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# Weibull tail-distributions revisited: a new look at some tail estimators

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## Abstract

In this paper, we propose to include Weibull tail-distributions in a more general family of distributions. In particular, the considered model also encompasses the whole Fréchet maximum domain of attraction as well as log-Weibull tail-distributions. The asymptotic normality of some tail estimators based on the log-spacings between the largest order statistics is established in a unified way within the considered family. This result permits to understand the similarity between most estimators of the Weibull tail-coefficient and the Hill estimator. Some different asymptotic properties, in terms of bias, rate of convergence, are also highlighted.

**AMS Subject Classifications:** 62G05, 62G20, 62G30.

**Keywords:** Weibull tail-distributions, extreme quantile, maximum domain of attraction, asymptotic normality.

## 1 Motivations

Weibull tail-distributions encompass a variety of light-tailed distributions, *i.e.* distributions in the Gumbel maximum domain of attraction, see [20] for further details. Weibull tail-distributions include for instance Weibull, Gaussian, gamma and logistic distributions. Let us recall that a cumulative distribution function  $F$  has a Weibull tail if its associated survival function  $\bar{F} = 1 - F$  satisfies the following property: There exists  $\theta > 0$  such that for all  $\lambda > 0$ ,

$$\lim_{t \rightarrow \infty} \frac{\log \bar{F}(\lambda t)}{\log \bar{F}(t)} = \lambda^{1/\theta}. \quad (1)$$

The parameter  $\theta$  is called the Weibull tail-coefficient. We refer to [7] for a general account on Weibull tail-distributions and to [6] for an application to the modeling of large claims in non-life insurance. Dedicated methods have been proposed to estimate the Weibull tail-coefficient since the relevant information is only contained in the extreme upper part of the sample denoted hereafter by  $X_1, \dots, X_n$ . A first direction was investigated in [8] where an estimator based on the record values is proposed. Another family of approaches [3, 4, 10, 13] consists of using the  $k_n$  upper order statistics  $X_{n-k_n+1,n} \leq \dots \leq X_{n,n}$  where  $(k_n)$  is an intermediate sequence of integers *i.e.* such that

$$\lim_{n \rightarrow \infty} k_n = \infty \text{ and } \lim_{n \rightarrow \infty} k_n/n = 0. \quad (2)$$

More specifically, most recent estimators are based on the log-spacings between the  $k_n$  upper order statistics [7, 11, 22, 23, 25, 26, 27]. All these estimators are thus similar to the Hill statistics [34] defined as

$$H_n(k_n) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}). \quad (3)$$

As an example, all three estimators proposed in [22] are proportional to  $H_n(k_n)$ . This similarity may be surprising since  $H_n(k_n)$  is dedicated to the estimation of the tail index  $\gamma$  for heavy-tailed distribution *i.e.* such that

$$\lim_{t \rightarrow \infty} \frac{\bar{F}(\lambda t)}{\bar{F}(t)} = \lambda^{-1/\gamma},$$

for all  $\lambda > 0$ . This property characterizes distributions belonging to the Fréchet maximum domain of attraction and sometimes called Pareto-type distributions.

The main goal of this work is therefore to explain why statistics based on log-spacings could be efficient in estimating tail parameters of both Weibull-tail and Pareto-type distributions. To this end, we introduce a family of distributions, indexed by two parameters  $\tau \in [0, 1]$  and  $\theta > 0$ , which includes these two type of distributions. The first parameter  $\tau$  allows to represent a large panel of distribution tails ranging from Weibull-type tails ( $\tau = 0$ ) to Pareto-type tails ( $\tau = 1$ ). The second parameter  $\theta$  is the parameter to be estimated. It coincides with the Weibull tail-coefficient when  $\tau = 0$  and with the tail index when  $\tau = 1$ .

An estimator  $\hat{\theta}_n(k_n)$  of  $\theta$  is then introduced for the new family of distributions and an estimator of extreme quantiles is derived. The asymptotic normality of these estimators is established in Section 3 in a unified way and illustrated on some simulated data in Section 4. Some concluding remarks are given in Section 5. Proofs are postponed to Section 6.

## 2 Model and estimators

### 2.1 Definition and first properties

Let us consider the family of survival distribution functions defined as

$(\mathbf{A}_1(\tau, \theta))$   $\bar{F}(x) = \exp(-K_\tau^-(\log H(x)))$  for  $x \geq x_*$  with  $x_* > 0$  and

- $K_\tau(x) = \int_1^x u^{\tau-1} du$  where  $\tau \in [0, 1]$ ,
- $H$  an increasing function such that  $H^-(t) = \inf\{x, H(x) \geq t\} = t^\theta \ell(t)$ , where  $\theta > 0$  and  $\ell$  is a slowly varying function *i.e.*  $\ell(\lambda x)/\ell(x) \rightarrow 1$  as  $x \rightarrow \infty$  for all  $\lambda \geq 1$ .

The function  $H^-$  is the so-called generalized inverse of  $H$ . Note that  $K_\tau^-$  coincides with the classical inverse since  $K_\tau$  is continuous. The expansion  $H^-(t) = t^\theta \ell(t)$  is equivalent to supposing that  $H^-$  is regularly varying at infinity with index  $\theta$ . This property is denoted by  $H^- \in \mathcal{R}_\theta$ , see [9] for more details on regular variations theory. Let us first highlight that the tail heaviness of  $\bar{F}$  is mainly driven by  $\tau \in [0, 1]$  and secondarily by  $\theta > 0$ :

**Proposition 1** *Let  $\bar{F}_{\tau_1, \theta_1}$  and  $\bar{F}_{\tau_2, \theta_2}$  be two survival distribution functions satisfying respectively  $(\mathbf{A}_1(\tau_1, \theta_1))$  and  $(\mathbf{A}_1(\tau_2, \theta_2))$ .*

- (i) *If  $\tau_1 < \tau_2$  then  $\bar{F}_{\tau_1, \theta_1}(x)/\bar{F}_{\tau_2, \theta_2}(x) \rightarrow 0$  as  $x \rightarrow \infty$  for all  $(\theta_1, \theta_2) \in (0, \infty)^2$ .*
- (ii) *If  $\tau_1 = \tau_2 = \tau$  and  $\theta_1 < \theta_2$  then  $\bar{F}_{\tau, \theta_1}(x)/\bar{F}_{\tau, \theta_2}(x) \rightarrow 0$  as  $x \rightarrow \infty$ .*

Thus, the larger is  $\tau$ , the heavier is the tail. Let us consider the two extremal cases  $\tau = 0$  and  $\tau = 1$ . Clearly, under  $(\mathbf{A}_1(0, \theta))$ ,  $\bar{F}(x) = \exp(-H(x))$  is the survival function of a Weibull-tail distribution, see (1). At the opposite,  $(\mathbf{A}_1(1, \theta))$  entails  $\bar{F}(x) = e^{1/H(x)} = x^{-1/\theta} \tilde{\ell}(x)$  where  $\tilde{\ell}$  is a slowly varying function. As a consequence,  $F$  belongs to the Fréchet maximum domain of attraction and  $\theta$  coincides with the tail index. In view of the above remarks, intermediate values of  $\tau \in (0, 1)$  correspond to distribution tails lighter than Pareto tails but heavier than Weibull tails. Indeed, we have  $\bar{F}(x) = \exp(-h(x))$  with  $h(x) \sim ((\tau/\theta) \log x)^{1/\tau}$  and thus  $h(x)/x^\beta \rightarrow 0$  for all  $\beta > 0$  while  $h(x)/\log(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , this property characterizing an “exponential type” distribution, see [32]. The next proposition provides a more precise characterization while examples are provided in Paragraph 2.2.

**Proposition 2**

- (i)  $F$  verifies  $(\mathbf{A}_1(0, \theta))$  if and only if  $F$  is a Weibull-tail distribution function with Weibull tail-coefficient  $\theta$ .
- (ii) If  $F$  verifies  $(\mathbf{A}_1(\tau, \theta))$ ,  $\tau \in [0, 1)$  and if  $H$  is twice differentiable then  $F$  belongs to the Gumbel maximum domain of attraction.
- (iii)  $F$  verifies  $(\mathbf{A}_1(1, \theta))$  if and only if  $F$  is in the Fréchet maximum domain of attraction with tail-index  $\theta$ .

**2.2 Examples**

In view of Proposition 2(i), Gaussian, gamma, Weibull, Benktander II, logistic and extreme-value distributions all verify  $(\mathbf{A}_1(0, \theta))$  since they are examples of Weibull tail-distributions (see [23], Table 1). Examples of distributions verifying  $(\mathbf{A}_1(\tau, \theta))$  with  $\tau \in (0, 1)$  include some log-Weibull tail-distributions. Let us recall that a random variable  $Y$  is distributed from a log-Weibull tail-distribution if  $\log(Y)$  follows a Weibull tail-distribution.

