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Weibull tail-distributions revisited: a new look at some tail estimators

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Abstract

In this paper, we propose to include Weibull tail-distributions in a more general family of distributions. In particular, the considered model also encompasses the whole Fréchet maximum domain of attraction as well as log-Weibull tail-distributions. The asymptotic normality of some tail estimators based on the log-spacings between the largest order statistics is established in a unified way within the considered family. This result permits to understand the similarity between most estimators of the Weibull tail-coefficient and the Hill estimator. Some different asymptotic properties, in terms of bias, rate of convergence, are also highlighted.

AMS Subject Classifications: 62G05, 62G20, 62G30.

Keywords: Weibull tail-distributions, extreme quantile, maximum domain of attraction, asymptotic normality.

1 Motivations

Weibull tail-distributions encompass a variety of light-tailed distributions, \textit{i.e.} distributions in the Gumbel maximum domain of attraction, see [20] for further details. Weibull tail-distributions include for instance Weibull, Gaussian, gamma and logistic distributions. Let us recall that a cumulative distribution function \(F\) has a Weibull tail if its associated survival function \(\bar{F} = 1 - F\) satisfies the following property: There exists \(\theta > 0\) such that for all \(\lambda > 0\),

\[
\lim_{t \to \infty} \frac{\log \bar{F}(\lambda t)}{\log \bar{F}(t)} = \lambda^{1/\theta}. \tag{1}
\]

The parameter \(\theta\) is called the Weibull tail-coefficient. We refer to [7] for a general account on Weibull tail-distributions and to [6] for an application to the modeling of large claims in non-life insurance. Dedicated methods have been proposed to estimate the Weibull tail-coefficient since the relevant information is only contained in the extreme upper part of the sample denoted hereafter by \(X_1, \ldots, X_n\). A first direction was investigated in [8] where an estimator based on the record values is proposed. Another family of approaches [3, 4, 10, 13] consists of using the \(k_n\) upper order statistics \(X_{n-k_n+1,n} \leq \ldots \leq X_{n,n}\) where \((k_n)\) is an intermediate sequence of integers \textit{i.e.} such that

\[
\lim_{n \to \infty} k_n = \infty \quad \text{and} \quad \lim_{n \to \infty} k_n/n = 0. \tag{2}
\]
More specifically, most recent estimators are based on the log-spacings between the \( k_n \) upper order statistics [7, 11, 22, 23, 25, 26, 27]. All these estimators are thus similar to the Hill statistics [34] defined as

\[
H_n(k_n) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}).
\]

As an example, all three estimators proposed in [22] are proportional to \( H_n(k_n) \). This similarity may be surprising since \( H_n(k_n) \) is dedicated to the estimation of the tail index \( \gamma \) for heavy-tailed distribution i.e. such that

\[
\lim_{t \to \infty} \frac{\tilde{F}(\lambda t)}{F(t)} = \lambda^{-1/\gamma},
\]

for all \( \lambda > 0 \). This property characterizes distributions belonging to the Fréchet maximum domain of attraction and sometimes called Pareto-type distributions.

The main goal of this work is therefore to explain why statistics based on log-spacings could be efficient in estimating tail parameters of both Weibull-tail and Pareto-type distributions. To this end, we introduce a family of distributions, indexed by two parameters \( \tau \in [0, 1] \) and \( \theta > 0 \), which includes these two type of distributions. The first parameter \( \tau \) allows to represent a large panel of distribution tails ranging from Weibull-type tails (\( \tau = 0 \)) to Pareto-type tails (\( \tau = 1 \)). The second parameter \( \theta \) is the parameter to be estimated. It coincides with the Weibull tail-coefficient when \( \tau = 0 \) and with the tail index when \( \tau = 1 \).

An estimator \( \bar{\theta}_n(k_n) \) of \( \theta \) is then introduced for the new family of distributions and an estimator of extreme quantiles is derived. The asymptotic normality of these estimators is established in Section 3 in a unified way and illustrated on some simulated data in Section 4. Some concluding remarks are given in Section 5. Proofs are postponed to Section 6.

## 2 Model and estimators

### 2.1 Definition and first properties

Let us consider the family of survival distribution functions defined as

\( (A_1(\tau, \theta)) \) \( \bar{F}(x) = \exp(-K^{-}(\log H(x))) \) for \( x \geq x_* \) with \( x_* > 0 \) and

- \( K_\tau(x) = \int_1^x u^{\tau-1} du \) where \( \tau \in [0, 1] \),
- \( H \) an increasing function such that \( H^{-}(t) = \inf\{x, \ H(x) \geq t\} = t^\theta \ell(t) \), where \( \theta > 0 \) and \( \ell \) is a slowly varying function i.e. \( \ell(\lambda x)/\ell(x) \to 1 \) as \( x \to \infty \) for all \( \lambda \geq 1 \).

The function \( H^{-} \) is the so-called generalized inverse of \( H \). Note that \( K^{-}_\tau \) coincides with the classical inverse since \( K_\tau \) is continuous. The expansion \( H^{-}(t) = t^\theta \ell(t) \) is equivalent to supposing that \( H^{-} \) is regularly varying at infinity with index \( \theta \). This property is denoted by \( H^{-} \in R_\theta \), see [9] for more details on regular variations theory. Let us first highlight that the tail heaviness of \( \bar{F} \) is mainly driven by \( \tau \in [0, 1] \) and secondarily by \( \theta > 0 \):

**Proposition 1** Let \( \bar{F}_{\tau_1, \theta_1} \) and \( \bar{F}_{\tau_2, \theta_2} \) be two survival distribution functions satisfying respectively \( (A_1(\tau_1, \theta_1)) \) and \( (A_1(\tau_2, \theta_2)) \).

(i) If \( \tau_1 < \tau_2 \) then \( \bar{F}_{\tau_1, \theta_1}(x)/\bar{F}_{\tau_2, \theta_2}(x) \to 0 \) as \( x \to \infty \) for all \( (\theta_1, \theta_2) \in (0, \infty)^2 \).

(ii) If \( \tau_1 = \tau_2 = \tau \) and \( \theta_1 < \theta_2 \) then \( \bar{F}_{\tau, \theta_1}(x)/\bar{F}_{\tau, \theta_2}(x) \to 0 \) as \( x \to \infty \).

2
Thus, the larger is $\tau$, the heavier is the tail. Let us consider the two extremal cases $\tau = 0$ and $\tau = 1$. Clearly, under $(A_1(0, \theta))$, $\tilde{F}(x) = \exp(-H(x))$ is the survival function of a Weibull-tail distribution, see (1). At the opposite, $(A_1(1, \theta))$ entails $\tilde{F}(x) = e^{1/H(x)} = x^{-1/\theta}h(x)$ where $\hat{\ell}$ is a slowly varying function. As a consequence, $F$ belongs to the Fréchet maximum domain of attraction and $\theta$ coincides with the tail index. In view of the above remarks, intermediate values of $\tau \in (0, 1)$ correspond to distribution tails lighter than Pareto tails but heavier than Weibull tails. Indeed, we have $\tilde{F}(x) = \exp(-h(x))$ with $h(x) \sim ((\tau/\theta) \log x)^{1/\tau}$ and thus $h(x)/x^\beta \to 0$ for all $\beta > 0$ while $h(x)/\log(x) \to \infty$ as $x \to \infty$, this property characterizing an “exponential type” distribution, see [32]. The next proposition provides a more precise characterization while examples are provided in Paragraph 2.2.

Proposition 2

(i) $F$ verifies $(A_1(0, \theta))$ if and only if $F$ is a Weibull-tail distribution function with Weibull tail-coefficient $\theta$.

(ii) If $F$ verifies $(A_1(\tau, \theta))$, $\tau \in [0, 1)$ and if $H$ is twice differentiable then $F$ belongs to the Gumbel maximum domain of attraction.

(iii) $F$ verifies $(A_1(1, \theta))$ if and only if $F$ is in the Fréchet maximum domain of attraction with tail-index $\theta$.

