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A case study of an Hamilton-Jacobi equation by the Adomian decompositional method

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Abstract

We present a study of the Adomian's Decomposition Method (ADM) applied to the Hamilton-Jacobi equations $u_t + H(u_x) = 0$.

We recall the well known characteristics methods in the case of this type of equations to justify the existence or not of solutions. This yields that the ADM gives efficient solutions in time only in $]0, T^[$, where T^* is the critical time of our equation.*

Key words: Adomian method, Hamilton-Jacobi equation, critical time, characteristic strips, viscosity solutions.

AMS subject Classification: 70H20

1 A presentation of the ADM

We consider in this work the Hamilton-Jacobi equation

$$P_u \begin{cases} u_t + H(u_x) & = 0 & x \in \mathbb{R}, t > 0 \\ u(0, x) & = u_0(x) & x \in \mathbb{R} \end{cases} \quad (1)$$

where $u = u(x, t)$ and $u_0 = u_0(x)$ are functions on a Hilbert spaces, and H a non linear operator.

Let take $L = \frac{\partial}{\partial t}$ as the linear operator. The principle of the ADM relies on the

assumption that the solution u of equation (1) can be set as a serie $u = \sum_{k=0}^{+\infty} u_k$. In [1] the authors considered the non linear part as a function of u_x . This yields to:

$$H\left(\frac{\partial u}{\partial x}\right) = \sum_{k=0}^{+\infty} A_k\left(\frac{\partial u_0}{\partial x}, \dots, \frac{\partial u_k}{\partial x}\right).$$

and

$$A_0 = H\left(\frac{\partial u_0}{\partial x}\right) \quad \text{with } u_0 = u(0, x)$$

$$A_{n+1} = \sum_{k=0}^n (k+1) u_{k+1}' \frac{\partial}{\partial u_k'} A_n \quad (2)$$

where $u_k' = \frac{\partial u_k}{\partial x}$.

Several authors give some formulas on the Adomian's polynomials A_k in the case of an operator depending exclusively on u (see for example [2, 4]). For more related works in this field one can see [3], [5], [8], [14], [15] and [16].

We present here other formulas more suited to our analysis.

Theorem 1 *Let's consider that the operator H of (1) is indefinitely differentiable in Frechet sense .*

We have:

$$A_0 = H(u_0') \quad \text{and} \quad A_n = \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} H^{(k)}(u_0') (u_{p_1}', u_{p_2}', \dots, u_{p_k}')$$

where $H^{(k)}(u_0')$ is the k -th derivative at u_0' , and $u_k' = \frac{\partial u_k}{\partial x}$.

Proof:

We note $\psi(\lambda) = \sum_{i=1}^n \lambda^i \frac{\partial u_i}{\partial x}$. Thus

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} H(\psi(\lambda)) = \frac{1}{n!} \frac{d^n}{d\lambda^n} [H \circ \psi](\lambda),$$

$$\text{but } [H \circ \psi](\lambda) = [H \circ \psi](\lambda_0) + \sum_{k=1}^n \frac{1}{k!} [H \circ \psi]^{(k)}(\lambda_0) \underbrace{(\lambda - \lambda_0, \lambda - \lambda_0, \dots, \lambda - \lambda_0)}_{k \text{ times}}$$

then we deduce that:

$$\begin{aligned} H(\psi(\lambda)) &= H(\psi(\lambda_0)) \\ &+ \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} \frac{1}{p_1! p_2! \dots p_k!} H^{(k)}(\psi(\lambda_0)) (\psi^{(p_1)}(\lambda_0)(\lambda - \lambda_0)^{p_1}, \dots, \psi^{(p_k)}(\lambda_0)(\lambda - \lambda_0)^{p_k}) \end{aligned}$$