**Proposition 3** *Suppose that  $F$  verifies  $(\mathbf{A}_1(0, \theta))$  with  $\theta \in (0, 1]$ . If, moreover, the slowly-varying function  $\ell$  is differentiable and  $\ell(t) \rightarrow \ell_\infty > 0$  as  $t \rightarrow \infty$  then  $F(\log \cdot)$  verifies  $(\mathbf{A}_1(\theta, \theta \ell_\infty))$ .*

As an example, the standard log-normal distribution can be looked at as a log-Weibull tail-distribution and thus verifies  $(\mathbf{A}_1(1/2, \sqrt{2}/2))$ . Similarly, the gamma distribution verifies  $(\mathbf{A}_1(0, 1))$  and the log-gamma distribution belongs to the Fréchet maximum domain of attraction, see for instance [16], Table 3.4.2. Finally, other examples of distributions satisfying  $(\mathbf{A}_1(1, \theta))$  can be found in the above mentioned table.

**2.3 Definition of the estimators**

Denoting by  $(k_n)$  an intermediate sequence of integers (see (2)), the following estimator of  $\theta$  is considered:

$$\hat{\theta}_n(k_n) = \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})), \tag{4}$$

with, for all  $t > 0$  and  $q \in \mathbb{N} \setminus \{0\}$ ,

$$\mu_{q,\tau}(t) = \int_0^\infty (K_\tau(x+t) - K_\tau(t))^q e^{-x} dx.$$

A crucial point is that the estimator (4) essentially consists in averaging the log-spacings between the upper-order statistics. Even more strongly,  $\widehat{\theta}_n(k_n)$  only differs from the Hill statistics (3) by a non-random normalizing sequence:  $\widehat{\theta}_n(k_n) = H_n(k_n)/\mu_{1,\tau}(\log(n/k_n))$ . This similarity can be intuitively understood by studying the log-spacing between two quantiles  $x_u$  and  $x_v$  of  $\overline{F}$ , with  $0 < u < v \leq 1$ . Under  $(\mathbf{A}_1(\tau, \theta))$  we have

$$\log x_u - \log x_v = \theta (K_\tau(-\log u) - K_\tau(-\log v)) + \log \left( \frac{\ell(\exp K_\tau(-\log u))}{\ell(\exp K_\tau(-\log v))} \right). \quad (5)$$

Now, since  $\ell$  is a slowly-varying function, if the orders  $u$  and  $v$  of the quantiles are small enough, the second term can be neglected in the right-hand side of (5) to obtain

$$\log x_u - \log x_v \simeq \theta (K_\tau(-\log u) - K_\tau(-\log v)), \quad (6)$$

which shows that log-spacings are approximately proportional to  $\theta$ . Since this key property holds for all  $\tau \in [0, 1]$ , it is thus shared by Pareto-type, Weibull tail and log-Weibull tail-distributions. Note that this property can be checked graphically on a sample by drawing a quantile-quantile plot. It consists in plotting the pairs  $(K_\tau(\log(n/i)), \log(X_{n-i+1,n}))$  for  $i = 1, \dots, k_n$ . From (6), the graph should be approximately linear. Following the same ideas, an estimator of the extreme quantile  $x_{p_n}$  can be deduced from (4) by:

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \exp \left( \widehat{\theta}_n(k_n) (K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))) \right). \quad (7)$$

Recall that an extreme quantile  $x_{p_n}$  of order  $p_n$  is defined by  $x_{p_n} = \overline{F}^{\leftarrow}(p_n)$  with  $p_n \rightarrow 0$  as  $n \rightarrow \infty$ . For instance, if  $np_n \rightarrow 0$  then  $x_{p_n}$  is larger than the maximum observation  $X_{n,n}$  of the sample (with probability tending to one). This requires to extrapolate sample results to areas where no data are observed and occurs in reliability [14], hydrology [36], finance [16],...

### 3 Asymptotic properties

We show in the next paragraph that the asymptotic normality of  $\widehat{\theta}_n(k_n)$  and  $\widehat{x}_{p_n}$  can be established for all  $\tau \in [0, 1]$  in a unified way. In this sense, the asymptotic behavior of these estimators is more a consequence of the log-spacings property than of a tail behavior (which can be exponential as well as polynomial). Paragraphs 3.2 and 3.3 illustrate our general result on the two extremal cases  $\tau = 0$  and  $\tau = 1$ .

#### 3.1 Main results

To establish the asymptotic normality of  $\widehat{\theta}_n(k_n)$ , a second-order condition on  $\ell$  is necessary:

$(\mathbf{A}_2(\rho))$  There exist  $\rho < 0$  and  $b(x) \rightarrow 0$  such that uniformly locally on  $\lambda \geq \lambda_0 > 0$

$$\log \left( \frac{\ell(\lambda x)}{\ell(x)} \right) \sim b(x) K_\rho(\lambda), \text{ when } x \rightarrow \infty.$$

It can be shown that necessarily  $|b| \in \mathcal{R}_\rho$  (see [24]). The second order parameter  $\rho < 0$  tunes the rate of convergence of  $\ell(\lambda x)/\ell(x)$  to 1. The closer is  $\rho$  to 0, the slower is the convergence. Condition  $(\mathbf{A}_2(\rho))$  is the cornerstone in all the proofs of asymptotic normality for extreme value estimators. It is used in [5, 33, 34] to prove the asymptotic normality of several estimators of the extreme value index.

**Theorem 1** Suppose that  $(\mathbf{A}_1(\tau, \theta))$  and  $(\mathbf{A}_2(\rho))$  hold. Let  $(k_n)$  be an intermediate sequence such that

$$\sqrt{k_n} b(\exp K_\tau(\log(n/k_n))) \rightarrow \lambda. \quad (8)$$

Then, introducing  $a_{\tau,\rho} = 1$  if  $\tau \in [0, 1)$  and  $a_{1,\rho} = 1/(1 - \rho)$ , we have

$$\sqrt{k_n} \left( \widehat{\theta}_n(k_n) - \theta - a_{\tau,\rho} b(\exp K_\tau(\log(n/k_n))) \right) \xrightarrow{d} \mathcal{N}(0, \theta^2). \quad (9)$$

It appears that the asymptotic variance of  $\widehat{\theta}_n(k_n)$  given by  $\mathcal{AV} = \theta^2/k_n$  is independent of  $\tau$ . In particular, it remains constant whatever the maximum domain of attraction of  $F$ . The asymptotic squared bias is given by  $\mathcal{ASB}(\tau, \rho) = a_{\tau,\rho}^2 b^2(\exp K_\tau(\log(n/k_n)))$ . If  $b^2$  is ultimately decreasing, then  $\mathcal{ASB}$  is a decreasing function of  $\tau \in [0, 1)$  with a jump at  $\tau = 1$ . These remarks are illustrated on simulated data in Section 4. The next result allows us to establish the rate of convergence of  $\widehat{\theta}_n(k_n)$  to  $\theta$  in (9).

**Proposition 4** Condition (8) with  $\lambda \neq 0$  implies  $\log(k_n) = -2\rho a_{\tau,2\rho} K_\tau(\log n)(1 + o(1))$ .

The rate of convergence is thus of order  $\exp(-\rho a_{\tau,2\rho} K_\tau(\log n)(1 + o(1)))$ . A geometrical rate of convergence is obtained only in the Fréchet maximum domain of attraction,  $\tau = 1$  yields  $\sqrt{k_n} = n^{-\rho/(1-2\rho)+o(1)}$  which is consistent with the conclusions of [31]. Weibull tail-distributions give rise to logarithmic rates of convergence,  $\tau = 0$  yields  $\sqrt{k_n} = (\log n)^{-\rho+o(1)}$  which is consistent with the results of [22]. More generally, the heavier is the tail, the better the rate of convergence is. The next result provides an extension of Statement 1 in [2], which was initially proved only for Weibull tail-distributions ( $\tau = 0$ ).

**Proposition 5** Suppose condition (8) holds with  $\lambda \neq 0$ . If  $\tau \in [0, 1/2]$  then

$$\mathcal{ASB}(\tau, \rho) = c_{\tau,\rho} b^2(\exp K_\tau(\log n))(1 + o(1)),$$

where  $c_{\tau,\rho} = 1$  if  $\tau \in [0, 1/2)$  and  $c_{1/2,\rho} = \exp(8\rho^2)$ .

It follows that, when  $\tau \in [0, 1/2]$ , the first order of the asymptotic bias is asymptotically independent of  $k_n$ . As a consequence, the asymptotic mean-squared error defined as  $\mathcal{ASB}(\tau, \rho) + \mathcal{AV}$  is eventually decreasing with respect to  $k_n$ . This remark, already made in [2] in the particular case  $\tau = 0$ , is only of theoretical interest. Indeed, in finite sample situations, condition (8) does not hold and the empirical mean-squared error is a convex function of  $k_n$ , see for instance [22]. Now, the asymptotic normality of the extreme quantile estimator (7) can be deduced from Theorem 1:

**Theorem 2** Suppose the assumptions of Theorem 1 hold with  $\lambda = 0$ . If, moreover,

$$(\log(n/k_n))^{1-\tau} (K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))) \rightarrow \infty \quad (10)$$

then,

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} \left( \frac{\widehat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

Let us now focus on the two particular cases  $\tau = 0$  (Weibull tail-distributions) and  $\tau = 1$  (Fréchet maximum domain of attraction).