2.2 Examples

In view of Proposition 2(i), Gaussian, gamma, Weibull, Benktander II, logistic and extreme-value distributions all verify $(A_1(0, \theta))$ since they are examples of Weibull tail-distributions (see [23], Table 1). Examples of distributions verifying $(A_1(\tau, \theta))$ with $\tau \in (0, 1)$ include some log-Weibull tail-distributions. Let us recall that a random variable $Y$ is distributed from a log-Weibull tail-distribution if $\log(Y)$ follows a Weibull tail-distribution.

Proposition 3 Suppose that $F$ verifies $(A_1(0, \theta))$ with $\theta \in (0, 1]$. If, moreover, the slowly-varying function $\ell$ is differentiable and $\ell(t) \to \ell_\infty > 0$ as $t \to \infty$ then $F(\log .)$ verifies $(A_1(\theta, \theta \ell_\infty))$.

As an example, the standard log-normal distribution can be looked at as a log-Weibull tail-distribution and thus verifies $(A_1(1/2, \sqrt{2}/2))$. Similarly, the gamma distribution verifies $(A_1(0, 1))$ and the log-gamma distribution belongs to the Fréchet maximum domain of attraction, see for instance [16], Table 3.4.2. Finally, other examples of distributions satisfying $(A_1(1, \theta))$ can be found in the above mentioned table.

2.3 Definition of the estimators

Denoting by $(k_n)$ an intermediate sequence of integers (see (2)), the following estimator of $\theta$ is considered:

$$\tilde{\theta}_n(k_n) = 1 - \frac{1}{\mu_1(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})),\quad (4)$$

with, for all $t > 0$ and $q \in \mathbb{N} \setminus \{0\}$,

$$\mu_{q,t}(s) = \int_0^\infty (K_\tau(x + t) - K_\tau(t))^q e^{-x} dx.$$
A crucial point is that the estimator (4) essentially consists in averaging the log-spacings between the upper-order statistics. Even more strongly, \( \hat{\theta}_n(k_n) \) only differs from the Hill statistics (3) by a non-random normalizing sequence: \( \hat{\theta}_n(k_n) = H_n(k_n)/\mu_{1,\tau}(\log(n/k_n)) \). This similarity can be intuitively understood by studying the log-spacing between two quantiles \( x_u \) and \( x_v \) of \( F \), with \( 0 < u < v \leq 1 \). Under \((A_1(\tau, \theta))\) we have

\[
\log x_u - \log x_v = \theta (K_\tau(-\log u) - K_\tau(-\log v)) + \log \left( \frac{\ell(\exp K_\tau(-\log u))}{\ell(\exp K_\tau(-\log v))} \right). \tag{5}
\]

Now, since \( \ell \) is a slowly-varying function, if the orders \( u \) and \( v \) of the quantiles are small enough, the second term can be neglected in the right-hand side of (5) to obtain

\[
\log x_u - \log x_v \approx \theta (K_\tau(-\log u) - K_\tau(-\log v)), \tag{6}
\]

which shows that log-spacings are approximately proportional to \( \theta \). Since this key property holds for all \( \tau \in [0,1] \), it is thus shared by Pareto-type, Weibull tail and log-Weibull tail-distributions. Note that this property can be checked graphically on a sample by drawing a quantile-quantile plot. It consists in plotting the pairs \((K_\tau(\log(n/2)), \log(X_{n-i+1,n}))\) for \( i = 1, \ldots, k_n \). From (6), the graph should be approximately linear. Following the same ideas, an estimator of the extreme quantile \( x_{p_n} \) can be deduced from (4) by:

\[
\hat{x}_{p_n} = X_{n-k_n+1,n} \exp \left( \hat{\theta}_n(k_n) (K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))) \right). \tag{7}
\]

Recall that an extreme quantile \( x_{p_n} \) of order \( p_n \) is defined by \( x_{p_n} = F^-(p_n) \) with \( p_n \rightarrow 0 \) as \( n \rightarrow \infty \). For instance, if \( np_n \rightarrow 0 \) then \( x_{p_n} \) is larger than the maximum observation \( X_{n,n} \) of the sample (with probability tending to one). This requires to extrapolate sample results to areas where no data are observed and occurs in reliability [14], hydrology [36], finance [16],...  

### 3 Asymptotic properties

We show in the next paragraph that the asymptotic normality of \( \hat{\theta}_n(k_n) \) and \( \hat{x}_{p_n} \) can be established for all \( \tau \in [0,1] \) in a unified way. In this sense, the asymptotic behavior of these estimators is more a consequence of the log-spacings property than of a tail behavior (which can be exponential as well as polynomial). Paragraphs 3.2 and 3.3 illustrate our general result on the two extremal cases \( \tau = 0 \) and \( \tau = 1 \).

#### 3.1 Main results

To establish the asymptotic normality of \( \hat{\theta}_n(k_n) \), a second-order condition on \( \ell \) is necessary:

\((A_2(\rho))\) There exist \( \rho < 0 \) and \( b(x) \rightarrow 0 \) such that uniformly locally on \( \lambda \geq \lambda_0 > 0 \)

\[
\log \left( \frac{\ell(\lambda x)}{\ell(x)} \right) \sim b(x)K_\rho(\lambda), \text{ when } x \rightarrow \infty.
\]

It can be shown that necessarily \( |b| \in R_\rho \) (see [24]). The second order parameter \( \rho < 0 \) tunes the rate of convergence of \( \ell(\lambda x)/\ell(x) \) to 1. The closer is \( \rho \) to 0, the slower is the convergence. Condition \((A_2(\rho))\) is the cornerstone in all the proofs of asymptotic normality for extreme value estimators. It is used in [5, 33, 34] to prove the asymptotic normality of several estimators of the extreme value index.
Theorem 1 Suppose that \((A_1(\tau, \theta))\) and \((A_2(\rho))\) hold. Let \((k_n)\) be an intermediate sequence such that
\[
\sqrt{k_n} b(\exp K_\tau(\log(n/k_n))) \to \lambda.
\] (8)
Then, introducing \(a_{\tau, \rho} = 1\) if \(\tau \in [0, 1)\) and \(a_{1, \rho} = 1/(1 - \rho)\), we have
\[
\sqrt{k_n} \left(\hat{\theta}_n(k_n) - \theta - a_{\tau, \rho} b(\exp K_\tau(\log(n/k_n)))\right) \overset{d}{\to} \mathcal{N}(0, \theta^2). \tag{9}
\]

It appears that the asymptotic variance of \(\hat{\theta}_n(k_n)\) given by \(\mathcal{A}V = \theta^2/k_n\) is independent of \(\tau\). In particular, it remains constant whatever the maximum domain of attraction of \(F\). The asymptotic squared bias is given by \(\mathcal{ASB}(\tau, \rho) = a_{\tau, \rho}^2 \rho^2(\exp K_\tau(\log(n/k_n)))\). If \(b^2\) is ultimately decreasing, then \(\mathcal{ASB}\) is a decreasing function of \(\tau \in [0, 1)\) with a jump at \(\tau = 1\). These remarks are illustrated on simulated data in Section 4. The next result allows us to establish the rate of convergence of \(\hat{\theta}_n(k_n)\) to \(\theta\) in (9).

Proposition 4 Condition (8) with \(\lambda \neq 0\) implies \(\log(k_n) = -2pa_{\tau, 2}\rho K_\tau(\log n)(1 + o(1))\). The rate of convergence is thus of order \(\exp(-pa_{\tau, 2}\rho K_\tau(\log n)(1 + o(1)))\). A geometrical rate of convergence is obtained only in the Fréchet maximum domain of attraction, \(\tau = 1\) yields \(\sqrt{k_n} = n^{-\rho/(1-2\rho)+o(1)}\) which is consistent with the conclusions of [31]. Weibull tail-distributions give rise to logarithmic rates of convergence, \(\tau = 0\) yields \(\sqrt{k_n} = (\log n)^{-\rho+o(1)}\) which is consistent with the results of [22]. More generally, the heavier is the tail, the better the rate of convergence is. The next result provides an extension of Statement 1 in [2], which was initially proved only for Weibull tail-distributions \((\tau = 0)\).

Proposition 5 Suppose condition (8) holds with \(\lambda \neq 0\). If \(\tau \in [0, 1/2]\) then
\[
\mathcal{ASB}(\tau, \rho) = c_{\tau, \rho} b^2(\exp K_\tau(\log n))(1 + o(1)) ,
\]
where \(c_{\tau, \rho} = 1\) if \(\tau \in [0, 1/2]\) and \(c_{1/2, \rho} = \exp(8\rho^2)\).

