x \otimes \xi - e_j^x \otimes \xi|| < \sqrt{2}.$$ 

Because $\hat{\varphi}(1)^{-1}(\hat{\varphi} \otimes \text{id})(U)$ is the projection on the $U$-invariant vectors, $\tilde{\xi} = (\hat{\varphi} \otimes \text{id})(U)\xi \in K$ is invariant. Let us show that $\tilde{\xi}$ is non-zero. Writing $U^x = \sum e_{ij}^x \otimes U_{ij}^x$ with $U_{ij}^x \in B(K)$, we have:

$$||U^x e_j^x \otimes \xi - e_j^x \otimes \xi||^2 = 2 - 2\Re(U_{ij}^x \xi, \xi), \quad \text{for all } x \in \text{Irred}(G), 1 \leq j \leq n_x.$$ 

It follows that $\Re(U_{ij}^x \xi, \xi) > 0$ for all $x \in \text{Irred}(G)$ and all $1 \leq j \leq n_x$. Thus,

$$\Re(\tilde{\xi}, \xi) = \sum_{x,i,j} \Re(\hat{\varphi}(e_{ij}^x)(U_{ij}^x \xi, \xi)) = \sum_{x,i} \frac{\dim_x(x)^2}{n_x} \Re((U_{ij}^x \xi, \xi)) > 0.$$ 

**Remark 4.** It is easy to see that a discrete quantum group is amenable and has property $T$ if and only if it is finite-dimensional. Indeed, the existence of almost invariant vectors for the regular representation is equivalent with amenability and it is well known that a discrete quantum group is finite dimensional if and only if the regular representation has a non-zero invariant vector. Moreover the previous proposition implies that all finite-dimensional discrete quantum groups have property $T$. 

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The main result of this paper is the following.

**Theorem 3.** Let $\hat{G}$ be discrete quantum group such that $L^\infty(\mathbb{G})$ is an infinite dimensional factor. The following assertions are equivalent:

1. $\hat{G}$ has property $T$.
2. $L^\infty(\mathbb{G})$ is a II$_1$ factor with property $T$.

**Proof.** We can suppose that $G$ is reduced, $C(\mathbb{G}) \subset B(L^2(\mathbb{G}))$ and $V \in l^\infty(\hat{G}) \otimes L^\infty(\mathbb{G})$. We denote by $M$ the von Neumann algebra $L^\infty(\mathbb{G})$. For each $x \in \text{Irred}(G)$ we choose an orthonormal basis $(e^x_i)_{1 \leq i \leq n_x}$ of $H_x$. When $\varphi$ is a trace we take $e^x_i = J_x(e^x_i)$. We put $u^x_i = (\omega e^x_i, e^x_i \otimes \text{id})(u^x_i)$.

1 $\Rightarrow$ 2: Suppose that $\hat{G}$ has property $T$. By Proposition 6, $M$ is finite factor. Thus, it is a II$_1$ factor. Let $(E, \epsilon)$ be a Kazhdan pair for $\hat{G}$. Let $K \in C(M)$ with morphisms $\pi_t : M \to B(K)$ and $\pi_r : M^{op} \to B(K)$. Let $\delta = \frac{1}{\max\{n_x \sqrt{n_x}, x \in E\}}$.

Suppose that there exists a unit vector $\xi' \in K$ such that:

$$||u^x_i \xi' - \xi' u^x_i|| < \delta, \quad \forall x \in E, \forall 1 \leq i, j \leq n_x.$$ 

Define $U = (\text{id} \otimes \pi_r)(V^*)(\text{id} \otimes \pi_l)(V)$. Because $V$ is a unitary representation of $\hat{G}$ and $\pi_r$ is an anti-homomorphism, it is easy to check that $U$ is a unitary representation of $\hat{G}$ on $K$. Moreover, for all $x \in E$, we have:

$$||U^x e^x_i \otimes \xi' - e^x_i \otimes \xi'|| = ||(\text{id} \otimes \pi_t)(u^x) e^x_i \otimes \xi' - (\text{id} \otimes \pi_r)(u^x) e^x_i \otimes \xi'||$$

$$= ||\sum_{k=1}^{n_x} e^x_k \otimes (u^x_k \xi' - \xi' u^x_k)||$$

$$\leq \sum_{k=1}^{n_x} ||e^x_k \otimes (u^x_k \xi' - \xi' u^x_k)||$$

$$< n_x \delta \leq \frac{\epsilon}{\sqrt{n_x}}.$$

It follows easily that for all $x \in E$ and all $\eta \in H_x$ we have $||U^x \eta \otimes \xi' - \eta \otimes \xi'|| < \epsilon||\eta||$. Thus there exists a non-zero $U$-invariant vector $\xi \in K$. It is easy to check that $\xi$ is a central vector.

2 $\Rightarrow$ 1: Suppose that $M$ is a II$_1$ factor with property $T$ and let $\epsilon > 0$ and $b_1, \ldots, b_n \in M$ be as in Proposition 8. Let $\varphi$ be the Haar state on $\mathbb{G}$. By Theorem 8, $\varphi$ is the unique tracial state on $M$. We can suppose that $||b_i||_2 = 1$.