### 3.2 Application to Weibull tail-distributions

If  $\tau = 0$ , the estimator (4) coincides with  $\widehat{\theta}_n^{(1)}$  introduced in [22], and

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \left( \frac{\log(1/p_n)}{\log(n/k_n)} \right)^{\widehat{\theta}_n(k_n)}$$

is the estimator proposed in [21]. As a consequence of Theorem 1 and Theorem 2, we obtain:

**Corollary 1** *Suppose that  $(\mathbf{A}_1(0, \theta))$  and  $(\mathbf{A}_2(\rho))$  hold. Let  $(k_n)$  be an intermediate sequence such that  $\sqrt{k_n} b(\log(n/k_n)) \rightarrow 0$ . Then,*

$$\sqrt{k_n} (\widehat{\theta}_n(k_n) - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

If, moreover

$$\log(n/k_n) (\log \log(1/p_n) - \log \log(n/k_n)) \rightarrow \infty \quad (11)$$

then,

$$\frac{\sqrt{k_n}}{\log \log(1/p_n) - \log \log(n/k_n)} \left( \frac{\widehat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

This result is very similar to Corollary 3.1 in [22] except that condition (11) is weaker than the one used in the above mentioned paper. Let us also note that estimators  $\widehat{\theta}_n^{(2)}$  and  $\widehat{\theta}_n^{(3)}$  in [22] can be respectively deduced from  $\widehat{\theta}_n$  by approximating  $\mu_{1,0}$  by a Riemann's sum or using the first order approximation  $\mu_{1,0}(t) \sim 1/t$  as  $t \rightarrow \infty$  given in Lemma 2(i).

### 3.3 Application to the Fréchet maximum domain of attraction

Letting  $\tau = 1$  and remarking that  $\mu_{q,1}(t) = q!$  for all  $t > 0$  and  $q \in \mathbb{N} \setminus \{0\}$ , the estimator (4) coincides with (3) which is the Hill estimator [34] of the tail index. Besides,

$$\widehat{x}_{p_n} = X_{n-k_n+1,n} \left( \frac{k_n}{np_n} \right)^{\widehat{\theta}_n(k_n)}$$

is the Weissman estimator [37]. A straightforward application of the above theorems gives back the classical results:

**Corollary 2** *Suppose that  $(\mathbf{A}_1(1, \theta))$  and  $(\mathbf{A}_2(\rho))$  hold. Let  $(k_n)$  be an intermediate sequence such that  $\sqrt{k_n} b(n/k_n) \rightarrow 0$ . Then,*

$$\sqrt{k_n} (\widehat{\theta}_n(k_n) - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

If, moreover  $k_n/(np_n) \rightarrow \infty$  then,

$$\frac{\sqrt{k_n}}{\log(k_n/(np_n))} \left( \frac{\widehat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

## 4 Illustration on simulations

The section is dedicated to the illustration of the conclusions drawn from Theorem 1 on simulated data. To this end, we consider a cumulative distribution function  $F_{\theta,\tau,\rho}$  verifying  $(\mathbf{A}_1(\tau, \theta))$  and  $(\mathbf{A}_2(\rho))$  with  $\theta = 1/2$ ,  $\tau \in \{0, 1/2, 1\}$  and  $\rho \in \{-1/2, -1/4\}$ . More specifically, the slowly-varying function is given by

$$\ell(x) = 1 - \frac{\theta}{\rho}(1+x)^\rho \left(1 + \frac{1}{x}\right)^\theta.$$

Following Proposition 2(i), it appears that the case  $\tau = 0$  corresponds to a Weibull tail-distribution (with Weibull tail-coefficient  $1/2$ ) similar to a Gaussian distribution. When  $\tau = 1/2$ , in view of Paragraph 2.1,  $\bar{F}(x) = \exp\{-\log x)^2(1 + o(1))\}$ , the distribution has a tail behavior similar to the log-normal distribution. Finally, Proposition 2(iii) shows that, when  $\tau = 1$ , the distribution belongs to the Fréchet maximum domain of attraction with tail-index  $1/2$ .

For each considered combination of  $\tau$  and  $\rho$ ,  $N = 500$  samples  $(\mathcal{X}_{n,j})_{j=1,\dots,N}$  of size  $n = 500$  were simulated from  $F_{1/2,\tau,\rho}$ . On each sample  $(\mathcal{X}_{n,j})$ , the estimate  $\hat{\theta}_{n,j}$  is computed for  $k = 2, \dots, 250$ , the associated empirical squared bias  $\mathcal{E}\mathcal{S}\mathcal{B}$  and empirical variance  $\mathcal{E}\mathcal{V}$  plots are built by plotting the pairs  $(k, (\bar{\theta}_n^{(1)}(k) - \theta)^2)$  and  $(k, \bar{\theta}_n^{(2)}(k) - (\bar{\theta}_n^{(1)}(k))^2)$  where for  $i \in \{1, 2\}$ ,

$$\bar{\theta}_n^{(i)}(k) = \frac{1}{N} \sum_{j=1}^N (\hat{\theta}_{n,j}(k))^i.$$

The empirical squared bias and the empirical variance are depicted on Figure 1 and Figure 2 respectively. Both graphs are represented on the same scale for the sake of comparison. As expected, the squared bias, for a fixed value of  $k$ , is an increasing function of  $\rho$  and a decreasing function of  $\tau$ . At the opposite, the variance seems to be independent of  $\rho$  and is not much dependent of  $\tau$ .

## 5 Concluding remarks

As illustrated in the previous sections, the model  $(\mathbf{A}_1(\tau, \theta))$  provides a new tool for the analysis of tail estimators based on log-spacings. It allows us to encompass Weibull tail-distributions in a more general framework and thus to explain why their dedicated tail estimators are very similar to Hill or Weissman statistics. The next step would be to estimate the parameter  $\tau$ . For instance, one can consider the following estimator based on the log-spacing between two Hill statistics

$$\hat{\tau}_n = 1 + \frac{\log H_n(k'_n) - \log H_n(k_n)}{\log \log(n/k'_n) - \log \log(n/k_n)},$$

where  $(k_n)$  and  $(k'_n)$  are two intermediate sequences such that

$$\liminf_{n \rightarrow \infty} \frac{\log(n/k'_n)}{\log(n/k_n)} > 1.$$

Let us note that

$$\begin{aligned} (\log \log(n/k'_n) - \log \log(n/k_n))(\hat{\tau}_n - \tau) &= \log(\hat{\theta}_n(k'_n)/\theta) - \log(\hat{\theta}_n(k_n)/\theta) \\ &+ \log\left(\frac{\mu_{1,\tau}(\log(n/k'_n))}{\log^{\tau-1}(n/k'_n)}\right) - \log\left(\frac{\mu_{1,\tau}(\log(n/k_n))}{\log^{\tau-1}(n/k_n)}\right). \end{aligned}$$



This implies that the consistency of  $\widehat{\tau}_n$  is a simple consequence of Theorem 1 and Lemma 2(i) whereas the asymptotic distribution is much more difficult to handle as it requires the joint distribution of  $\widehat{\theta}_n(k'_n)$  and  $\widehat{\theta}_n(k_n)$ . Also in practice, the choice of the parameters  $k_n$  and  $k'_n$  is an open question. These two points are currently under investigation.

Other extensions are possible, among others bias correction based on the estimation of the second-order parameter [28, 29]. To this end, an exponential regression model for these tail distributions extending [5, 11, 12, 18] would be of interest. We also plan to adapt our results to the case  $\tau > 1$  and to investigate the possible links with super-heavy tails [19]. Finally, this work could be further extended to random variables  $Y = \psi(X)$  where  $X$  has a parent distribution satisfying  $(\mathbf{A}_1(\tau, \theta))$ . For instance, choosing  $\psi(x) = x^* - 1/x$  would allow to consider distributions (with finite endpoint  $x^*$ ) in the Weibull maximum domain of attraction. This may help for including the negative Hill estimator (see for instance [17] or [30], paragraph 3.6.2) in our framework.

## 6 Proofs

We first give some preliminary lemmas. Their proofs are postponed to the appendix.

### 6.1 Preliminary lemmas

The first lemma provides some uniform approximations based on  $(\mathbf{A}_1(\tau, \theta))$  and  $(\mathbf{A}_2(\rho))$ .