It follows that, when \(\tau \in [0, 1/2]\), the first order of the asymptotic bias is asymptotically independent of \(k_n\). As a consequence, the asymptotic mean-squared error defined as \(\mathcal{ASB}(\tau, \rho) + \mathcal{A}V\) is eventually decreasing with respect to \(k_n\). This remark, already made in [2] in the particular case \(\tau = 0\), is only of theoretical interest. Indeed, in finite sample situations, condition (8) does not hold and the empirical mean-squared error is a convex function of \(k_n\), see for instance [22]. Now, the asymptotic normality of the extreme quantile estimator (7) can be deduced from Theorem 1:

Theorem 2 Suppose the assumptions of Theorem 1 hold with \(\lambda = 0\). If, moreover,
\[
(\log(n/k_n))^{1-\tau}(K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))) \to \infty \tag{10}
\]
then,
\[
\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} \left(\frac{x_{p_n}}{x_{p_n}} - 1\right) \overset{d}{\to} \mathcal{N}(0, \theta^2).
\]

Let us now focus on the two particular cases \(\tau = 0\) (Weibull tail-distributions) and \(\tau = 1\) (Fréchet maximum domain of attraction).
3.2 Application to Weibull tail-distributions

If $\tau = 0$, the estimator (4) coincides with $\hat{\theta}_n^{(1)}$ introduced in [22], and

$$\hat{x}_{p_n} = X_{n-k_n+1,n} \left( \frac{\log(1/p_n)}{\log(n/k_n)} \right)^{\hat{\theta}_n(k_n)}$$

is the estimator proposed in [21]. As a consequence of Theorem 1 and Theorem 2, we obtain:

**Corollary 1** Suppose that $(A_1(0, \theta))$ and $(A_2(\rho))$ hold. Let $(k_n)$ be an intermediate sequence such that $\sqrt{k_n} b(\log(n/k_n)) \rightarrow 0$. Then,

$$\sqrt{k_n} \left( \hat{\theta}_n(k_n) - \theta \right) \xrightarrow{d} N(0, \theta^2).$$

If, moreover

$$\log(n/k_n) (\log(1/p_n) - \log(n/k_n)) \rightarrow \infty$$

then,

$$\sqrt{k_n} \left( \frac{\hat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} N(0, \theta^2).$$

This result is very similar to Corollary 3.1 in [22] except that condition (11) is weaker than the one used in the above mentioned paper. Let us also note that estimators $\hat{\theta}_n^{(2)}$ and $\hat{\theta}_n^{(3)}$ in [22] can be respectively deduced from $\hat{\theta}_n$ by approximating $\mu_{1,0}$ by a Riemann’s sum or using the first order approximation $\mu_{1,0}(t) \sim 1/t$ as $t \rightarrow \infty$ given in Lemma 2(i).

3.3 Application to the Fréchet maximum domain of attraction

Letting $\tau = 1$ and remarking that $\mu_{q,1}(t) = q!$ for all $t > 0$ and $q \in \mathbb{N} \setminus \{0\}$, the estimator (4) coincides with (3) which is the Hill estimator [34] of the tail index. Besides,

$$\hat{x}_{p_n} = X_{n-k_n+1,n} \left( \frac{k_n}{np_n} \right)^{\hat{\theta}_n(k_n)}$$

is the Weissman estimator [37]. A straightforward application of the above theorems gives back the classical results:

**Corollary 2** Suppose that $(A_1(1, \theta))$ and $(A_2(\rho))$ hold. Let $(k_n)$ be an intermediate sequence such that $\sqrt{k_n} b(n/k_n) \rightarrow 0$. Then,

$$\sqrt{k_n} \left( \hat{\theta}_n(k_n) - \theta \right) \xrightarrow{d} N(0, \theta^2).$$

If, moreover $k_n/(np_n) \rightarrow \infty$ then,

$$\sqrt{k_n} \left( \frac{\hat{x}_{p_n}}{x_{p_n}} - 1 \right) \xrightarrow{d} N(0, \theta^2).$$
4 Illustration on simulations

The section is dedicated to the illustration of the conclusions drawn from Theorem 1 on simulated data. To this end, we consider a cumulative distribution function $F_{\theta,\tau,\rho}$ verifying (A\(_1\)(\(\tau, \theta\))) and (A\(_2\)(\(\rho\))) with $\theta = 1/2$, $\tau \in \{0, 1/2, 1\}$ and $\rho \in \{-1/2, -1/4\}$. More specifically, the slowly-varying function is given by

$$\ell(x) = 1 - \frac{\theta}{\rho} (1 + x)^{\theta} \left( 1 + \frac{1}{x} \right)^{\theta}.$$ 

Following Proposition 2(i), it appears that the case $\tau = 0$ corresponds to a Weibull tail-distribution (with Weibull tail-coefficient $1/2$) similar to a Gaussian distribution. When $\tau = 1/2$, in view of Paragraph 2.1, $\bar{F}(x) = \exp\{-\log x^{2} (1 + o(1))\}$, the distribution has a tail behavior similar to the log-normal distribution. Finally, Proposition 2(iii) shows that, when $\tau = 1$, the distribution belongs to the Fréchet maximum domain of attraction with tail-index $1/2$.

For each considered combination of $\tau$ and $\rho$, $N = 500$ samples $(X_{n,j})_{j=1,...,N}$ of size $n = 500$ were simulated from $F_{1/2,\tau,\rho}$. On each sample $(X_{n,j})$, the estimate $\hat{\theta}_{n,j}$ is computed for $k = 2, \ldots, 250$, the associated empirical squared bias $\text{ESB}$ and empirical variance $\text{EV}$ plots are built by plotting the pairs $\left(k, (\bar{\theta}^{(1)}_{n}(k) - \theta)^{2}\right)$ and $\left(k, (\bar{\theta}^{(2)}_{n}(k) - (\bar{\theta}^{(1)}_{n}(k))^{2}\right)$ where for $i \in \{1, 2\}$,

$$\bar{\theta}^{(i)}_{n}(k) = \frac{1}{N} \sum_{j=1}^{N} (\hat{\theta}_{n,j}(k))^{i}.$$ 

The empirical squared bias and the empirical variance are depicted on Figure 1 and Figure 2 respectively. Both graphs are represented on the same scale for the sake of comparison. As expected, the squared bias, for a fixed value of $k$, is an increasing function of $\rho$ and a decreasing function of $\tau$. At the opposite, the variance seems to be independent of $\rho$ and is not much dependent of $\tau$.

5 Concluding remarks

As illustrated in the previous sections, the model (A\(_1\)(\(\tau, \theta\))) provides a new tool for the analysis of tail estimators based on log-spacings. It allows us to encompass Weibull tail-distributions in a more general framework and thus to explain why their dedicated tail estimators are very similar to Hill or Weisssman statistics. The next step would be to estimate the parameter $\tau$. For instance, one can consider the following estimator based on the log-spacing between two Hill statistics

$$\hat{\tau}_{n} = 1 + \frac{\log H_{n}(k'_{n}) - \log H_{n}(k_{n})}{\log \log(n/k'_{n}) - \log \log(n/k_{n})},$$

where $(k_{n})$ and $(k'_{n})$ are two intermediate sequences such that

$$\liminf_{n \to \infty} \frac{\log(n/k'_{n})}{\log(n/k_{n})} > 1.$$ 

Let us note that

$$\log \log(n/k'_{n}) - \log \log(n/k_{n})) (\hat{\tau}_{n} - \tau) = \log((\hat{\theta}_{n}(k'_{n})/\theta) - \log(\hat{\theta}_{n}(k_{n})/\theta) + \log \left( \frac{\mu_{1,\tau}(\log(n/k'_{n}))}{\log \log(n/k'_{n})} \right) - \log \left( \frac{\mu_{1,\tau}(\log(n/k_{n}))}{\log \log(n/k_{n})} \right).$$
This implies that the consistency of \( \hat{\tau}_n \) is a simple consequence of Theorem 1 and Lemma 2(i) whereas the asymptotic distribution is much more difficult to handle as it requires the joint distribution of \( \hat{\tau}_n(k'_n) \) and \( \hat{\theta}_n(k_n) \). Also in practice, the choice of the parameters \( k_n \) and \( k'_n \) is an open question. These two points are currently under investigation.