For $\lambda_0 = 0$, we have:

$$\begin{aligned} H(\psi(\lambda)) &= H\left(\frac{\partial u_0}{\partial x}\right) \\ &+ \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} \frac{1}{p_1! p_2! \dots p_k!} H^{(k)}\left(\frac{\partial u_0}{\partial x}\right) \left(\lambda^{p_1} p_1! \frac{\partial u_{p_1}}{\partial x}, \dots, \lambda^{p_k} p_k! \frac{\partial u_{p_k}}{\partial x}\right) \\ &= H\left(\frac{\partial u_0}{\partial x}\right) \\ &+ \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} \frac{p_1! p_2! \dots p_k!}{p_1! p_2! \dots p_k!} H^{(k)}\left(\frac{\partial u_0}{\partial x}\right) \left(\frac{\partial u_{p_1}}{\partial x}, \dots, \frac{\partial u_{p_k}}{\partial x}\right) \lambda^{\overbrace{p_1 + \dots + p_k}^n} \\ &= H\left(\frac{\partial u_0}{\partial x}\right) \\ &+ \lambda^n \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} H^{(k)}\left(\frac{\partial u_0}{\partial x}\right) \left(\frac{\partial u_{p_1}}{\partial x}, \dots, \frac{\partial u_{p_k}}{\partial x}\right) \end{aligned}$$

$$\text{Thus: } \frac{d^n}{d\lambda^n} H(\psi(\lambda)) = n! \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} H^{(k)}\left(\frac{\partial u_0}{\partial x}\right) \left(\frac{\partial u_{p_1}}{\partial x}, \dots, \frac{\partial u_{p_k}}{\partial x}\right)$$

This ends our demonstration. ■

Theorem 2 For the problem P_u , we have $A_{n-1} = \tilde{u}_n(x) \frac{t^{n-1}}{(n-1)!}$

and $u_n(x, t) = \tilde{u}_n(x) \frac{t^n}{n!} \quad \forall n \geq 1$, with

$$\tilde{u}_n = \sum_{k=1}^{n-1} \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n-1} \frac{n!}{p_1!p_2!\dots p_k!} H^{(k)}(u_0) (\tilde{u}_{p_1}, \tilde{u}_{p_2}, \dots, \tilde{u}_{p_k})$$

Proof:

This will be done by recurrence.

In fact, $A_0 = H(u_0)$,

then $u_1 = \int_0^t H(u_0) ds = H(u_0)t$, and $\tilde{u}_1(x) = H(u_0(x))$

Let's assume that $\forall k \leq n$ $A_{k-1} = \tilde{u}_k(x) \frac{t^{k-1}}{(k-1)!}$ and $u_k(x, t) = \tilde{u}_k(x) \frac{t^k}{k!}$.

From Theorem 1, we have: $A_n = \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} H^{(k)}(u_0) (u_{p_1}, u_{p_2}, \dots, u_{p_k})$.

Then:

$$\begin{aligned} A_n &= \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} H^{(k)}(u_0) (u_{p_1}, u_{p_2}, \dots, u_{p_k}) \\ &= \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} H^{(k)}(u_0) (\tilde{u}_{p_1}, \tilde{u}_{p_2}, \dots, \tilde{u}_{p_k}) \times \frac{t^{p_1}}{p_1!} \frac{t^{p_2}}{p_2!} \dots \frac{t^{p_k}}{p_k!} \\ &= \frac{t^n}{n!} \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} \frac{n!}{p_1!p_2!\dots p_k!} H^{(k)}(u_0) (\tilde{u}_{p_1}, \tilde{u}_{p_2}, \dots, \tilde{u}_{p_k}) \end{aligned}$$

We have:

$$\tilde{u}_{n+1} = \sum_{k=1}^n \frac{1}{k!} \sum_{p_1+p_2+\dots+p_k=n} \frac{n!}{p_1!p_2!\dots p_k!} H^{(k)}(u_0) (\tilde{u}_{p_1}, \tilde{u}_{p_2}, \dots, \tilde{u}_{p_k}) ,$$

and, $u_{n+1} = \int_0^t A_n ds = \tilde{u}_{n+1}(x) \int_0^t \frac{s^n}{n!} ds$

this leads to the result. ■

These two formulas show that u_n closely depends on the derivatives $H^{(k)} \left(\frac{\partial u_0}{\partial x} \right)$,

consequently on $\frac{\partial u_0}{\partial x}$. On the other side the form of $u_n(x, t) = \tilde{u}_n(x) \frac{t^n}{n!}$ can help us

to set some conjectures about the convergence of the serie $\sum_{k=0}^{\infty} u_k(x, t)$.

We have: $\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{\tilde{u}_{n+1}(x)}{\tilde{u}_n(x)} \right| \frac{|t|}{n+1} \leq \left| \frac{\tilde{u}_{n+1}(x)}{\tilde{u}_n(x)} \right| |t|$

Using the well known rule of d'Alembert, one can see that the convergence of $\sum_{k=0}^{\infty} u_k(x, t)$ depends on $H^{(k)} \left(\frac{\partial}{\partial x} u_0 \right)$ and $\frac{\partial}{\partial x} u_0$.

