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HAMILTONIANS WITH PURELY DISCRETE SPECTRUM

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ABSTRACT. We discuss criteria for a self-adjoint operator on $L^2(X)$ to have empty essential spectrum. We state a general result for the case of a locally compact abelian group $X$ and give examples for $X = \mathbb{R}^n$.

1. Let $\Delta$ be the positive Laplacian on $\mathbb{R}^n$. We set $B_a(r) = \{ x \in \mathbb{R}^n \mid |x - a| \leq r \}$ and $B_a = B_a(1)$.

Proposition 1. Let $V$ be a real locally integrable function on $\mathbb{R}^n$ such that:

(i) if $\lambda > 0$ then the measure $\omega_\lambda(a)$ of the set $\{ x \in B_a \mid V(x) < \lambda \}$ satisfies $\lim_{a \to \infty} \omega_\lambda(a) = 0$, 
(ii) the negative part of $V$ satisfies $V_- \leq \mu \Delta + \nu$ for some positive real numbers $\mu, \nu$ with $0 < \mu < 1$. 

Then the spectrum of the self-adjoint operator $H$ associated to the form sum $\Delta + V$ is purely discrete.

Remark 2. Let $V_k = \max\{ V, 0 \}$ and for each $\lambda > 0$ let $\Omega_\lambda = \{ x \mid V_+(x) < \lambda \}$. Then $\omega_\lambda(a)$ is the measure of the set $B_a \cap \Omega_\lambda$. From Lemma 5 it follows that the condition (i) is equivalent to

$$\lim_{a \to \infty} \int_{B_a} \frac{dx}{1 + V(x)} = 0. \quad (1)$$

Remark 3. From Lemma 7 we get $\lim_{a \to \infty} \omega_\lambda(a) = 0$ if $\int_{\Omega_\lambda} \omega^p dx < \infty$ for some $p > 0$. Thus Theorems 1 and 3 from [S] are consequences of Proposition 1. In the case $V \geq 0$ Proposition 1 is a consequence of Theorem 2.2 from [MS]. More general results will be obtained below. Note, however, that our techniques are not applicable in the framework considered in Theorem 2 from [S] and in [WW].

Proposition 1 is very easy to prove if condition (1) is replaced by $\lim_{x \to \infty} V_+(x) = \infty$. In fact, let us consider an arbitrary locally compact space $X$ and let $\mathcal{H}$ be a Hilbert $X$-module, i.e. $\mathcal{H}$ is a Hilbert space and a nondegenerate $*$-morphism $\phi \mapsto \phi(Q)$ of $C_0(X)$ into $B(\mathcal{H})$ is given. For example, one may take $\mathcal{H} = L^2(X, \mu)$ for some Radon measure $\mu$. Then we have the following simple compactness criterion: if $R$ is a bounded self-adjoint operator on $\mathcal{H}$ such that (i) if $\phi \in C_0(X)$ then $\phi(Q)R$ is a compact operator, (ii) one has $\pm R \leq \theta(Q)$ for some $\theta \in C_0(X)$, then $R$ is a compact operator. Indeed, note first that the operator $R \phi \equiv \phi(Q)$ will also be compact for all $\phi \in C_0(X)$. Let $\varepsilon > 0$ and let us choose $\phi$ such that $0 \leq \phi \leq 1$ and $\theta \phi = \varepsilon$, where $\phi^\perp = 1 - \phi$. Then $\pm R \phi \phi = R \phi \phi \leq \varepsilon$ which implies $\| R \phi \phi \| \leq \varepsilon$. So we have $\| R - R \phi \phi \| \leq \varepsilon$ and $\phi R + \phi R \phi$ is a compact operator. Now let us say that a self-adjoint operator $H$ on $\mathcal{H}$ is locally compact if $\phi(Q)(H+i)^{-1}$ is compact for all $\phi \in C_0(X)$. Then we get: If $H$ is a locally compact self-adjoint operator on $\mathcal{H}$ and if there is a continuous function $\Theta : X \to \mathbb{R}$ such that $\lim_{x \to \infty} \Theta(x) = +\infty$ and $H \geq \Theta(Q)$, then the spectrum of $H$ is purely discrete (the nondegeneracy of the morphism is needed for the definition of $\Theta(Q)$ for unbounded $\Theta$).