**Lemma 1** *If  $(\mathbf{A}_1(\tau, \theta))$  and  $(\mathbf{A}_2(\rho))$  hold then*

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) \right| = o(b(x)), \text{ when } x \rightarrow \infty.$$

Let us define for all  $q \in \mathbb{N} \setminus \{0\}$ ,  $\tau \in [0, 1]$  and  $t > 0$ ,  $\sigma_{q,\tau}^2(t) = \mu_{2q,\tau}(t) - \mu_{q,\tau}^2(t)$ . The following lemma is of analytical nature. It provides first-order expansions which will be useful in the sequel.

**Lemma 2** *For all  $q \in \mathbb{N} \setminus \{0\}$  and  $\tau \in [0, 1]$ , when  $t \rightarrow \infty$ :*

- (i)  $\mu_{q,\tau}(t) \sim q! t^{(\tau-1)q}$ ,
- (ii)  $\sigma_{q,\tau}^2(t)/\mu_{q,\tau}^2(t) \rightarrow (2q)!/(q!)^2 - 1$ ,
- (iii)  $\mu'_{1,\tau}(t)/\mu_{1,\tau}(t) \rightarrow 0$ .

The next lemma presents an expansion of  $\widehat{\theta}_n(k_n)$ .

**Lemma 3** *Let  $(k_n)$  be an intermediate sequence. Then, under  $(\mathbf{A}_1(\tau, \theta))$ , the following expansions hold:*

$$\widehat{\theta}_n(k_n) = \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \left( \theta \theta_{n,1}^{(1)}(E_{n-k_n+1,n}) + \theta_{n,2}(E_{n-k_n+1,n}) \right),$$

with, for all  $q \in \mathbb{N} \setminus \{0\}$ ,

$$\begin{aligned} \theta_{n,1}^{(q)}(t) &= \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (K_\tau(F_i + t) - K_\tau(t))^q, \\ \theta_{n,2}(t) &= \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left( \frac{\ell(\exp K_\tau(F_i + t))}{\ell(\exp K_\tau(t))} \right), \end{aligned}$$

and where  $E_{n-k_n+1,n}$  is the  $(n - k_n + 1)$ th order statistic associated to  $n$  independent standard exponential variables and  $\{F_1, \dots, F_{k_n-1}\}$  are independent standard exponential variables and independent from  $E_{n-k_n+1,n}$ .

The asymptotic behavior of the  $(n - k_n + 1)$ th standard exponential order statistic is described in the following lemma.

**Lemma 4** *Let  $(k_n)$  be an intermediate sequence. Then, for all differentiable function  $g$ , we have*

$$\sqrt{k_n}(g(E_{n-k_n+1,n}) - g(\log(n/k_n))) = O_{\mathbb{P}}(1)g'(\log(n/k_n))(1 + o_{\mathbb{P}}(1)).$$

The next two lemmas provide the key results for establishing the asymptotic distribution of  $\widehat{\theta}_n(k_n)$ . They describe the asymptotic behavior of the random terms appearing in Lemma 3.

**Lemma 5** *Let  $(k_n)$  be an intermediate sequence. Then, for all  $q \in \mathbb{N} \setminus \{0\}$ ,*

$$\frac{\sqrt{k_n}}{\sigma_{q,\tau}(E_{n-k_n+1,n})} \left( \theta_{n,1}^{(q)}(E_{n-k_n+1,n}) - \mu_{q,\tau}(E_{n-k_n+1,n}) \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

**Lemma 6** *Suppose that  $(\mathbf{A}_1(\tau, \theta))$  and  $(\mathbf{A}_2(\rho))$  hold. Let  $(k_n)$  be an intermediate sequence. Then,*

$$\theta_{n,2}(E_{n-k_n+1,n}) = b(\exp K_{\tau}(E_{n-k_n+1,n}))\theta_{n,3}(E_{n-k_n+1,n})(1 + o_{\mathbb{P}}(1)),$$

where

$$\left| \theta_{n,3}(E_{n-k_n+1,n}) - \theta_{n,1}^{(1)}(E_{n-k_n+1,n}) \right| \leq -\frac{\rho}{2}\theta_{n,1}^{(2)}(E_{n-k_n+1,n}).$$

Moreover, if  $\tau = 1$ , then  $\theta_{n,3}(E_{n-k_n+1,n}) \xrightarrow{P} 1/(1 - \rho)$ .

## 6.2 Proofs of the main results

**Proof of Proposition 1** – Assumptions  $(\mathbf{A}_1(\tau_1, \theta_1))$  and  $(\mathbf{A}_1(\tau_2, \theta_2))$  entail

$$\frac{\bar{F}_{\tau_1, \theta_1}(x)}{\bar{F}_{\tau_2, \theta_2}(x)} = \exp \left[ -K_{\tau_1}^{\leftarrow}(\log H_1(x)) \left( 1 - \frac{K_{\tau_2}^{\leftarrow}(\log H_2(x))}{K_{\tau_1}^{\leftarrow}(\log H_1(x))} \right) \right], \quad (12)$$

where  $H_1 \in \mathcal{R}_{1/\theta_1}$  and  $H_2 \in \mathcal{R}_{1/\theta_2}$ . As a consequence, for all  $q \in \{1, 2\}$ ,  $\log H_q(x) \sim \log(x)/\theta_q$  when  $x \rightarrow \infty$ , see [9], Proposition 1.3.6. Let us first prove (i):  $0 < \tau_1 < \tau_2$  implies

$$K_{\tau_q}^{\leftarrow}(\log H_q(x)) \sim (\tau_q/\theta_q)^{1/\tau_q} (\log x)^{1/\tau_q} \rightarrow \infty, \quad (13)$$

and thus

$$\frac{K_{\tau_2}^{\leftarrow}(\log H_2(x))}{K_{\tau_1}^{\leftarrow}(\log H_1(x))} \sim \frac{(\tau_2/\theta_2)^{1/\tau_2}}{(\tau_1/\theta_1)^{1/\tau_1}} (\log x)^{1/\tau_2 - 1/\tau_1} \rightarrow 0. \quad (14)$$

Collecting (12), (13) and (14) gives the result:  $\bar{F}_{\tau_1, \theta_1}(x)/\bar{F}_{\tau_2, \theta_2}(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Similarly, if  $\tau_1 = 0$ , then

$$\frac{K_{\tau_2}^{\leftarrow}(\log H_2(x))}{K_0^{\leftarrow}(\log H_1(x))} \sim \frac{(\tau_2/\theta_2)^{1/\tau_2}}{H_1(x)} (\log x)^{1/\tau_2} \rightarrow 0,$$

which concludes the first part of the proof. Let us now focus on (ii) and suppose  $\theta_1 < \theta_2$ . If  $\tau > 0$  then

$$\frac{K_{\tau}^{\leftarrow}(\log H_2(x))}{K_{\tau}^{\leftarrow}(\log H_1(x))} \rightarrow \left( \frac{\theta_1}{\theta_2} \right)^{1/\tau} < 1,$$

as  $x \rightarrow \infty$ , while, if  $\tau = 0$ ,

$$\frac{K_0^{\leftarrow}(\log H_2(x))}{K_0^{\leftarrow}(\log H_1(x))} = \frac{H_2(x)}{H_1(x)} \rightarrow 0,$$

as  $x \rightarrow \infty$ . In both cases, for  $x$  large enough,

$$1 - \frac{K_{\tau}^{\leftarrow}(\log H_2(x))}{K_{\tau}^{\leftarrow}(\log H_1(x))} > 0, \quad (15)$$

and collecting (12), (13) and (15) concludes the proof:  $\bar{F}_{\tau, \theta_1}(x)/\bar{F}_{\tau, \theta_2}(x) \rightarrow 0$  as  $x \rightarrow \infty$ .  $\blacksquare$

**Proof of Proposition 2** – Proofs of (i) and (iii) are straightforward consequences of Paragraph 2.1. Let us focus on (ii). In view of the characterization (3.35) in [16] of the Gumbel maximum domain of attraction, it is sufficient to prove that there exists a positive function  $a$ , differentiable with  $a'(t) \rightarrow 0$  as  $t \rightarrow \infty$ , such that

$$\bar{F}(x) = \exp \left\{ - \int_{x_*}^x \frac{dt}{a(t)} \right\}, \quad x \geq x_*. \quad (16)$$

Letting  $a = 1/(K_\tau^\leftarrow(\log H))'$ , it thus remains to prove that  $a'(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all  $\tau \in [0, 1)$ . To this end, let us remark that

$$\begin{aligned} a'(t) &= \frac{1}{K_\tau^\leftarrow(\log H(t))} \left( \tau - 1 + \left( 1 - \frac{H''(t)H(t)}{H'(t)^2} \right) (1 + \tau \log H(t)) \right) \\ &= \frac{1}{K_\tau^\leftarrow(\log H(t))} (\tau - 1 + (\theta + o(1))(1 + \tau \log H(t))), \end{aligned}$$

since  $H' \in \mathcal{R}_{1/\theta-1}$  implies  $H''(t)H(t)/H'(t)^2 \rightarrow 1 - \theta$  as  $t \rightarrow \infty$ . Two cases arise:

- If  $\tau \in (0, 1)$  then  $a'(t) \sim \theta(\tau \log H(t))^{1-1/\tau} \rightarrow 0$  as  $t \rightarrow \infty$ .
- Otherwise, when  $\tau = 0$ , we have  $a'(t) = (\theta - 1 + o(1))/H(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

In both situations, the conclusion follows.  $\blacksquare$

**Proof of Proposition 3** – Let us suppose that  $F$  verifies  $(\mathbf{A}_1(0, \theta))$  with  $\theta \in (0, 1]$ . Then, introducing  $W(x) = \exp K_\theta(H(\log x))$ , we have  $\bar{F}(\log x) = \exp(-K_\theta^\leftarrow(\log W(x)))$ . It thus remains to prove that  $W^\leftarrow \in \mathcal{R}_{\theta\ell_\infty}$ . Simple calculations show that

$$\begin{aligned} W^\leftarrow(t) &= \exp \{ H^\leftarrow(K_\theta^\leftarrow(\log t)) \} \\ &= \exp \{ (1 + \theta \log t) \ell(K_\theta^\leftarrow(\log t)) \} \\ &= e^{\ell_\infty} t^{\theta\ell_\infty} \varphi(t), \end{aligned}$$

where we have defined  $\varphi(t) = \psi(\log t)$  with  $\psi(x) = \exp\{(1 + \theta x)[\ell(K_\theta^\leftarrow(x)) - \ell_\infty]\}$ . As a consequence,

$$\begin{aligned} t(\log \varphi(t))' &= (\log \psi)'(\log t) \\ &= \theta(\ell(K_\theta^\leftarrow(\log t)) - \ell_\infty) + K_\theta^\leftarrow(\log t) \ell'(K_\theta^\leftarrow(\log t)) \\ &= o(1), \end{aligned}$$

since, from [9], p. 15,  $u\ell'(u)/\ell(u) \rightarrow 0$  as  $u \rightarrow \infty$ . Using again [9], p. 15, it follows that  $\varphi$  is a slowly varying function. Thus,  $W^\leftarrow \in \mathcal{R}_{\theta\ell_\infty}$  and  $F(\log \cdot)$  verifies  $(\mathbf{A}_1(\theta, \theta\ell_\infty))$ .  $\blacksquare$

**Proof of Theorem 1** – Lemma 5 states that for  $q \in \{1, 2\}$ ,

$$\frac{\sqrt{k_n}}{\sigma_{q,\tau}(E_{n-k_n+1,n})} \left( \theta_{n,1}^{(q)}(E_{n-k_n+1,n}) - \mu_{q,\tau}(E_{n-k_n+1,n}) \right) = \xi_n^{(q)}$$

where  $\xi_n^{(q)} \xrightarrow{d} \mathcal{N}(0, 1)$ . Then, by Lemma 3,

$$\begin{aligned} \sqrt{k_n} (\hat{\theta}_n(k_n) - \theta - a_{\tau,\rho} b(\exp K_\tau(\log(n/k_n)))) &= \sqrt{k_n} \theta \left( \frac{\mu_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} - 1 \right) + \theta \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} \xi_n^{(1)} \\ &+ \sqrt{k_n} \left( \frac{\theta_{n,2}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} - a_{\tau,\rho} b(\exp K_\tau(\log(n/k_n))) \right) \\ &\stackrel{\text{def}}{=} T_n^{(1)} + T_n^{(2)} + T_n^{(3)}, \end{aligned}$$

and the three terms are studied separately. First, applying Lemma 4 to  $g = \mu_{1,\tau}$  yields

$$T_n^{(1)} = O_{\mathbb{P}}(1) \frac{\mu'_{1,\tau}(\log(n/k_n)(1 + o_{\mathbb{P}}(1)))}{\mu_{1,\tau}(\log(n/k_n))} = o_{\mathbb{P}}(1), \quad (17)$$

in view of Lemma 2(i, iii). Second,

$$T_n^{(2)} = \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \left( 1 + \frac{T_n^{(1)}}{\theta\sqrt{k_n}} \right) \theta\xi_n^{(1)} = \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \theta\xi_n^{(1)} (1 + o_{\mathbb{P}}(1))$$

and, from Lemma 2(ii),  $\sigma_{1,\tau}(E_{n-k_n+1,n})/\mu_{1,\tau}(E_{n-k_n+1,n}) \xrightarrow{P} 1$ . As a preliminary conclusion,

$$T_n^{(2)} = \theta\xi_n^{(1)} (1 + o_{\mathbb{P}}(1)). \quad (18)$$

From Lemma 6,  $T_n^{(3)}$  can be expanded as

$$\begin{aligned} T_n^{(3)} &= \sqrt{k_n} b(\exp K_{\tau}(\log(n/k_n))) \left( \frac{b(\exp K_{\tau}(E_{n-k_n+1,n}))}{b(\exp K_{\tau}(\log(n/k_n)))} \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) \\ &= \lambda \left( \frac{b(\exp K_{\tau}(E_{n-k_n+1,n}))}{b(\exp K_{\tau}(\log(n/k_n)))} \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) (1 + o(1)). \end{aligned}$$

Introducing  $T_n^{(3,1)} = K_{\tau}(E_{n-k_n+1,n}) - K_{\tau}(\log(n/k_n))$  and applying Lemma 4 with  $g = K_{\tau}$  yield

$$\exp T_n^{(3,1)} = \exp \left( O_{\mathbb{P}}(1) \frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}} \right) \xrightarrow{P} 1, \quad (19)$$

since  $\tau \in [0, 1]$ . Therefore,  $b$  being regularly varying,

$$b(\exp K_{\tau}(E_{n-k_n+1,n})/b(\exp K_{\tau}(\log(n/k_n))) \xrightarrow{P} 1$$

as well, and consequently

$$\begin{aligned} T_n^{(3)} &= \lambda \left( \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) (1 + o(1)) \\ &= \lambda \left( \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \left( 1 + \frac{T_n^{(1)}}{\theta\sqrt{k_n}} \right) (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) (1 + o(1)) \\ &= \lambda \left( \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} (1 + o_{\mathbb{P}}(1)) - a_{\tau,\rho} \right) (1 + o(1)), \end{aligned}$$

from (17). Two situations occur. If  $\tau = 1$ , then, in view of Lemma 6,  $\theta_{n,3}(E_{n-k_n+1,n}) \xrightarrow{P} a_{1,\rho} = 1/(1-\rho)$ ,  $\mu_{1,1}(E_{n-k_n+1,n}) = 1$  and thus  $T_n^{(3)} \xrightarrow{P} 0$ . If  $\tau \in [0, 1)$ ,  $T_n^{(3)}$  can be rewritten as

$$T_n^{(3)} = \lambda \left( (T_n^{(3,2)} + T_n^{(3,3)})(1 + o_{\mathbb{P}}(1)) - 1 \right) (1 + o(1)),$$

where

$$T_n^{(3,2)} \stackrel{\text{def}}{=} \frac{\theta_{n,1}^{(1)}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} = 1 + \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \frac{\xi_n^{(1)}}{\sqrt{k_n}} = 1 + o_{\mathbb{P}}(1)$$

$$\begin{aligned}
|T_n^{(3,3)}| &\stackrel{\text{def}}{=} \frac{|\theta_{n,3}(E_{n-k_n+1,n}) - \theta_{n,1}^{(1)}(E_{n-k_n+1,n})|}{\mu_{1,\tau}(E_{n-k_n+1,n})} \\
&\leq \frac{\rho \theta_{n,1}^{(2)}(E_{n-k_n+1,n}) \mu_{2,\tau}(E_{n-k_n+1,n})}{2 \mu_{2,\tau}(E_{n-k_n+1,n}) \mu_{1,\tau}(E_{n-k_n+1,n})} \\
&\stackrel{d}{=} -\rho(\log(n/k_n))^{\tau-1}(1 + o_{\mathbb{P}}(1)) \left(1 + \frac{\sigma_{2,\tau}(E_{n-k_n+1,n}) \xi_n^{(2)}}{\mu_{2,\tau}(E_{n-k_n+1,n}) \sqrt{k_n}}\right) \\
&= O_{\mathbb{P}}(\log(n/k_n))^{\tau-1},
\end{aligned}$$

in view of Lemma 2, Lemma 5 and Lemma 6. Thus, for all  $\tau \in [0, 1)$ ,  $T_n^{(3)} \xrightarrow{P} 0$ . Taking (17) and (18) into account concludes the proof.  $\blacksquare$