2 Characteristics method

We still consider the problem P_u in (1).

By deriving formally in space (P_u) and taking $v = \frac{\partial u}{\partial x}$, we have the conservation

law equation

$$v_t + \frac{\partial}{\partial x} [H(v)] = 0 \quad (3)$$

with the initial condition $v(x, 0) = v_0(x) = \frac{\partial u_0}{\partial x}$.

Assuming that v is derivable in space, we then have:

$$v_t + H'(v) v_x = 0 \quad (4)$$

Using $a(v) = H'(v)$, we have: $v_t + a(v) v_x = 0$

Let's consider in the plan (x, t) the characteristic strips $x \mapsto x(t)$ solutions of the equation

$$\begin{cases} \frac{dx}{dt} = a(v(x(t), t)) \\ x(0) = x_0 \end{cases} \quad (5)$$

Writing $w(t) = v(x(t), t)$, we have:

$$\begin{aligned} \frac{d}{dt} w(t) &= \frac{d}{dt} v((x, t), t) = \left(v_t + \frac{dx}{dt} \frac{\partial v}{\partial x} \right) ((x, t), t) \\ &= \left(v_t + a \frac{\partial v}{\partial x} \right) ((x, t), t) = 0 \end{aligned}$$

This leads to: $w(t) = w_0$; i.e. v is constant on the the characteristic strips.

From (5) we have $\frac{dx}{dt} = a(v_0(x_0))$, then the characteristic strip from x_0 is a straight line with equation given by:

$$x(t) = a(v_0(x_0)) t + x_0 \quad (6)$$

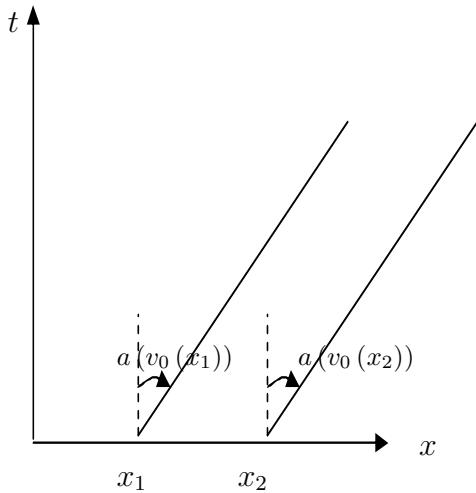


Figure 1: *Characteristics with non decreasing $a(v_0(\cdot))$*

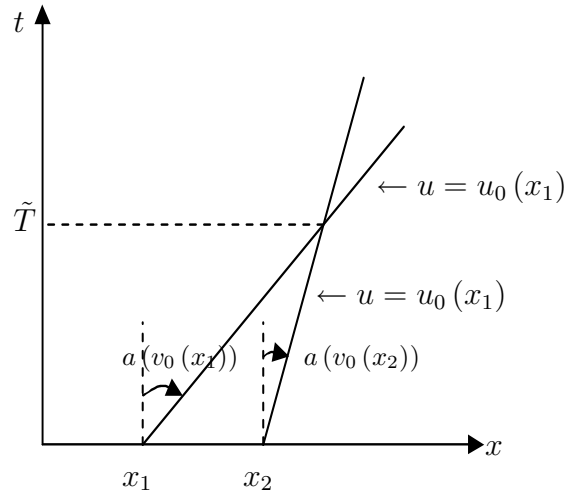


Figure 2: *Characteristics with decreasing $a(v_0(\cdot))$*

If $a(v_0(\cdot))$ is non decreasing, we obtain regular solutions, else one can observe losses of regularity (see fig.1), i.e. the solution lost in regularity in time at \tilde{T} .

The time $T^* = \inf\{\tilde{T}\}$ is called critical time of the equation (3) and is calculated by [6, 7, 12]:

$$T^* = -\frac{1}{\min_{x \in \mathbb{R}} \left(H' \circ \frac{\partial u_0}{\partial x} \right)'(x)} = -\frac{1}{\min_{x \in \mathbb{R}} (a(v_0))'(x)}. \quad (7)$$

We can see that $H' \circ \frac{\partial u_0}{\partial x}$ is one of the terms $H^{(k)} \circ \frac{\partial u_0}{\partial x}$ in the Adomian polynomials A_n .