2. On the other hand, Proposition 1 can be significantly generalized. For example, $\Delta$ may be replaced by a higher order operator with matrix valued coefficients and $V$ does not have to be a function. These results are consequences of the following "abstract" fact. We fix a locally compact abelian group $X$, choose a finite dimensional Hilbert space $E$, and define $\mathcal{H} = L^2(X) \otimes E$. For $a \in X$ and $k \in X^*$ (the dual locally compact abelian group) we denote $U_a$ and $V_k$ the unitary operators on $\mathcal{H}$ given by

$$(U_a f)(x) = f(x + a) \quad \text{and} \quad (V_k f)(x) = k(x) f(x).$$

We denote additively the operations both in $X$ and in $X^*$ and denote 0 their neutral elements.
Theorem 4. Let $H$ be a self-adjoint operator on $\mathcal{H}$ such that for some (hence for all) $z \in \mathbb{C}$ not in the spectrum of $H$ the operator $R = (H - z)^{-1}$ satisfies
\[
\lim_{k \to 0} ||V_k R V_k^* - R|| = 0, \quad \lim_{\alpha \to 0} ||U_{\alpha} - 1|| = 0.
\tag{2}
\]
Then $H$ has purely discrete spectrum if and only if $w\text{-}\lim_{\alpha \to \infty} U_{\alpha} R U_{\alpha}^* = 0$.

**Proof:** If the spectrum of $H$ is purely discrete then $R$ is compact so $w\text{-}\lim_{\alpha \to \infty} U_{\alpha} R U_{\alpha}^* = 0$. The reciprocal assertion follows from Theorem 1.2 from [GI]. Indeed, with the terminology used there, all the localizations at infinity of $H$ will be equal to $\infty$ hence the essential spectrum of $H$ will be empty. \(\square\)

Some notations: if $\phi$ is a $B(E)$-valued Borel function on $X$ then $\phi(Q)$ is the operator of multiplication by $\phi$ on $\mathcal{H}$; if $\psi$ is a similar function on $X^*$ then $\psi(P) = \mathcal{F}^{-1} M_\psi \mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation and $M_\phi$ is the operator of multiplication by $\psi$ on $L^2(X^*) \otimes E$. Note that $V_k \psi(P) V_k^* = \psi(P + k)$.

If $\phi \in L^\infty(X)$ and $\phi \geq 0$ then it is easy to check that $w\text{-}\lim_{\alpha \to \infty} U_{\alpha} \phi(Q) U_{\alpha}^* = 0$ if and only if $s\text{-}\lim_{\alpha \to \infty} \phi(Q) U_{\alpha} = 0$ and also if and only if there is a compact neighborhood of the origin $W$ such that $\lim_{\alpha \to \infty} \int_{a + W} \phi \, dx = 0$. Then we say that $\phi$ is weakly vanishing (at infinity). See Section 6 in [GG] for further properties of this class of functions. Below $W$ is a compact neighborhood of the origin, $W = a + W$, and we denote $|M|$ the Haar measure of a set $M$.

**Lemma 5.** A positive function $\phi \in L^\infty(X)$ is weakly vanishing if and only if for any number $\lambda > 0$ the set $\Omega^\lambda = \{ x \mid \phi(x) > \lambda \}$ has the property $\lim_{\alpha \to \infty} |W_{\alpha} \cap \Omega^\lambda| = 0$.

This follows from the estimates
\[ \lambda |W_{\alpha} \cap \Omega^\lambda| \leq \int_{W_{\alpha}} \phi \, dx \leq ||\phi||_{L^\infty} |W_{\alpha} \cap \Omega^\lambda| + \lambda |W|. \]

**Proposition 6.** Let $H$ be an invertible self-adjoint operator satisfying (2) and such that $\pm H^{-1} \leq \phi(Q)$ for some weakly vanishing function $\phi$. Then $H$ has purely discrete spectrum.

Indeed, we may take $R = H^{-1}$ and then for any $f \in \mathcal{H}$ we have $|\langle f, U_{\alpha} R U_{\alpha}^* \rangle| \leq ||f||_{L^\infty} |\langle f, U_{\alpha} \phi(Q) U_{\alpha}^* \rangle|$.

**Proof of Proposition 1:** Here $X = \mathbb{R}^n$ and we identify as usual $X$ with its dual by setting $k(x) = e^{ikx}$ for $x, k \in X$. Then if $P_j = -i \partial_j$ and $P = (P_1, \ldots, P_n)$ we get $V_k P V_k^* = P + k$. To simplify notations we write $H$ for $\Delta + V + 1 + \nu$, so that $H \geq (1 - \mu) \Delta + V + 1 \geq V + 1 \geq 1$. Then observe that the local form domain of $H$ is $\mathcal{G} \equiv D(H^{1/2}) = \{ f \in \mathcal{H}^1 \mid V^{1/2} f \in L^2 \}$ where $\mathcal{H}^1$ is the first order Sobolev space. Thus $R = H^{-1} : L^2 \to \mathcal{H}$ is continuous and this implies the second part of condition (2). On the other hand, $H$ extends to a continuous bijection $\mathcal{G} \to \mathcal{G}$ whose inverse is an extension of $R$ to a continuous map $\mathcal{G} \to \mathcal{G}$. We keep the notations $H, R$ for these extensions. Clearly $V_k$ leaves invariant $\mathcal{G}$ and extends to a continuous operator on $\mathcal{G}$ and the groups of operators $\{V_k\}$ are of class $C_0$ in both spaces. Now $H_k := V_k H V_k^* = (P + k)^2 + V = H + 2kP + k^2$ in $B(\mathcal{G}, \mathcal{G})$ so if $R_k := V_k R V_k^*$ then
\[ R_k - R = R_k(H - H_k)R = -R_k(2kP + k^2)R \]
in $B(\mathcal{G}, \mathcal{G})$. Now clearly the first part of (2) is fulfilled. Finally, it suffices to show that $H^{-1} \leq \phi(Q)$ for a weakly vanishing function $\phi$. But $H \geq 1 + V_{\alpha}$ and we may take $\phi = (1 + V_{\alpha})^{-1}$ due to (1). \(\square\)