Other extensions are possible, among others bias correction based on the estimation of the second-order parameter [28, 29]. To this end, an exponential regression model for these tail distributions extending [5, 11, 12, 18] would be of interest. We also plan to adapt our results to the case \( \tau > 1 \) and to investigate the possible links with super-heavy tails [19]. Finally, this work could be further extended to random variables \( Y = \psi(X) \) where \( X \) has a parent distribution satisfying \( (A_1(\tau, \theta)) \). For instance, choosing \( \psi(x) = x^*-1/x \) would allow to consider distributions (with finite endpoint \( x^* \)) in the Weibull maximum domain of attraction. This may help for including the negative Hill estimator (see for instance [17] or [30], paragraph 3.6.2) in our framework.

### 6 Proofs

We first give some preliminary lemmas. Their proofs are postponed to the appendix.

#### 6.1 Preliminary lemmas

The first lemma provides some uniform approximations based on \( (A_1(\tau, \theta)) \) and \( (A_2(\rho)) \).

**Lemma 1** If \( (A_1(\tau, \theta)) \) and \( (A_2(\rho)) \) hold then

\[
\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) \right| = o(b(x)), \quad \text{when } x \to \infty.
\]

Let us define for all \( q \in \mathbb{N} \setminus \{0\} \), \( \tau \in [0, 1] \) and \( t > 0 \), \( \sigma^2_{q,\tau}(t) = \mu_{2q,\tau}(t) - \mu^2_{q,\tau}(t) \). The following lemma is of analytical nature. It provides first-order expansions which will be useful in the sequel.

**Lemma 2** For all \( q \in \mathbb{N} \setminus \{0\} \) and \( \tau \in [0, 1] \), when \( t \to \infty \):

(i) \( \mu_{q,\tau}(t) \sim q! t^{(\tau-1)q} \),

(ii) \( \sigma^2_{q,\tau}(t)/\mu^2_{q,\tau}(t) \to (2q)!/(q!)^2 - 1 \),

(iii) \( \mu^2_{1,\tau}(t)/\mu_{1,\tau}(t) \to 0 \).

The next lemma presents an expansion of \( \hat{\theta}_n(k_n) \).

**Lemma 3** Let \( (k_n) \) be an intermediate sequence. Then, under \( (A_1(\tau, \theta)) \), the following expansions hold:

\[
\hat{\theta}_n(k_n) = \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \left( \theta_{n,1}^{(1)}(E_{n-k_n+1,n}) + \theta_{n,2}(E_{n-k_n+1,n}) \right),
\]

with, for all \( q \in \mathbb{N} \setminus \{0\} \),

\[
\theta_{n,1}^{(q)}(t) = \frac{1}{k_n-1} \sum_{i=1}^{k_n-1} (K_\tau(F_i + t) - K_\tau(t))^q,
\]

\[
\theta_{n,2}(t) = \frac{1}{k_n-1} \sum_{i=1}^{k_n-1} \log \left( \frac{\ell(K_\tau(F_i + t))}{\ell(K_\tau(t))} \right),
\]

and where \( E_{n-k_n+1,n} \) is the \((n-k_n+1)th\) order statistic associated to \( n \) independent standard exponential variables and \( \{F_1, \ldots, F_{k_n-1}\} \) are independent standard exponential variables and independent from \( E_{n-k_n+1,n} \).
The asymptotic behavior of the \((n - k_n + 1)\)th standard exponential order statistic is described in the following lemma.

**Lemma 4** Let \((k_n)\) be an intermediate sequence. Then, for all differentiable function \(g\), we have

\[
\sqrt{k_n} (g(E_{n-k_n+1,n}) - g(\log(n/k_n))) = O_P(1)g'(\log(n/k_n)(1 + o_P(1))).
\]

The next two lemmas provide the key results for establishing the asymptotic distribution of \(\hat{\theta}_n(k_n)\). They describe the asymptotic behavior of the random terms appearing in Lemma 3.

**Lemma 5** Let \((k_n)\) be an intermediate sequence. Then, for all \(q \in \mathbb{N} \setminus \{0\}\),

\[
\frac{\sqrt{k_n}}{\sigma_{q,\tau}(E_{n-k_n+1,n})} \left( \theta_{n,1}(E_{n-k_n+1,n}) - \mu_{q,\tau}(E_{n-k_n+1,n}) \right) \overset{d}{\to} \mathcal{N}(0, 1).
\]

**Lemma 6** Suppose that \((A_1(\tau, \theta))\) and \((A_2(\rho))\) hold. Let \((k_n)\) be an intermediate sequence. Then,

\[
\theta_{n,2}(E_{n-k_n+1,n}) = b(\exp K_{\tau}(E_{n-k_n+1,n}))\theta_{n,3}(E_{n-k_n+1,n})(1 + o_P(1)),
\]

where

\[
\left| \theta_{n,3}(E_{n-k_n+1,n}) - \theta_{n,3}^{(1)}(E_{n-k_n+1,n}) \right| \leq -\frac{\rho}{2}\theta_{n,1}^{(2)}(E_{n-k_n+1,n}).
\]

Moreover, if \(\tau = 1\), then \(\theta_{n,3}(E_{n-k_n+1,n}) \overset{p}{\to} 1/(1 - \rho)\).

### 6.2 Proofs of the main results

**Proof of Proposition 1** – Assumptions \((A_1(\tau_1, \theta_1))\) and \((A_1(\tau_2, \theta_2))\) entail

\[
\frac{F_{\tau_1,\theta_1}(x)}{F_{\tau_2,\theta_2}(x)} = \exp \left[ -K_{\tau_1}^{-}\left(\log H_{1}(x)\right) \left( 1 - \frac{K_{\tau_2}^{-}\left(\log H_{2}(x)\right)}{K_{\tau_1}^{-}\left(\log H_{1}(x)\right)} \right) \right],
\]

(12)

where \(H_{1} \in \mathcal{R}_{1/\theta_1}\) and \(H_{2} \in \mathcal{R}_{1/\theta_2}\). As a consequence, for all \(q \in \{1, 2\}\), \(\log H_{q}(x) \sim \log(x)/\theta_q\) when \(x \to \infty\), see [9], Proposition 1.3.6. Let us first prove (i): \(0 < \tau_1 < \tau_2\) implies

\[
K_{\tau_1}^{-}\left(\log H_{q}(x)\right) \sim (\tau_q/\theta_q)^{1/\tau_q} (\log x)^{1/\tau_q} \to \infty,
\]

(13)

and thus

\[
\frac{K_{\tau_2}^{-}\left(\log H_{2}(x)\right)}{K_{\tau_1}^{-}\left(\log H_{1}(x)\right)} \sim \frac{(\tau_2/\theta_2)^{1/\tau_2}}{(\tau_1/\theta_1)^{1/\tau_1}} (\log x)^{1/\tau_1 - 1/\tau_2} \to 0.
\]

(14)

Collecting (12), (13) and (14) gives the result: \(\hat{F}_{\tau_1,\theta_1}(x)/\hat{F}_{\tau_2,\theta_2}(x) \to 0\) as \(x \to \infty\). Similarly, if \(\tau_1 = 0\), then

\[
\frac{K_{\tau_2}^{-}\left(\log H_{2}(x)\right)}{K_{0}^{-}\left(\log H_{1}(x)\right)} \sim \frac{(\tau_2/\theta_2)^{1/\tau_2}}{H_{1}(x)} (\log x)^{1/\tau_2} \to 0,
\]

which concludes the first part of the proof. Let us now focus on (ii) and suppose \(\theta_1 < \theta_2\). If \(\tau > 0\) then

\[
\frac{K_{\tau}^{-}\left(\log H_{2}(x)\right)}{K_{\tau}^{-}\left(\log H_{1}(x)\right)} \to \left(\frac{\theta_1}{\theta_2}\right)^{1/\tau} < 1,
\]

as \(x \to \infty\), while, if \(\tau = 0\),

\[
\frac{K_{0}^{-}\left(\log H_{2}(x)\right)}{K_{0}^{-}\left(\log H_{1}(x)\right)} = \frac{H_{2}(x)}{H_{1}(x)} \to 0,
\]

as \(x \to \infty\). In both cases, for \(x\) large enough,

\[
1 - \frac{K_{\tau}^{-}\left(\log H_{2}(x)\right)}{K_{\tau}^{-}\left(\log H_{1}(x)\right)} > 0,
\]