This leads us to say that the radius of convergence in time of the series $\sum_{k=0}^{\infty} u_k(x, t)$ links closely to the critical time T^* .

3 Applications

3.1 Example 1

Let's consider the equation:

$$\begin{cases} u_t + \frac{1}{2}(u_x)^2 = 0 \\ u(x, 0) = u_0(x) = -x^2 \end{cases}$$

The associate conservation law problem is the well-known Burger's equation [11]:

$$\begin{cases} v_t + \left(\frac{v^2}{2} \right)_x = 0 \\ v(x, 0) = v_0(x) = -2x \end{cases}$$

We then have $H(v) = \frac{1}{2}v^2$. Thus $H'(v) = v \equiv a(v)$, and:

$$T^* = -\frac{1}{\min_{x \in \mathbb{R}} (a(v_0))'(x)} = -\frac{1}{\min_{x \in \mathbb{R}} (v_0)'(x)} = \frac{1}{2}$$

Consider Theorem 1, the ADM gives:

$$\begin{aligned} u_0 &= -x^2 \\ A_0 &= \frac{1}{2}(u_0)^2 = 2x^2 \\ u_1 &= -\int A_0 dt = -2tx^2 \\ A_1 &= u_0 u_1 \\ u_2 &= -4t^2 x^2 \\ \dots &\dots \\ u_n &= -x^2 (2t)^n \end{aligned}$$

So:

$$u(t, x) = -x^2 \sum_{n=0}^{\infty} (2t)^n.$$

For $|t| < 1/2$ we have:

$$u(t, x) = \frac{x^2}{1 - 2t}$$

One can easily verify that after $t = 1/2$, $u(t, x) = \frac{x^2}{1 - 2t}$ remains a solution of our equation. But this is not the physically true. As we can see, $u(t, x)$ isn't regular at $t = 1/2$. Recalling the fact that this equation model frontier evolution (for example in mathematical morphology [17, 8]), we know that this irregularity spreads forward for $t > 1/2$. At this stage, only generalized or entropic solutions survived after $T^* = \frac{1}{2}$. We get these solutions by using Kruskov's formula (see for example [6]). Numerically these solutions can be computed with Godunov or Hamilton-Jacobi schemes.(see [12, 9])

3.2 Example 2

$$\begin{cases} u_t + \sqrt{1 + u_x^2} = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

Case 1: $u_0(x) = ax + b$, $a, b \in \mathbb{R}$.

We can notice that the characteristic strips, in this case, are parallel lines. So this equation has a regular solution at all time. The viscosity solutions theory (see [10, 13]) stipulates that the solutions are parallel lines to the initial condition. One can verify then $H'(u'_0)$ is constant. Then $T^* = +\infty$.

With the ADM, we have:

$$\begin{aligned} u_o &= ax + b \\ A_0 &= \sqrt{1 + a^2} \\ u_1 &= t\sqrt{1 + a^2} \\ A_1 &= 0 \\ u_2 &= 0 \\ \dots &\dots \dots \\ u_n &= 0 \quad \forall n \geq 2 \end{aligned}$$

Thus:

$$u(t, x) = ax + b + t\sqrt{1 + a^2}$$

This solution is exactly the viscosity solution given by the well-known Lax formula [10, 13].

Case 2: $u_0(x) = \sin x$

We then have $H(v) = \sqrt{1 + v^2}$.

Thus $H'(v) = \frac{v}{\sqrt{1 + v^2}} \equiv a(v)$,

$$\text{and: } T^* = -\frac{1}{\min_{x \in \mathbb{R}} (a(v_0))'(x)} = -\frac{1}{\min_{x \in \mathbb{R}} \frac{d}{dx} \left(\frac{v_0}{\sqrt{1 + v_0^2}} \right) (x)}$$

$$\frac{d}{dx} \left(\frac{v_0}{\sqrt{1 + v_0^2}} \right) (x) = \frac{d}{dx} \left(\frac{\cos x}{\sqrt{1 + \cos^2 x}} \right) = \frac{-\sin x}{(1 + \cos^2 x) \sqrt{1 + \cos^2 x}}$$

It is easy to see that:

$$\min_{x \in \mathbb{R}} \frac{-\sin x}{(1 + \cos^2 x) \sqrt{1 + \cos^2 x}} = \left[\frac{-\sin x}{(1 + \cos^2 x) \sqrt{1 + \cos^2 x}} \right]_{x=\frac{\pi}{2}}.$$

Thus $T^* = 1$.