**Proof of Proposition 4** – From (8), we have

$$\frac{1}{2} \log k_n + \log |b(\exp K_\tau(\log(n/k_n)))| \rightarrow \log |\lambda|$$

as  $n \rightarrow \infty$ , and since  $K_\tau(\log(n/k_n)) \rightarrow \infty$  as  $n \rightarrow \infty$ , it follows that

$$\frac{\log k_n}{2K_\tau(\log(n/k_n))} + \frac{\log |b(\exp K_\tau(\log(n/k_n)))|}{K_\tau(\log(n/k_n))} \rightarrow 0$$

as  $n \rightarrow \infty$ . Now,  $|b|$  is a regularly-varying function with index  $\rho$  and thus  $\log |b(x)|/\log x \rightarrow \rho$  for all  $x \rightarrow \infty$ , see [9], Proposition 1.3.6. As a consequence, we obtain

$$\frac{\log k_n}{K_\tau(\log(n/k_n))} \rightarrow -2\rho \tag{20}$$

as  $n \rightarrow \infty$ . Let us first remark that, if  $\tau = 1$  then (20) implies

$$\log k_n = \frac{2\rho}{2\rho - 1}(\log n)(1 + o(1)) = \frac{2\rho}{2\rho - 1}K_1(\log n)(1 + o(1))$$

and the conclusion follows. Otherwise, if  $\tau \in [0, 1)$ , condition (20) can be rewritten as

$$\frac{\log k_n}{\log n} \frac{\log n}{K_\tau(\log(n/k_n))} \rightarrow -2\rho. \tag{21}$$

Besides, since  $K_\tau$  is non-decreasing,

$$\frac{\log n}{K_\tau(\log(n/k_n))} \geq \frac{\log n}{K_\tau(\log n)} \rightarrow \infty$$

for all  $\tau \in [0, 1)$  and thus, in view of (21), necessarily  $\log k_n/\log n \rightarrow 0$  as  $n \rightarrow \infty$ . As a consequence,  $\log(n/k_n)$  is asymptotically equivalent to  $\log n$  and thus  $K_\tau(\log(n/k_n))$  is asymptotically equivalent to  $K_\tau(\log(n))$  as well. Replacing in (20), the conclusion follows.  $\blacksquare$

**Proof of Proposition 5** – Let us consider  $\tau \in [0, 1/2)$  and suppose that (8) holds with  $\lambda \neq 0$ . Following Proposition 4,  $\log(k_n) = -2\rho K_\tau(\log n)(1 + o(1))$  and thus  $\log(k_n)/\log(n) \rightarrow 0$  as  $n \rightarrow \infty$ . A first order Taylor expansion shows that there exists  $\eta_n \in [0, 1]$  such that

$$\begin{aligned}
\Delta_n &\stackrel{\text{def}}{=} \exp\{K_\tau(\log(n/k_n)) - K_\tau(\log n)\} = \exp\{-(\log k_n)K'_\tau(\log(n) - \eta_n \log(k_n))\} \\
&= \exp\{-(\log k_n)K'_\tau(\log n)(1 + o(1))\},
\end{aligned}$$

since  $K'_\tau$  is regularly-varying. As a consequence,

$$\Delta_n = \exp\{2\rho K_\tau(\log n)K'_\tau(\log n)(1 + o(1))\}$$

and thus  $\Delta_n \rightarrow 1$  if  $\tau \in [0, 1/2)$  or  $\Delta_n \rightarrow \exp(4\rho)$  if  $\tau = 1/2$ . Since  $b^2$  is regularly varying with index  $2\rho$  it follows that

$$\begin{aligned} ASB(\tau, \rho) &= b^2(\exp K_\tau(\log n)) \frac{b^2(\Delta_n \exp K_\tau(\log n))}{b^2(\exp K_\tau(\log n))} \\ &= c_{\tau, \rho} b^2(\exp K_\tau(\log n))(1 + o(1)), \end{aligned}$$

and the conclusion follows. ■

**Proof of Theorem 2** – From (7), one can infer that

$$\begin{aligned} \log \widehat{x}_{p_n} - \log x_{p_n} &= (\log(X_{n-k_n+1, n}) - \log \bar{F}^{\leftarrow}(k_n/n)) \\ &+ (\widehat{\theta}_n(k_n) - \theta)(K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))) \\ &+ \log \frac{\ell(\exp K_\tau(\log(n/k_n)))}{\ell(\exp K_\tau(\log(1/p_n)))} \\ &\stackrel{\text{def}}{=} Q_n^{(1)} + Q_n^{(2)} + Q_n^{(3)}. \end{aligned}$$

The three terms are studied separately. First, note that in view of  $(\mathbf{A}_1(\tau, \theta))$  and  $(\mathbf{A}_2(\rho))$ ,  $Q_n^{(1)}$  can be expanded as

$$\begin{aligned} Q_n^{(1)} &= \log H^{\leftarrow}(\exp K_\tau(E_{n-k_n+1, n})) - \log H^{\leftarrow}(\exp K_\tau(\log(n/k_n))) \\ &= \theta(K_\tau(E_{n-k_n+1, n}) - K_\tau(\log(n/k_n))) + \log \frac{\ell(\exp K_\tau(E_{n-k_n+1, n}))}{\ell(\exp K_\tau(\log(n/k_n)))} \\ &\stackrel{\text{def}}{=} \theta T_n^{(3,1)} + Q_n^{(1,2)}, \end{aligned}$$

where  $T_n^{(3,1)}$  is defined in the proof of Theorem 1 as

$$T_n^{(3,1)} = K_\tau(E_{n-k_n+1, n}) - K_\tau(\log(n/k_n)) = O_{\mathbb{P}}(1) \frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}}, \quad (22)$$

in view of (19). Moreover,  $Q_n^{(1,2)} \stackrel{\text{def}}{=} \log \ell(\lambda_n x_n) - \log \ell(x_n)$ , where  $x_n = \exp K_\tau(\log(n/k_n)) \rightarrow \infty$  and  $\lambda_n = \exp T_n^{(3,1)} \xrightarrow{P} 1$ . Thus, from  $(\mathbf{A}_2(\rho))$  we have

$$\begin{aligned} Q_n^{(1,2)} &= b(\exp K_\tau(\log(n/k_n))) K_\rho(\lambda_n)(1 + o_{\mathbb{P}}(1)) \\ &= b(\exp K_\tau(\log(n/k_n))) \log(\lambda_n)(1 + o_{\mathbb{P}}(1)) \\ &= O_{\mathbb{P}}(1) b(\exp K_\tau(\log(n/k_n))) \frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}}, \end{aligned}$$

in view of (22). Since  $b(x) \rightarrow 0$  as  $x \rightarrow \infty$ , it follows that

$$Q_n^{(1,2)} = o_{\mathbb{P}}\left(\frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}}\right),$$

entailing

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} Q_n^{(1)} = O_{\mathbb{P}}\left(\frac{(\log(n/k_n))^{\tau-1}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))}\right) = o_{\mathbb{P}}(1),$$

from (10). Now, concerning the second term, Theorem 1 entails that

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} Q_n^{(2)} = \sqrt{k_n} \left( \hat{\theta}_n(k_n) - \theta \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

Finally,  $Q_n^{(3)} = \log \ell(x_n^*) - \log \ell(\lambda_n^* x_n^*)$  where  $\lambda_n^* = \exp[K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))] \geq 1$  in view of (10) and  $x_n^* = \exp K_\tau(\log(n/k_n)) \rightarrow \infty$ . Thus, Lemma 1 entails

$$\frac{\sqrt{k_n} Q_n^{(3)}}{\log \lambda_n^*} \sim -\sqrt{k_n} b(x_n^*) \frac{K_\rho(\lambda_n^*)}{\log \lambda_n^*} = o\left(\frac{K_\rho(\lambda_n^*)}{\log \lambda_n^*}\right),$$

since  $\sqrt{k_n} b(x_n^*) = \sqrt{k_n} b(\exp K_\tau(\log(n/k_n))) \rightarrow 0$ . Taking account of the inequality  $K_\rho(x) \leq \log x$  for all  $x \geq 1$  yields

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} Q_n^{(3)} = o(1).$$

Combining the above results, Theorem 2 follows. ■

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## Appendix: Proof of auxiliary results

**Proof of Lemma 1** – From  $(\mathbf{A}_1(\tau, \theta))$  and  $(\mathbf{A}_2(\rho))$ , it is easy to infer that, for any constant  $\tilde{C} > 0$ , we have

$$\begin{aligned} \frac{1}{\tilde{C}b(x)} \left( \frac{H^\leftarrow(\lambda x) - H^\leftarrow(x)}{\theta H^\leftarrow(x)(1 + b(x)/\theta)} - \frac{\lambda^\theta - 1}{\theta} \right) &= \frac{\lambda^\theta}{\tilde{C}\theta} K_\rho(\lambda) - \frac{1}{\tilde{C}\theta} \frac{\lambda^\theta - 1}{\theta} + o(1) \\ &= \frac{\theta + \rho}{\tilde{C}\theta} \frac{1}{\rho} [K_{\theta+\rho}(\lambda) - K_\theta(\lambda)] + o(1). \end{aligned}$$