(15)

and collecting (12), (13) and (15) concludes the proof: \(\hat{F}_{\tau,\theta_1}(x)/\hat{F}_{\tau,\theta_2}(x) \to 0\) as \(x \to \infty\).
Proof of Proposition 2 — Proofs of (i) and (iii) are straightforward consequences of Paragraph 2.1. Let us focus on (ii). In view of the characterization (3.35) in [16] of the Gumbel maximum domain of attraction, it is sufficient to prove that there exists a positive function $a$, differentiable with $a'(t) \to 0$ as $t \to \infty$, such that

$$F(x) = \exp \left\{ - \int_{x_*}^{x} \frac{dt}{a(t)} \right\}, \quad x \geq x_*.$$  (16)

Letting $a = 1/(K_{\tau}^- (\log H))'$, it thus remains to prove that $a'(t) \to 0$ as $t \to \infty$ for all $\tau \in [0, 1)$. To this end, let us remark that

$$a'(t) = \frac{1}{K_{\tau}^- (\log H(t))} \left( \tau - 1 + \left( 1 - \frac{H''(t)H(t)}{H'(t)^2} \right) (1 + \tau \log H(t)) \right)$$

$$= \frac{1}{K_{\tau}^- (\log H(t))} (\tau - 1 + (\theta + o(1))(1 + \tau \log H(t))),$$

since $H' \in R_{1/\theta - 1}$ implies $H''(t)H(t)/H'(t)^2 \to 1 - \theta$ as $t \to \infty$. Two cases arise:

- If $\tau \in (0, 1)$ then $a'(t) \sim \theta(\tau \log H(t))^{-1/\tau} \to 0$ as $t \to \infty$.
- Otherwise, when $\tau = 0$, we have $a'(t) = (\theta - 1 + o(1))/H(t) \to 0$ as $t \to \infty$.

In both situations, the conclusion follows. ■

Proof of Proposition 3 — Let us suppose that $F$ verifies $(A_1(0, \theta))$ with $\theta \in (0, 1]$. Then, introducing $W(x) = \exp K_{\theta}(H(\log x))$, we have $F(\log x) = \exp(-K_{\theta}^- (\log W(x)))$. It thus remains to prove that $W^{-} \in R_{\theta, \infty}$. Simple calculations show that

$$W^{-}(t) = \exp \{ H^-(K_{\theta}^- (\log t)) \}$$

$$= \exp \{ (1 + \theta \log t) \ell (K_{\theta}^- (\log t)) \}$$

$$= e^{\ell \varphi(t)},$$

where we have defined $\varphi(t) = \psi(\log t)$ with $\psi(x) = \exp \{ (1 + \theta x)[\ell(K_{\theta}^- (x)) - \ell_{\infty}] \}$. As a consequence,

$$t'(\log \varphi(t))' = (\log \psi)'(\log t)$$

$$= \theta(\ell(K_{\theta}^- (\log t)) - \ell_{\infty}) + K_{\theta}^- (\log t) \ell'(K_{\theta}^- (\log t))$$

$$= o(1),$$

since, from [9], p. 15, $u'(u)/\ell(u) \to 0$ as $u \to \infty$. Using again [9], p. 15, it follows that $\varphi$ is a slowly varying function. Thus, $W^{-} \in R_{\theta, \infty}$ and $F(\log \cdot)$ verifies $(A_1(\theta, \theta \ell_{\infty}))$. ■

Proof of Theorem 1 — Lemma 5 states that for $q \in \{1, 2\}$,

$$\sqrt{k_n} \left( \frac{\theta_q(E_n-k_n+1, n)}{\sigma_q,\tau(E_n-k_n+1, n)} - \mu_q,\tau(E_n-k_n+1, n) \right) = \xi^{(q)}_n$$

where $\xi^{(q)}_n \overset{d}{\to} N(0, 1)$. Then, by Lemma 3,

$$\sqrt{k_n} \left( \theta_n(k_n) - \theta - a_{\tau, \rho}b(\exp K_{\tau}(\log(n/k_n))) \right) = \sqrt{k_n} \theta \left( \frac{\mu_1,\tau(E_n-k_n+1, n)}{\mu_1,\tau(\log(n/k_n))} - 1 \right) + \theta \frac{\sigma_1,\tau(E_n-k_n+1, n)}{\mu_1,\tau(\log(n/k_n))} \xi^{(1)}_n$$

$$+ \sqrt{k_n} \left( \frac{\theta_2(E_n-k_n+1, n)}{\mu_1,\tau(\log(n/k_n))} - a_{\tau, \rho}b(\exp K_{\tau}(\log(n/k_n))) \right) \overset{def}{=} T^{(1)}_n + T^{(2)}_n + T^{(3)}_n,$$
and the three terms are studied separately. First, applying Lemma 4 to \( g = \mu_1,\tau \) yields

\[
T_n^{(1)} = O_P(1) \frac{\mu'_{1,\tau}(\log(n/k_n))(1 + o_P(1))}{\mu_{1,\tau}(\log(n/k_n))} = o_P(1),
\]  

(17)
in view of Lemma 2(i, iii). Second,

\[
T_n^{(2)} = \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \left( 1 + \frac{T_n^{(1)}}{\theta/k_n} \right) \theta_k^{(1)} = \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \theta_k^{(1)}(1 + o_P(1))
\]

and, from Lemma 2(ii), \( \sigma_{1,\tau}(E_{n-k_n+1,n})/\mu_{1,\tau}(E_{n-k_n+1,n}) \xrightarrow{P} 1 \). As a preliminary conclusion,

\[
T_n^{(2)} = \theta_k^{(1)}(1 + o_P(1)).
\]

(18)

From Lemma 6, \( T_n^{(3)} \) can be expanded as

\[
T_n^{(3)} = \sqrt{k_n} b(\exp K_{\tau}(\log(n/k_n))) \left( \frac{b(\exp K_{\tau}(E_{n-k_n+1,n}))}{b(\exp K_{\tau}(\log(n/k_n)))} \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_P(1)) - a_{\tau,\rho} \right)
\]

\[
= \lambda \left( \frac{b(\exp K_{\tau}(E_{n-k_n+1,n}))}{b(\exp K_{\tau}(\log(n/k_n)))} \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_P(1)) - a_{\tau,\rho} \right)(1 + o(1)).
\]

Introducing \( T_n^{(3,1)} = K_{\tau}(E_{n-k_n+1,n}) - K_{\tau}(\log(n/k_n)) \) and applying Lemma 4 with \( g = K_{\tau} \) yield

\[
\exp T_n^{(3,1)} = \exp \left( O_P(1) \frac{(\log(n/k_n))^{\tau-1}}{\sqrt{k_n}} \right) \xrightarrow{P} 1,
\]

(19)
since \( \tau \in [0,1] \). Therefore, \( b \) being regularly varying,

\[
b(\exp K_{\tau}(E_{n-k_n+1,n})/b(\exp K_{\tau}(\log(n/k_n))) \xrightarrow{P} 1
\]
as well, and consequently

\[
T_n^{(3)} = \lambda \left( \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(\log(n/k_n))} (1 + o_P(1)) - a_{\tau,\rho} \right)(1 + o(1))
\]

\[
= \lambda \left( \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \left( 1 + \frac{T_n^{(1)}}{\theta/k_n} \right) (1 + o_P(1)) - a_{\tau,\rho} \right)(1 + o(1))
\]

\[
= \lambda \left( \frac{\theta_{n,3}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} (1 + o_P(1)) - a_{\tau,\rho} \right)(1 + o(1)),
\]

from (17). Two situations occur. If \( \tau = 1 \), then, in view of Lemma 6, \( \theta_{n,3}(E_{n-k_n+1,n}) \xrightarrow{P} a_{1,\rho} = 1/(1 - \rho) \), \( \mu_{1,1}(E_{n-k_n+1,n}) = 1 \) and thus \( T_n^{(3)} \xrightarrow{P} 0 \). If \( \tau \in [0,1) \), \( T_n^{(3)} \) can be rewritten as

\[
T_n^{(3)} = \lambda \left( (T_n^{(3,2)} + T_n^{(3,3)})(1 + o_P(1)) - 1 \right)(1 + o(1)),
\]

where

\[
T_n^{(3,2)} \overset{\text{def}}{=} \frac{\theta_{n,1}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} = 1 + \frac{\sigma_{1,\tau}(E_{n-k_n+1,n})}{\mu_{1,\tau}(E_{n-k_n+1,n})} \frac{\xi_n}{\sqrt{k_n}} = 1 + o_P(1)
\]