The ADM gives:

$$u_0 = \sin x$$

$$u_1 = t\sqrt{1 + \cos^2 x}$$

$$u_2 = -\frac{t^2}{2} \frac{\cos^2 x \sin x}{1 + \cos^2 x}$$

$$u_3 = \frac{t^3}{6(1 + \cos^2 x)^{7/2}} (2 \cos^2 x \sin^2 x + \cos^4 x \sin^2 x - \cos^4 x - 2 \cos^6 x - \cos^8 x)$$

$$u_4 = \frac{t^4}{24(1 + \cos^2 x)^5} (10 \cos^4 x \sin^2 x + 13 \cos^4 x \sin x - 4 \cos^3 x \sin^2 x \\ + 24 \cos^6 x \sin x + 8 \cos^8 x \sin x + 7 \cos^{10} x \sin x - 6 \cos^2 x \sin^3 x \\ - 6 \cos^4 x \sin^3 x + 3 \cos^2 x \sin x)$$

... ..

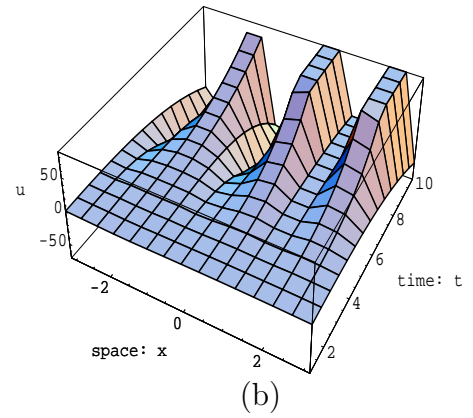
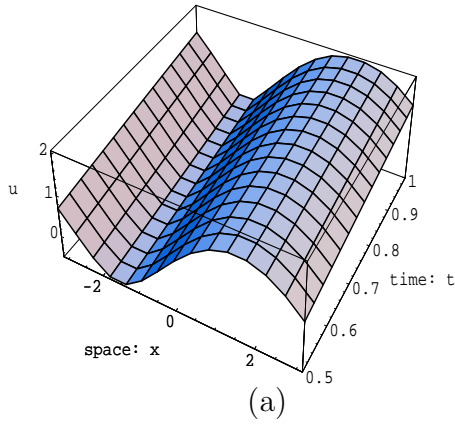


Figure 3: $\sum_{k=0}^4 u_k(x, t)$ for (a) $0.5 \leq t \leq 1$ and (b) $1 \leq t \leq 10$

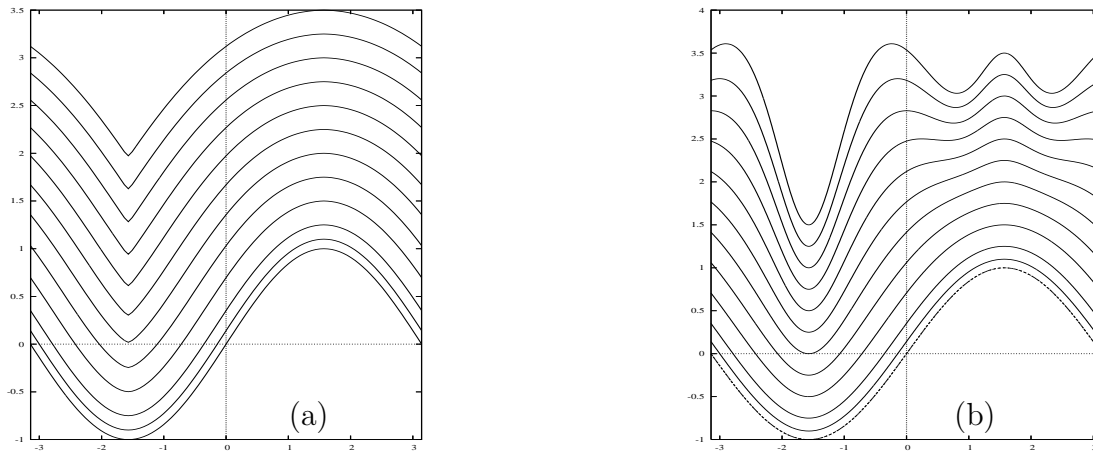


Figure 4: (a) Numerical solution with Hamilton-Jacobi scheme and (b) $\sum_{k=0}^4 u_k(x, t)$ at different times (upwards : $t = 0, 0.1, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, 2.25, 2.5$).