Then, choosing  $\tilde{C}$  such that  $(\theta + \rho)/(\tilde{C}\theta) = 1$ , a direct application of Lemma 5.2 in [15] yields, for any  $\varepsilon > 0$  and  $\lambda \geq 1$ ,

$$\begin{aligned} &\min(1, \lambda^{-\rho-\varepsilon}) \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta}b^2(x) [K_\rho(\lambda) - K_{-\theta}(\lambda)] \right| \\ &\leq \varepsilon \tilde{C}\theta |b(x)| |1 + b(x)/\theta| \min(1, \lambda^{-\rho-\varepsilon}) [\lambda^{-\theta} + 1 + 2\lambda^{\rho+\varepsilon}] \\ &\leq 4\varepsilon \tilde{C}\theta |b(x)| \min(1, \lambda^{-\rho-\varepsilon}) [1 + \lambda^{\rho+\varepsilon}] \\ &\leq 8\varepsilon \tilde{C}\theta |b(x)| \end{aligned}$$

for  $x$  large enough. Moreover, letting  $0 < \varepsilon < -\rho$  yields

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta}b^2(x) [K_\rho(\lambda) - K_{-\theta}(\lambda)] \right| = o(b(x)). \quad (23)$$

Besides,  $K_\rho(\lambda) - K_{-\theta}(\lambda)$  is bounded when  $\rho < 0$ , and therefore (23) can be simplified as

$$\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) \right| = o(b(x)), \text{ as } x \rightarrow \infty,$$

and the conclusion follows. ■

**Proof of Lemma 2** – (i) Let us consider for  $t > 1$  and  $q \in \mathbb{N} \setminus \{0\}$ ,

$$Q_q(t) = \int_0^\infty \left( \frac{K_\tau(x+t) - K_\tau(t)}{K'_\tau(t)} \right)^q e^{-x} dx.$$

There exists  $\eta \in (0, 1)$  such that

$$\left| \frac{K_\tau(x+t) - K_\tau(t)}{xK'_\tau(t)} \right| = \left( 1 + \frac{\eta x}{t} \right)^{\tau-1} \leq 1.$$

Thus, Lebesgue Theorem implies that

$$\lim_{t \rightarrow \infty} Q_q(t) = \int_0^\infty \lim_{t \rightarrow \infty} \left( 1 + \frac{\eta x}{t} \right)^{q(\tau-1)} x^q e^{-x} dx = \int_0^\infty x^q e^{-x} dx = q!$$

which concludes the first part of the proof.

(ii) is a straightforward consequence of (i).

(iii) We have

$$\begin{aligned} \mu'_{1,\tau}(t) &= \int_0^\infty (K'_\tau(x+t) - K'_\tau(t)) e^{-x} dx \\ &= \int_0^\infty K'_\tau(x+t) e^{-x} dx - K'_\tau(t) \\ &= \int_0^\infty K_\tau(x+t) e^{-x} dx - K_\tau(t) - K'_\tau(t) \\ &= \mu_{1,\tau}(t) - t^{\tau-1}. \end{aligned}$$

Finally, (i) states that  $t^{\tau-1}/\mu_{1,\tau}(t) \rightarrow 1$  as  $t \rightarrow \infty$  which entails  $\mu'_{1,\tau}(t)/\mu_{1,\tau}(t) \rightarrow 0$  as  $t \rightarrow \infty$ . ■

**Proof of Lemma 3** – Recall that

$$\begin{aligned} \hat{\theta}_n &= \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \\ &\stackrel{d}{=} \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left( \frac{H^{\leftarrow}(\exp K_\tau(E_{n-i+1,n}))}{H^{\leftarrow}(\exp K_\tau(E_{n-k_n+1,n}))} \right), \end{aligned}$$

where  $E_{1,n}, \dots, E_{n,n}$  are ordered statistics generated by  $n$  independent standard exponential random variables. The Rényi representation of the Exp(1) ordered statistics (see [1], p. 72) yields

$$\{E_{n-i+1,n}\}_{i=1,\dots,k_n-1} \stackrel{d}{=} \{F_{k_n-i,k_n-1} + E_{n-k_n+1,n}\}_{i=1,\dots,k_n-1}, \quad (24)$$

where  $\{F_{1,k_n-1}, \dots, F_{k_n-1,k_n-1}\}$  are ordered statistics independent from  $E_{n-k_n+1,n}$  and generated by  $k_n - 1$  independent standard exponential variables  $\{F_1, \dots, F_{k_n-1}\}$ . We thus have

$$\hat{\theta}_n(k_n) \stackrel{d}{=} \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left( \frac{H^{\leftarrow}(\exp K_\tau(F_{k_n-i,k_n-1} + E_{n-k_n+1,n}))}{H^{\leftarrow}(\exp K_\tau(E_{n-k_n+1,n}))} \right)$$

$$\begin{aligned}
&\stackrel{d}{=} \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left( \frac{H^{\leftarrow}(\exp K_\tau(F_i + E_{n-k_n+1,n}))}{H^{\leftarrow}(\exp K_\tau(E_{n-k_n+1,n}))} \right) \\
&\stackrel{d}{=} \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \left( \theta_{n,1}^{(1)}(E_{n-k_n+1,n}) + \theta_{n,2}(E_{n-k_n+1,n}) \right)
\end{aligned}$$

in view of  $(\mathbf{A}_1(\tau, \theta))$  and the conclusion follows.  $\blacksquare$

**Proof of Lemma 4** – A first order expansion of the function  $g$  leads to,

$$\sqrt{k_n}(g(E_{n-k_n+1,n}) - g(\log n/k_n)) = \sqrt{k_n}(E_{n-k_n+1,n} - \log(n/k_n))g'(\tilde{\eta}_n),$$

with  $\tilde{\eta}_n \in [\min(E_{n-k_n+1,n}, \log(n/k_n)), \max(E_{n-k_n+1,n}, \log(n/k_n))]$ . Now, Lemma 1 in [25] shows that  $\sqrt{k_n}(E_{n-k_n+1,n} - \log(n/k_n)) \xrightarrow{d} \mathcal{N}(0, 1)$  which implies that  $\tilde{\eta}_n = \log(n/k_n)(1 + o_{\mathbb{P}}(1))$  and the result follows.  $\blacksquare$

**Proof of Lemma 5** – Let us introduce for all  $t \geq 1$  and  $q \in \mathbb{N} \setminus \{0\}$ ,

$$S_n^{(q)}(t) = \frac{(k_n - 1)^{1/2}}{\sigma_{q,\tau}(t)} (\theta_{n,1}^{(q)}(t) - \mu_{q,\tau}(t)) = \frac{(k_n - 1)^{-1/2}}{\sigma_{q,\tau}(t)} \sum_{i=1}^{k_n-1} Y_i^{(q)}(t),$$

where  $Y_i^{(q)}(t) \stackrel{\text{def}}{=} (K_\tau(F_i + t) - K_\tau(t))^q - \mu_{q,\tau}(t)$ ,  $i = 1, \dots, k_n - 1$  are centered, independent and identically distributed random variables with variance  $\sigma_{q,\tau}^2(t)$ . Clearly, in view of the Central Limit Theorem, for all  $t \geq 1$  and  $q \in \mathbb{N} \setminus \{0\}$ ,  $S_n^{(q)}(t)$  converges in distribution to a standard Gaussian distribution. Our goal is to prove that, for all  $x \in \mathbb{R}$  and  $q \in \mathbb{N} \setminus \{0\}$ ,

$$\mathbb{P}(S_n^{(q)}(E_{n-k_n+1,n}) \leq x) \rightarrow \Phi(x) \text{ as } n \rightarrow \infty,$$

where  $\Phi$  is the cumulative distribution function of the standard Gaussian distribution. Lemma 2(i) implies that for all  $\varepsilon \in (0, 1)$ , and  $r \in \mathbb{N} \setminus \{0\}$ , there exists  $T_\varepsilon \geq 1$  such that for all  $t \geq T_\varepsilon$ ,

$$(1 - \varepsilon) r! t^{r(\tau-1)} \leq \mu_{r,\tau}(t) \leq (1 + \varepsilon) r! t^{r(\tau-1)}. \quad (25)$$