Using the classical G.N.S. construction $(L^2(\mathbb{G}), \Omega)$ for $\varphi$ we have, for all $a \in M$,

$$a\Omega = \sum_{x,k,l} n_x \varphi((u^x_k)^* a) u^x_k \Omega.$$

In particular, $||b_i||_2^2 = \sum n_x |\varphi((u^x_k)^* b_i)|^2 = 1$. Fix $\delta > 0$ then there exists a finite subset $E \subset \text{Irred}(\mathbb{G})$ such that, for all $1 \leq i \leq n$,

$$\sum_{x \notin E, k, l} n_x |\varphi((u^x_k)^* b_i)|^2 < \delta^2.$$
Let $U$ be a unitary representation of $\hat{G}$ on $K$ having almost invariant vectors and $\xi \in K$ an $(E, \delta)$-invariant unit vector. Turn $L^2(\hat{G}) \otimes K$ into a correspondence from $M$ to $M$ using the morphisms $\pi_1 : M \to B(L^2(\hat{G}) \otimes K)$, $\pi_1(a) = U(\alpha \otimes 1)U^*$ and $\pi_r : M^{\text{op}} \to B(L^2(\hat{G}) \otimes K)$, $\pi_r(a) = J a^* J \otimes 1$, where $J$ is the modular conjugation of $\varphi$. Let $\hat{\xi} = \Omega \otimes \xi$. It is easy to see that $\pi_1(u_{kl}^x) = \sum_s u_{kl}^x \otimes U_{sl}^x$ and, for all $a \in M$,

$$a\hat{\xi} = \sum n_x \varphi((u_{kl}^x)^* b_i) u_{kl}^x \Omega \otimes U_{sl}^x \xi.$$ 

Note that, because $\varphi$ is a trace, $\Omega$ is a central vector in $L^2(\hat{G})$ and we have, for all $a \in M$, $\hat{\xi} a = a \Omega \otimes \xi$. It follows that, for all $1 \leq i \leq n$, we have

$$||b_i \hat{\xi} - \hat{\xi} b_i||^2 = || \sum_{x, k, l, s} n_x \varphi((u_{kl}^x)^* b_i) u_{kl}^x \Omega \otimes U_{sl}^x \xi - \sum_{x, k, l} n_x \varphi((u_{kl}^x)^* b_i) u_{kl}^x \Omega \otimes \xi ||^2$$

$$= || \sum_{x, k, l} n_x \varphi((u_{kl}^x)^* b_i) \left( \sum_s u_{kl}^x \Omega \otimes U_{sl}^x \xi - u_{kl}^x \Omega \otimes \xi \right) ||^2$$

$$= || \sum_{x, k, l} \sqrt{n_x} \varphi((u_{kl}^x)^* b_i) \left( \sum_s e_s^x \otimes J_x(e_k^x) \otimes U_{sl}^x \xi - e^x_s \otimes J_x(e_k^x) \otimes \xi \right) ||^2$$

$$= || \sum_{x, k, l} \sqrt{n_x} \varphi((u_{kl}^x)^* b_i) J_x(e_k^x) \otimes \left( \sum_s e_s^x \otimes U_{sl}^x \xi - e^x_s \otimes \xi \right) ||^2$$

$$= || \sum_{x, k, l} \sqrt{n_x} \varphi((u_{kl}^x)^* b_i) J_x(e_k^x) \otimes (U^x e^x_s \otimes \xi - e^x_s \otimes \xi) ||^2$$

$$= \sum_{x, k} n_x \left| \sum_l \varphi((u_{kl}^x)^* b_i) (U^x e^x_s \otimes \xi - e^x_s \otimes \xi) \right|^2$$

$$= \sum_{x, k} n_x \left| U^x \eta^x_k \otimes \xi - \eta^x_k \otimes \xi \right|^2,$$ where $\eta^x_k = \sum_l \varphi((u_{kl}^x)^* b_i) e^x_l$

$$= \sum_{x, k} n_x \left| U^x \eta^x_k \otimes \xi - \eta^x_k \otimes \xi \right|^2 + \sum_{x \not\in E, k} n_x \left| U^x \eta^x_k \otimes \xi - \eta^x_k \otimes \xi \right|^2 < \delta^2 \sum_{x \in E, k} n_x \left| \eta^x_k \right|^2 + 4 \sum_{x \not\in E, k} n_x \left| \eta^x_k \right|^2$$

$$< \delta^2 \sum_{x \in E, k} n_x \left| \varphi((u_{kl}^x)^* b_i) \right|^2 + 4 \sum_{x \not\in E, k, l} n_x \left| \varphi((u_{kl}^x)^* b_i) \right|^2$$

$$< \delta^2 + 4 \delta^2 = 5 \delta^2.$$
Let \( Q \) be the orthogonal projection on \( H_x \otimes H_x \). Using \( u^k_{ij}(e^x_i \otimes e^y_j) \subset \bigoplus_{z \subset x \otimes y} H_z \otimes H_z \), and \( x \subset x \otimes y \) if and only if \( y = 1 \), we find:

\[
Qu^k_{ij}(e^x_i \otimes e^y_j) = \delta_{y,1} \frac{1}{\sqrt{n_x}} e^x_k \otimes e^y_i.
\]