(11)
\[ |T_n^{(3)}| \overset{\text{def}}{=} \frac{\mid \theta_{n,3}(E_{n-k_{n}+1,n}) - \theta_{n,1}^{(1)}(E_{n-k_{n}+1,n}) \mid}{\mu_{1,\tau}(E_{n-k_{n}+1,n})} \leq -\rho \frac{\theta_{n,1}^{(2)}(E_{n-k_{n}+1,n})}{2 \mu_{2,\tau}(E_{n-k_{n}+1,n})} \frac{\mu_{1,\tau}(E_{n-k_{n}+1,n})}{\mu_{2,\tau}(E_{n-k_{n}+1,n})} \] 

\[ \overset{d}{=} -\rho (\log(n/k_{n}))^{-1} \left( 1 + o_{\rho}(1) \right) \left( 1 + \frac{\sigma_{2,\tau}(E_{n-k_{n}+1,n})}{\mu_{2,\tau}(E_{n-k_{n}+1,n})} \xi_{n}^{(2)} \right) \] 

\[ = O_{\rho}(\log(n/k_{n}))^{-1}, \] 

in view of Lemma 2, Lemma 5 and Lemma 6. Thus, for all \( \tau \in [0,1) \), \( T_n^{(3)} \overset{P}{\rightarrow} 0 \). Taking (17) and (18) into account concludes the proof. \( \blacksquare \)

**Proof of Proposition 4** — From (8), we have

\[ \frac{1}{2} \log k_{n} + \log |b(\exp K_{\tau}(\log(n/k_{n})))| \rightarrow \log |\lambda| \] 

as \( n \rightarrow \infty \), and since \( K_{\tau}(\log(n/k_{n})) \rightarrow \infty \) as \( n \rightarrow \infty \), it follows that

\[ \frac{\log k_{n}}{2K_{\tau}(\log(n/k_{n}))} + \frac{\log |b(\exp K_{\tau}(\log(n/k_{n})))|}{K_{\tau}(\log(n/k_{n}))} \rightarrow 0 \] 

as \( n \rightarrow \infty \). Now, \( |b| \) is a regularly-varying function with index \( \rho \) and thus \( \log |b(x)| / \log x \rightarrow \rho \) for all \( x \rightarrow \infty \), see [9], Proposition 1.3.6. As a consequence, we obtain

\[ \frac{\log k_{n}}{K_{\tau}(\log(n/k_{n}))} \rightarrow -2\rho \] 

(20)

as \( n \rightarrow \infty \). Let us first remark that, if \( \tau = 1 \) then (20) implies

\[ \log k_{n} = \frac{2\rho}{2\rho - 1}(\log n)(1 + o(1)) = \frac{2\rho}{2\rho - 1}K_{1}(\log n)(1 + o(1)) \] 

and the conclusion follows. Otherwise, if \( \tau \in [0,1) \), condition (20) can be rewritten as

\[ \frac{\log k_{n}}{\log n} \frac{\log n}{K_{\tau}(\log(n/k_{n}))} \rightarrow -2\rho. \] 

(21)

Besides, since \( K_{\tau} \) is non-decreasing,

\[ \frac{\log n}{K_{\tau}(\log(n/k_{n}))} \geq \frac{\log n}{K_{\tau}(\log n)} \rightarrow \infty \] 

for all \( \tau \in [0,1) \) and thus, in view of (21), necessarily \( \log k_{n} / \log n \rightarrow 0 \) as \( n \rightarrow \infty \). As a consequence, \( \log(n/k_{n}) \) is asymptotically equivalent to \( \log n \) and thus \( K_{\tau}(\log(n/k_{n})) \) is asymptotically equivalent to \( K_{\tau}(\log(n)) \) as well. Replacing in (20), the conclusion follows. \( \blacksquare \)

**Proof of Proposition 5** — Let us consider \( \tau \in [0,1/2) \) and suppose that (8) holds with \( \lambda \neq 0 \). Following Proposition 4, \( \log(k_{n}) = -2\rho K_{\tau}(\log n)(1 + o(1)) \) and thus \( \log(k_{n}) / \log(n) \rightarrow 0 \) as \( n \rightarrow \infty \). A first order Taylor expansion shows that there exists \( \eta_{n} \in [0,1] \) such that

\[ \Delta_{n} \overset{\text{def}}{=} \exp\{ K_{\tau}(\log(k_{n})) - K_{\tau}(\log n) \} = \exp\{ -\log k_{n}K_{\tau}^{\prime}(\log n) - \eta_{n} \log(k_{n}) \} \] 

\[ = \exp\{ -\log k_{n}K_{\tau}^{\prime}(\log n)(1 + o(1)) \}, \]
since $K'_r$ is regularly-varying. As a consequence,

$$\Delta_n = \exp\{2\rho K_r(\log n)K'_r(\log n)(1 + o(1))\}$$

and thus $\Delta_n \to 1$ if $\tau \in [0, 1/2)$ or $\Delta_n \to \exp(4\rho)$ if $\tau = 1/2$. Since $b^2$ is regularly varying with index $2\rho$ it follows that

$$\mathcal{A}_B(\tau, \rho) = b^2(\exp K_r(\log n)) \frac{b^2(\Delta_n \exp K_r(\log n))}{b^2(\exp K_r(\log n))} = c_{\tau, \rho} b^2(\exp K_r(\log n))(1 + o(1)),$$

and the conclusion follows.

**Proof of Theorem 2** — From (7), one can infer that

$$\log \hat{x}_{\rho}n - \log x_{\rho} = (\log(X_{n-k_n+1,n}) - \log \Phi^{-}(k_n/n))$$

$$+ (\hat{\theta}_n(k_n) - \theta)(K_r(\log(1/p_n)) - K_r(\log(n/k_n)))$$

$$+ \log \ell(\exp K_r(\log(n/k_n)))$$

$$= Q^{(1)} + Q^{(2)} + Q^{(3)}.$$ 

The three terms are studied separately. First, note that in view of (A$_1$, $\theta$) and (A$_2$(\rho)), $Q^{(1)}$ can be expanded as

$$Q^{(1)} = \log H^{-}(\exp K_r(E_{n-k_n+1,n}) - \log K_r(\log(n/k_n)))$$

$$= \theta(K_r(E_{n-k_n+1,n}) - K_r(\log(n/k_n))) + \log \ell(\exp K_r(E_{n-k_n+1,n})/\ell(\exp K_r(\log(n/k_n))))$$

$$\overset{\text{def}}{=} \theta T^{(3,1)}_n + Q^{(1,2)},$$

where $T^{(3,1)}_n$ is defined in the proof of Theorem 1 as

$$T^{(3,1)}_n = K_r(E_{n-k_n+1,n}) - K_r(\log(n/k_n)) = O_P(1) \frac{(\log(n/k_n))^{\tau - 1}}{\sqrt{k_n}}, \tag{22}$$

in view of (19). Moreover, $Q^{(1,2)}_n \overset{\text{def}}{=} \log \ell(\lambda_n x_n) - \log \ell(x_n)$, where $x_n = \exp K_r(\log(n/k_n)) \to \infty$ and $\lambda_n = \exp T^{(3,1)}_n \overset{P}{\to} 1$. Thus, from (A$_2$(\rho)) we have

$$Q^{(1,2)}_n = b(\exp K_r(\log(n/k_n)))K_r(\lambda_n)(1 + o_P(1))$$

$$= b(\exp K_r(\log(n/k_n)))\log(\lambda_n)(1 + o_P(1))$$

$$= O_P(1)b(\exp K_r(\log(n/k_n)))\frac{(\log(n/k_n))^{\tau - 1}}{\sqrt{k_n}},$$

in view of (22). Since $b(x) \to 0$ as $x \to \infty$, it follows that

$$Q^{(1,2)}_n = o_P\left(\frac{(\log(n/k_n))^{\tau - 1}}{\sqrt{k_n}}\right),$$

entailing

$$\frac{\sqrt{k_n}}{K_r(\log(1/p_n)) - K_r(\log(n/k_n))}Q^{(1)}_n = O_P\left(\frac{(\log(n/k_n))^{\tau - 1}}{K_r(\log(1/p_n)) - K_r(\log(n/k_n))}\right) = o_P(1),$$