One can see as shown on figures 3 and 4 (a)-(b) that for $0 \leq t \leq T^*$ (with $T^* = 1$) the ADM gives an accurate approximation of the solution.

This confirms that the ADM, for the case of Hamilton-Jacobi equations, gives a good approximation of the solution for only $t \leq T^*$ even with few terms u_n . Nevertheless at $t > T^*$ the computation of several terms of the series (u_n) doesn't ameliorate the solution.

4 Conclusion

We show in this paper that, for Hamilton-Jacobi equations and generally hyperbolic equations the ADM gives accurate solution or a good approximation of the solution only for $t \leq T^*$. We deduce from there that ADM doesn't gives global solution in time for this type of equation.

References

- [1] B. BIRREGAH, B. SOME and B. Mampassi, New formulation of Adomian Polynomials Applied to Hamilton-Jacobi Equations, *Far East J. Appl. Math.* 11(1) pp. 103-108, 2003
- [2] K. ABBAOUI and Y. CHERRUAULT, New Ideas for Proving Convergence of Decomposition Methods, *Comp. Math. Applic.* Vol. 29, No. 7, pp. 103-108, 1995.
- [3] ADOMIAN, Solving frontier problems of physics. The Decomposition Method, *Kluwer Academic Press*, (1994).
- [4] E. BABOLIAN, Sh. JAVADI, New method for calculating Adomian polynomials, *Applied Mathematics and Computation* 153 253259, (2004)

- [5] MUSTAFA INC, On numerical solutions of partial differential equations by the decomposition method, *Kragujevac J. Math.*, 26, 153-164, (2004)
- [6] A. Y. LE ROUX, Approximation de quelques problèmes hyperboliques non linéaires, *Thèse, Université de Rennes*, 1979
- [7] STANLEY H. BENTON, JR. The Hamilton- Jacobi Equation: A global Approach, *Mathematics in Science and Engineering, Vol. 131, Academic Press New York San Fransco London* 1977.
- [8] B. BIRREGAH, Morphologie mathématique en traitement d'images: Application la résolution de quelques EDP. non linéaires, Mémoire de DEA- Univ. Ouagadougou, 2001
- [9] S. OSHER and J.A. SETHIAN, Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations, reprinted from *Journal of Computational Physics*, Vol 79 N 1 November 1988.
- [10] T. STRÖMBERG, The Hopf-Lax formula gives the unique viscosity solution, *Departement of Mathematics, LULEA University of Technology , SE-981 87 LULEA, SWEDEN*.
- [11] J.M. BURGER, A mathematical model illustrating the theory of turbulence, *Adv. Appl. Mech. I*, (1948) 171-199.
- [12] S.OSHER et J.A. SETHIAN, Front propagating with Curvature-Dependent Speed: Algorithm Based on Hamilton-Jacobi Formulations, *J. Comp. Phys.*, vol.79, (1988), pp. 12-49
- [13] M. G. CRANDALL, H. ISHII and P.-L. LIONS, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* 27 (1992), pp. 1-67.
[http : //www.ams.org/bull/pre-1996-data/199227-1/199227-1TOC.html](http://www.ams.org/bull/pre-1996-data/199227-1/199227-1TOC.html)
- [14] D. KAYA, The use of Adomian decomposition method for solving a specific nonlinear partial differential equations, *Bulletin of the Belgian Mathematical Society*, V. 9, No 3, pp 343-349 (Sept. 2002)
- [15] G. ADOMIAN, A review of the decomposition method in applied mathematics, *J. Math. Anal. Appl* 135 (1988) 501-544.
- [16] Y. CHERRUAULT, Modèles et méthodes mathématiques pour les sciences du vivant, Presses Universitaires de France, (1998).
- [17] R. BOOMGAARD and A. SMEULDER, The morphological structure of Images; The differential equations of morphological scale-space, *IEEE Transaction on Pattern Analysis and Machine Intelligence*, Vol. 16. No. 11. Nov. 1994.