Using the same arguments and the fact that \( J = \bigoplus (J_x \otimes J_x) \) we find:

\[
QJ(u^k_{ij})^*J(e^x_i \otimes e^y_j) = \delta_{y,1} \frac{1}{\sqrt{n_x}} e^x_j \otimes e^y_i.
\]

Applying \( Q \otimes 1 \) to Eq. (4) we obtain:

\[
\sum_k e^x_k \otimes e^y_k \otimes U^x_{kj} \eta = \sum_k e^x_k \otimes \eta, \quad \text{for all } x \in \text{Irred}(G), \ 1 \leq i, j \leq n_x.
\]

Thus, for all \( x \in \text{Irred}(G) \) and all \( 1 \leq j \leq n_x \), we have:

\[
U^x(e^x_j \otimes \eta) = \sum_k e^x_k \otimes U^x_{kj} \eta = e^x_j \otimes \eta.
\]

Thus \( \eta \) is a non-zero \( U \)-invariant vector. \( \square \)

The preceding theorem admits the following corollary about the persistance of property \( T \) by twisting.

**Corollary 2.** Let \( G \) be a compact quantum group such that \( L^\infty(G) \) is an infinite dimensional factor. Suppose that \( K \) is an abelian co-subgroup of \( G \) (see [8]). Let \( \sigma \) be a continuous bicharacter on \( \hat{K} \) and denote by \( \hat{G}^\sigma \) the twisted quantum group. If \( \hat{G} \) has property \( T \) then \( \hat{G}^\sigma \) is a discrete quantum group with property \( T \).

**Proof.** If \( \hat{G} \) has property \( T \) then the Haar state \( \varphi \) on \( G \) is a trace. Thus the co-subgroup \( K \) is stable (in the sense of [3]) and the Haar state \( \varphi_\sigma \) on \( \hat{G}^\sigma \) is the same, i.e. \( \varphi = \varphi_\sigma \). It follows that \( \hat{G}^\sigma \) is a compact quantum group with \( L^\infty(\hat{G}^\sigma) = L^\infty(\hat{G}) \). Thus \( L^\infty(\hat{G}^\sigma) \) is a \( \Pi_1 \) factor with property \( T \) and \( \hat{G}^\sigma \) has property \( T \). \( \square \)

**Example 1.** The group \( SL_{2n+1}(\mathbb{Z}) \) is I.C.C. and has property \( T \) for all \( n \geq 1 \). Let \( K_n \) be the subgroup of diagonal matrices in \( SL_{2n+1}(\mathbb{Z}) \). We have \( K_n = \mathbb{Z}^{2n} = \langle t_1, \ldots, t_{2n} | t_i^2 = 1 \forall i, \ t_it_j = t_jt_i \forall i, j \rangle \) and \( K_n \) is an abelian co-subgroup
of $G_{2n+1} = (C^*(SL_{2n+1}(\mathbb{Z})), \Delta)$. Consider the following bicharacter on $\widehat{K}_n = K_n$: $\sigma$ is the unique bicharacter such that $\sigma(t_i, t_j) = -1$ if $i \leq j$ and $\sigma(t_i, t_j) = 1$ if $i > j$. By the preceding Corollary, the twisted quantum group $\widehat{G}_{2n+1}^\sigma$ has property $T$ for all $n \geq 1$. When $n$ is even, $SL_n(\mathbb{Z})$ is not I.C.C. and $I$ and $-I$ lie in the centre of $SL_n(\mathbb{Z})$. We consider the group $PSL_n(\mathbb{Z}) = SL_n(\mathbb{Z})/\{I, -I\}$ in place of $SL_n(\mathbb{Z})$ in the even case. It is well known that $PSL_{2n}(\mathbb{Z})$ is I.C.C. and has property $T$ for $n \geq 2$. The group of diagonal matrices in $SL_{2n}(\mathbb{Z})$ is $\mathbb{Z}^{2n-1}_2$ which contains $\{I, -I\}$. We consider the following abelian subgroup of $PSL_{2n}(\mathbb{Z})$: $L_n = \mathbb{Z}^{2n-1}_2/\{I, -I\} = \mathbb{Z}^{2n-2}_2 = K_{n-1}$ and the same bicharacter $\sigma$ on $K_{n-1}$. Let $G_{2n} = (C^*(PSL_{2n}(\mathbb{Z})), \Delta)$. By the preceding Corollary, the twisted quantum group $\widehat{G}_{2n}^\sigma$ has property $T$ for all $n \geq 2$.

References