Furthermore, for  $x \in \mathbb{R}$ ,

$$\begin{aligned}
\mathbb{P}(S_n^{(q)}(E_{n-k_n+1,n}) \leq x) - \Phi(x) &= \int_0^{T_\varepsilon} (\mathbb{P}(S_n^{(q)}(t) \leq x) - \Phi(x)) h_n(t) dt \\
&+ \int_{T_\varepsilon}^\infty (\mathbb{P}(S_n^{(q)}(t) \leq x) - \Phi(x)) h_n(t) dt \stackrel{\text{def}}{=} A_n + B_n,
\end{aligned}$$

where  $h_n$  is the density of the random variable  $E_{n-k_n+1,n}$ . First, let us focus on the term  $A_n$ . We have,

$$|A_n| \leq 2\mathbb{P}(E_{n-k_n+1,n} \leq T_\varepsilon).$$

Since  $E_{n-k_n+1,n}/\log(n/k) \xrightarrow{P} 1$  (see [25], Lemma 1), it is easy to show that  $A_n \rightarrow 0$ . Now, let us consider the term  $B_n$ . For all  $t \geq T_\varepsilon$ ,

$$\begin{aligned}
\mathbb{E}(|Y_1^{(q)}(t)|^3) &\leq \mathbb{E}((K_\tau(F_1 + t) - K_\tau(t))^q + \mu_{q,\tau}(t))^3 \\
&= \mu_{3q,\tau}(t) + 3\mu_{q,\tau}(t)\mu_{2q,\tau}(t) + 4\mu_{q,\tau}^3(t) \\
&\leq C_1(\varepsilon) t^{3q(\tau-1)} < \infty,
\end{aligned}$$

from (25). Here, and in the following,  $C_1(\varepsilon)$ ,  $C_2$ ,  $C_3(\varepsilon)$  and  $C_4(\varepsilon)$  are positive constants independent of  $t$ . Thus, from Berry-Esséen's inequality (see [35], Theorem 3), we have:

$$\sup_x |\mathbb{P}(S_n^{(q)}(t) \leq x) - \Phi(x)| \leq C_2 L_n \quad \text{with} \quad L_n = \frac{(k_n - 1)^{-1/2}}{\sigma_{q,\tau}^3(t)} \mathbb{E}(|Y_1^{(q)}(t)|^3).$$

From (25), since  $t \geq T_\varepsilon$ ,

$$\sigma_{q,\tau}^2(t) = \mu_{2q,\tau}(t) - \mu_{q,\tau}^2(t) \geq C_3(\varepsilon) t^{2q(\tau-1)}.$$

Thus,  $L_n \leq C_4(\varepsilon)(k_n - 1)^{-1/2}$  and therefore

$$|B_n| \leq C_2 C_4(\varepsilon) (k_n - 1)^{-1/2} \mathbb{P}(E_{n-k_n+1,n} \geq T_\varepsilon) \leq C_2 C_4(\varepsilon) (k_n - 1)^{-1/2} \rightarrow 0,$$

which concludes the proof.  $\blacksquare$

**Proof of Lemma 6** – Let us consider the random variables  $x_n = \exp[K_\tau(E_{n-k_n+1,n})]$  and  $\lambda_{i,n} = \exp[K_\tau(F_i + E_{n-k_n+1,n}) - K_\tau(E_{n-k_n+1,n})]$ ,  $i = 1, \dots, k_n - 1$ . It is clear that  $x_n \xrightarrow{P} \infty$  in view of Lemma 1 in [25] and  $\lambda_{i,n} \geq 1$ . Thus, letting

$$\theta_{n,3}(E_{n-k_n+1,n}) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} K_\rho[\exp(K_\tau(F_i + E_{n-k_n+1,n}) - K_\tau(E_{n-k_n+1,n}))],$$

Lemma 1 entails

$$\theta_{n,2}(E_{n-k_n+1,n}) \stackrel{d}{=} b(\exp K_\tau(E_{n-k_n+1,n})) \theta_{n,3}(E_{n-k_n+1,n}) (1 + o_{\mathbb{P}}(1)).$$

Since  $|K_\rho(\exp u) - u| \leq -\rho u^2/2$  for all  $u \geq 0$ , we have

$$\left| \theta_{n,3}(E_{n-k_n+1,n}) - \theta_{n,1}^{(1)}(E_{n-k_n+1,n}) \right| \leq -\frac{\rho}{2} \theta_{n,1}^{(2)}(E_{n-k_n+1,n}).$$

Moreover, if  $\tau = 1$ , then

$$\theta_{n,3}(E_{n-k_n+1,n}) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} K_\rho(\exp F_i) \xrightarrow{P} \int_0^{+\infty} K_\rho(\exp u) \exp(-u) du = \frac{1}{1 - \rho},$$

in view of the law of large numbers, and the conclusion follows.  $\blacksquare$

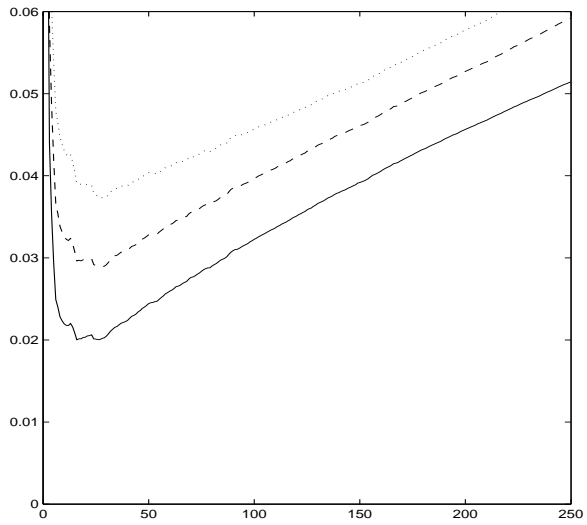
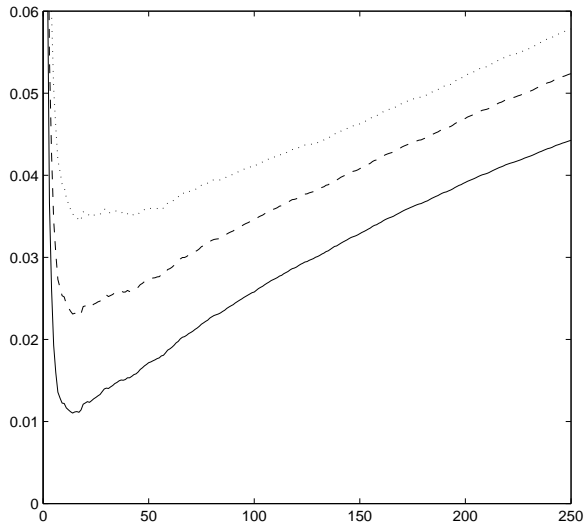


Figure 1: Empirical squared bias as a function of  $k$  obtained with  $\widehat{\theta}_n(k_n)$  computed on 500 samples of size 500 from  $F_{1/2,\tau,\rho}$ . Up:  $\rho = -1/2$ , down:  $\rho = -1/4$ , solid line:  $\tau = 1$ , dashed line:  $\tau = 1/2$ , dotted line:  $\tau = 0$ .

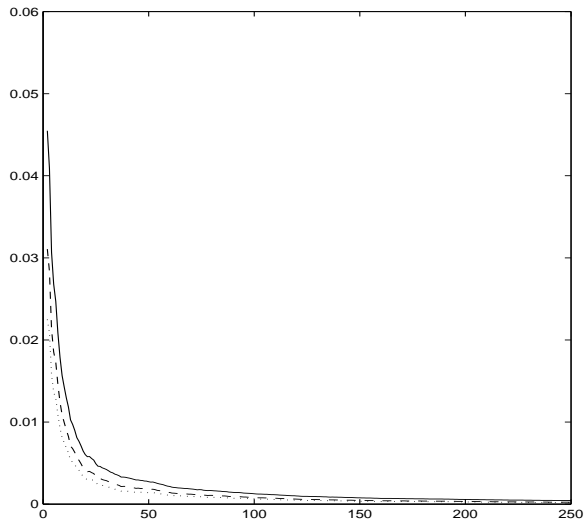
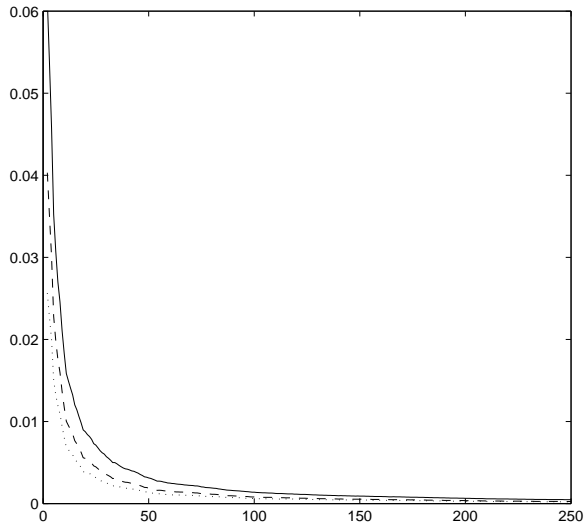


Figure 2: Empirical variance as a function of  $k$  obtained with  $\widehat{\theta}_n(k_n)$  computed on 500 samples of size 500 from  $F_{1/2,\tau,\rho}$ . Up:  $\rho = -1/2$ , down:  $\rho = -1/4$ , solid line:  $\tau = 1$ , dashed line:  $\tau = 1/2$ , dotted line:  $\tau = 0$ .