13
from (10). Now, concerning the second term, Theorem 1 entails that

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} Q_n^{(2)} = \sqrt{k_n} \left( \hat{\theta}_n(k_n) - \theta \right) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

Finally, $Q_n^{(3)} = \log \ell(x_n^*) - \log \ell(\lambda_n^* x_n^*)$ where $\lambda_n^* = \exp[K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))] \geq 1$ in view of (10) and $x_n^* = \exp K_\tau(\log(n/k_n)) \to \infty$. Thus, Lemma 1 entails

$$\frac{\sqrt{k_n}}{\log \lambda_n^*} Q_n^{(3)} \sim -\sqrt{k_n} b(x_n^*) \frac{K_{\rho}(\lambda_n^*)}{\log \lambda_n^*} = o \left( \frac{K_{\rho}(\lambda_n^*)}{\log \lambda_n^*} \right),$$

since $\sqrt{k_n} b(x_n^*) = \sqrt{k_n} b(\exp K_\tau(\log(n/k_n))) \to 0$. Taking account of the inequality $K_{\rho}(x) \leq \log x$ for all $x \geq 1$ yields

$$\frac{\sqrt{k_n}}{K_\tau(\log(1/p_n)) - K_\tau(\log(n/k_n))} Q_n^{(3)} = o(1).$$

Combining the above results, Theorem 2 follows.

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References


Appendix: Proof of auxiliary results

Proof of Lemma 1 — From \((A_1(\tau, \theta))\) and \((A_2(\rho))\), it is easy to infer that, for any constant \(\tilde{C} > 0\), we have

\[
\frac{1}{C\theta(x)} \left( \frac{H^-(\lambda x) - H^-(x)}{\theta H^-(x)(1 + b(x)/\theta)} - \theta \right) = \frac{\theta}{C\theta} K_\rho(\lambda) - \frac{1}{C\theta} + o(1)
\]

\[
= \frac{\theta + \rho}{C\theta} \left( K_{\rho+\theta} - K_{\theta} \right) + o(1).
\]

Then, choosing \(\tilde{C}\) such that \((\theta + \rho)/(\tilde{C}\theta) = 1\), a direct application of Lemma 5.2 in [15] yields, for any \(\varepsilon > 0\) and \(\lambda \geq 1\),

\[
\min(1, \lambda^{-\rho-\varepsilon}) \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta} b^2(x) \left( K_\rho(\lambda) - K_{-\theta}(\lambda) \right) \right|
\]

\[
\leq \varepsilon \lambda^{\theta}[b(x)] \left| \frac{1}{\theta} + 1 + 2 \lambda^{\rho+\varepsilon} \right|
\]

\[
\leq 4 \varepsilon \lambda^{\theta}[b(x)] \min(1, \lambda^{-\rho-\varepsilon}) \left[ 1 + \lambda^{\rho+\varepsilon} \right]
\]

\[
\leq 8 \varepsilon \lambda^{\theta}[b(x)]
\]

for \(x\) large enough. Moreover, letting \(0 < \varepsilon < -\rho\) yields

\[
\sup_{\lambda \geq 1} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) - \frac{1}{\theta} b^2(x) \left( K_\rho(\lambda) - K_{-\theta}(\lambda) \right) \right| = o(b(x)).
\]
Besides, $K_\rho(\lambda) - K_{-\theta}(\lambda)$ is bounded when $\rho < 0$, and therefore (23) can be simplified as
\[
\sup_{\lambda \geq 1} \left\| \frac{\ell(\lambda x)}{\ell(x)} - 1 - b(x)K_\rho(\lambda) \right\| = o(b(x)), \text{ as } x \to \infty,
\]
and the conclusion follows.

\section*{Proof of Lemma 2}
(i) Let us consider for $t > 1$ and $q \in \mathbb{N} \setminus \{0\}$,
\[
Q_q(t) = \int_0^\infty \left( \frac{K_\tau(x + t) - K_\tau(t)}{K_\tau'(t)} \right)^q e^{-x} dx.
\]
There exists $\eta \in (0, 1)$ such that
\[
\left| \frac{K_\tau(x + t) - K_\tau(t)}{xK_\tau'(t)} \right| = \left( 1 + \frac{\eta x \tau}{t} \right)^{-1} \leq 1.
\]
Thus, Lebesgue Theorem implies that
\[
\lim_{t \to \infty} Q_q(t) = \lim_{t \to \infty} \int_0^\infty \left( 1 + \frac{\eta x \tau}{t} \right)^{q(\tau - 1)} x^q e^{-x} dx = \int_0^\infty x^q e^{-x} dx = q!
\]
which concludes the first part of the proof.

(ii) is a straightforward consequence of (i).

(iii) We have
\[
\mu^\prime_{1,\tau}(t) = \int_0^\infty (K_\tau'(x + t) - K_\tau'(t)) e^{-x} dx
\]
\[
= \int_0^\infty K_\tau'(x + t)e^{-x} dx - K_\tau'(t)
\]
\[
= \int_0^\infty K_\tau(x + t)e^{-x} dx - K_\tau(t) - K_\tau'(t)
\]
\[
= \mu_{1,\tau}(t) - t^{\tau - 1}.
\]
Finally, (i) states that $t^{\tau - 1}/\mu_{1,\tau}(t) \to 1$ as $t \to \infty$ which entails $\mu^\prime_{1,\tau}(t)/\mu_{1,\tau}(t) \to 0$ as $t \to \infty$.

\section*{Proof of Lemma 3}
Recall that
\[
\hat{\theta}_n = \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n - 1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n}))
\]
\[
= \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n - 1} \log \left( \frac{H^-(\exp K_\tau(E_{n-i+1,n}))}{H^-(\exp K_\tau(E_{n-k_n+1,n}))} \right),
\]
where $E_{1,n}, \ldots, E_{n,n}$ are ordered statistics generated by $n$ independent standard exponential random variables. The Rényi representation of the Exp(1) ordered statistics (see [1], p. 72) yields
\[
\{E_{n-i+1,n}\}_{i=1}^{k_n} \overset{d}{=} \{F_{k_n-i,k_n-1} + E_{n-k_n+1,n}\}_{i=1}^{k_n-1},
\]
where $\{F_{k_n-1}, \ldots, F_{k_n-1}\}$ are ordered statistics independent from $E_{n-k_n+1,n}$ and generated by $k_n - 1$ independent standard exponential variables $\{F_1, \ldots, F_{k_n-1}\}$. We thus have
\[
\hat{\theta}_n(k_n) = \frac{1}{\mu_{1,\tau}(\log(n/k_n))} \frac{1}{k_n - 1} \sum_{i=1}^{k_n - 1} \log \left( \frac{H^-(\exp K_\tau(F_{k_n-i,k_n-1} + E_{n-k_n+1,n}))}{H^-(\exp K_\tau(E_{n-k_n+1,n}))} \right)
\]
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\[
\begin{align*}
    \frac{1}{\mu_{1,\tau}(\log(n/k_n))} &= \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left( \frac{H^{-}((\exp K_{\tau} F_i + E_{n-k_n+1,n}))}{H^{-}(\exp K_{\tau} E_{n-k_n+1,n})} \right) \\
    \frac{1}{\mu_{1,\tau}(\log(n/k_n))} &= \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} \log \left( \frac{H^{-}((\exp K_{\tau} F_i + E_{n-k_n+1,n}))}{H^{-}(\exp K_{\tau} E_{n-k_n+1,n})} \right)
\end{align*}
\]

in view of (A$_1(\tau, \theta)$) and the conclusion follows.

**Proof of Lemma 4** – A first order expansion of the function $g$ leads to,
\[
\sqrt{k_n}(g(E_{n-k_n+1,n}) - g(\log(n/k_n))) = \sqrt{k_n}(E_{n-k_n+1,n} - \log(n/k_n))g'(\hat{\eta}_n),
\]
with $\hat{\eta}_n \in [\min(E_{n-k_n+1,n}, \log(n/k_n)), \max(E_{n-k_n+1,n}, \log(n/k_n))]$. Now, Lemma 1 in [25] shows that $\sqrt{k_n}(E_{n-k_n+1,n} - \log(n/k_n)) \overset{d}{\to} \mathcal{N}(0,1)$ which implies that $\hat{\eta}_n = \log(n/k_n)(1 + O_p(1))$ and the result follows.

**Proof of Lemma 5** – Let us introduce for all $t \geq 1$ and $q \in \mathbb{N} \setminus \{0\},$
\[
S^{(q)}_n(t) = \frac{(k_n - 1)^{1/2}}{\sigma_{q,\tau}(t)}(\hat{\theta}^{(q)}_{n,1}(t) - \mu_{q,\tau}(t)) = \frac{(k_n - 1)^{-1/2}}{\sigma_{q,\tau}(t)} \sum_{i=1}^{k_n-1} Y_i^{(q)}(t),
\]
where $Y_i^{(q)}(t) \overset{def}{=} (K_{\tau}(F_i + t) - K_{\tau}(t))^q - \mu_{q,\tau}(t)$, $i = 1, \ldots, k_n - 1$ are centered, independent and identically distributed random variables with variance $\sigma_{q,\tau}^2(t)$. Clearly, in view of the Central Limit Theorem, for all $t \geq 1$ and $q \in \mathbb{N} \setminus \{0\}$, $S^{(q)}_n(t)$ converges in distribution to a standard Gaussian distribution. Our goal is to prove that, for all $x \in \mathbb{R}$ and $q \in \mathbb{N} \setminus \{0\}$,
\[
\mathbb{P}(S^{(q)}_n(E_{n-k_n+1,n}) \leq x) \to \Phi(x) \text{ as } n \to \infty,
\]
where $\Phi$ is the cumulative distribution function of the standard Gaussian distribution. Lemma 2(i) implies that for all $\varepsilon \in (0,1)$, and $r \in \mathbb{N} \setminus \{0\}$, there exists $T_\varepsilon \geq 1$ such that for all $t \geq T_\varepsilon$,
\[
(1 - \varepsilon) r! t^{(r-1)} \leq \mu_{r,\tau}(t) \leq (1 + \varepsilon) r! t^{(r-1)}. \tag{25}
\]
Furthermore, for $x \in \mathbb{R},$
\[
\mathbb{P}(S^{(q)}_n(E_{n-k_n+1,n}) \leq x) - \Phi(x) = \int_0^{T_\varepsilon} (\mathbb{P}(S^{(q)}_n(t) \leq x) - \Phi(x)) h_n(t)dt + \int_{T_\varepsilon}^{\infty} (\mathbb{P}(S^{(q)}_n(t) \leq x) - \Phi(x)) h_n(t)dt \overset{def}{=} A_n + B_n,
\]

where $h_n$ is the density of the random variable $E_{n-k_n+1,n}$. First, let us focus on the term $A_n$. We have,
\[
|A_n| \leq 2 \mathbb{P}(E_{n-k_n+1,n} \leq T_\varepsilon).
\]
Since $E_{n-k_n+1,n}/\log(n/k) \overset{p}{\to} 1$ (see [25], Lemma 1), it is easy to show that $A_n \to 0$. Now, let us consider the term $B_n$. For all $t \geq T_\varepsilon$,
\[
\mathbb{E}(|Y_1^{(q)}(t)|^3) \leq \mathbb{E}((K_{\tau}(F_1 + t) - K_{\tau}(t))^q + \mu_{q,\tau}(t))^3
\]
\[
= \mu_{3q,\tau}(t) + 3\mu_{q,\tau}(t)\mu_{2q,\tau}(t) + 4\mu_{q,\tau}^3(t)
\]
\[
\leq C_1(\varepsilon) r^3 t^{3(r-1)} < \infty,
\]

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from (25). Here, and in the following, \( C_1(\varepsilon), C_2, C_3(\varepsilon) \) and \( C_4(\varepsilon) \) are positive constants independent of \( t \). Thus, from Berry-Esséen’s inequality (see [35], Theorem 3), we have:

\[
\sup_x |\mathbb{P}(S_n^{(q)}(t) \leq x) - \Phi(x)| \leq C_2 L_n \quad \text{with} \quad L_n = \frac{(k_n - 1)^{-1/2}}{\sigma_{q,\tau}(t)} \mathbb{E}(|Y_1^{(q)}(t)|^3).
\]

From (25), since \( t \geq T_\varepsilon \),

\[
\sigma_{q,\tau}^2(t) = \mu_{q,\tau}(t) - \mu_{q,\tau}(t) \geq C_3(\varepsilon) t^{2(\tau - 1)}.
\]

Thus, \( L_n \leq C_4(\varepsilon)(k_n - 1)^{-1/2} \) and therefore

\[
|B_n| \leq C_2 C_4(\varepsilon) (k_n - 1)^{-1/2} \mathbb{P}(E_{n-k_n+1, n} \geq T_\varepsilon) \leq C_2 C_4(\varepsilon) (k_n - 1)^{-1/2} \to 0,
\]

which concludes the proof.

**Proof of Lemma 6** – Let us consider the random variables \( x_n = \exp[K_\tau(E_{n-k_n+1, n})] \) and \( \lambda_{i,n} = \exp[K_\tau(F_i + E_{n-k_n+1, n}) - K_\tau(E_{n-k_n+1, n})], i = 1, \ldots, k_n - 1 \). It is clear that \( x_n \overset{p}{\to} \infty \) in view of Lemma 1 in [25] and \( \lambda_{i,n} \geq 1 \). Thus, letting

\[
\theta_{n,3}(E_{n-k_n+1, n}) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} K_\rho \left[ \exp(K_\tau(F_i + E_{n-k_n+1, n}) - K_\tau(E_{n-k_n+1, n})) \right],
\]

Lemma 1 entails

\[
\theta_{n,2}(E_{n-k_n+1, n}) \overset{d}{=} b(\exp K_\tau(E_{n-k_n+1, n}) \theta_{n,3}(E_{n-k_n+1, n})(1 + o_{\mathbb{P}}(1)).
\]

Since \( |K_\rho(\exp u) - u| \leq -\rho u^2/2 \) for all \( u \geq 0 \), we have

\[
\left| \theta_{n,3}(E_{n-k_n+1, n}) - \theta_{n,3}^{(1)}(E_{n-k_n+1, n}) \right| \leq \frac{\rho \theta_{n,3}^{(2)}(E_{n-k_n+1, n})}{2}.
\]

Moreover, if \( \tau = 1 \), then

\[
\theta_{n,3}(E_{n-k_n+1, n}) = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} K_\rho(\exp F_i) \overset{p}{\to} \int_0^{+\infty} K_\rho(\exp u) \exp(-u) du = \frac{1}{1 - \rho},
\]

in view of the law of large numbers, and the conclusion follows.
Figure 1: Empirical squared bias as a function of $k$ obtained with $\hat{\theta}_n(k_n)$ computed on 500 samples of size 500 from $F_{1/2, \tau, \rho}$. Up: $\rho = -1/2$, down: $\rho = -1/4$, solid line: $\tau = 1$, dashed line: $\tau = 1/2$, dotted line: $\tau = 0$. 
Figure 2: Empirical variance as a function of $k$ obtained with $\hat{\theta}_n(k_n)$ computed on 500 samples of size 500 from $F_{1/2,\tau,\rho}$. Up: $\rho = -1/2$, down: $\rho = -1/4$, solid line: $\tau = 1$, dashed line: $\tau = 1/2$, dotted line: $\tau = 0$. 

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