**Remark 3:** is a consequence of the next result.

**Lemma 7.** Let $\Omega \subset \mathbb{R}^n$ be a Borel set and let $\omega : \mathbb{R}^n \to \mathbb{R}$ be defined by $\omega(a) = |B_a \cap \Omega|$. If $\omega^p$ is integrable on $\Omega$ for some $p > 0$ then $\omega(a) \to 0$ as $a \to \infty$.

**Proof:** The main point is the following observation due to Hans Henrik Rugh: let $\nu$ be the minimal number of (closed) balls of radius $1/2$ needed to cover a ball of radius one; then for any $a$ there is a Borel set $A_a \subset B_a \subset \Omega$ with $|A_a| \geq \omega(a)/\nu$ such that $\omega(x) \geq \omega(a)/\nu$ if $x \in A_a$. Indeed, let $N$ be a set of $\nu$ points such that $B_a \subset \cup_{n \in N} B_{B_a(1/2)}$. If $D_a = B_a \cap B_a(1/2)$ then $\omega(a) \leq \sum_{b} |D_b \cap \Omega|$ hence there is $b(a)$ such that $A_a = D_{b(a)} \cap \Omega$ satisfies $|A_a| \geq \omega(a)/\nu$. Since $A_a$ has diameter smaller than one, for
If \( R \in B(\mathcal{H}) \) satisfies the first part of (2) we say that \( R \) is a regular operator (or \( Q \)-regular). The regularity of the resolvent of a differential operators on \( \mathbb{R}^n \) is easy to check because \( V_k P V_k^* = P + k \), cf. the proof of Proposition 1. The second part of (2) is equivalent to the existence of a factorization \( R = \psi(P)S \) with \( \psi \in C_0(X^*) \) and \( S \in B(\mathcal{H}) \). If \( X = \mathbb{R}^n \) then it suffices that the domain of \( H \) be included in some Sobolev space \( \mathcal{H}^m \) with \( m > 0 \) real. We now give an extension of Proposition 1 which is proved in essentially the same way. We assume \( X = \mathbb{R}^n \) and work with Sobolev spaces but a similar statement holds for an arbitrary \( X \): it suffices to replace the function \( \langle k \rangle^m \) which defines \( \mathcal{H}^m \) by an arbitrary weight \([GI]\) and the ball \( B_a \) by \( a + W \) where \( W \) is a compact neighborhood of the origin.

**Proposition 8.** Let \( H_0 \) be a bounded from below self-adjoint operator on \( \mathcal{H} \) with form domain equal to \( \mathcal{H}^m \) for some real \( m > 0 \) and satisfying \( \lim_{k \to 0} V_k H_0 V_k^* = H_0 \) in norm in \( B(\mathcal{H}^m, \mathcal{H}^{-m}) \). Let \( V \) be a positive locally integrable function such that \( \lim_{\lambda \to +\infty} |\{ x \in B_a | V(x) < \lambda \}| = 0 \) for each \( \lambda > 0 \). Then the self-adjoint operator \( H \) associated to the form sum \( H_0 + V \) has purely discrete spectrum.

Let \( h : X \to B(E) \) be a continuous symmetric operator valued function with \( c' |p|^{2m} \leq h(p) \leq c'' |p|^{2m} \) (as operators on \( E \)) for some constants \( c', c'' > 0 \) and all large \( p \). Let \( W : \mathcal{H}^m \to \mathcal{H}^{-m} \) be a symmetric operator such that \( W \geq -\mu h(P) - \nu \) with \( \mu < 1 \) and such that \( V_k W V_k^* \to W \) in norm in \( B(\mathcal{H}^m, \mathcal{H}^{-m}) \) as \( k \to 0 \). Then the form sum \( h(P) + W \) is bounded from below and closed on \( \mathcal{H}^m \) and the self-adjoint operator \( H_0 \) associated to it satisfies the conditions of Proposition 8.

Assume that \( m \geq 1 \) is an integer and let \( L = \sum_{\alpha, \beta} P^\alpha a_{\alpha, \beta}(Q) P^\beta : \mathcal{H}^m \to \mathcal{H}^{-m} \) where \( \alpha, \beta \) are multi-indices of length \( \leq m \) and \( a_{\alpha, \beta} \) are functions \( X \to B(E) \) such that \( a_{\alpha, \beta}(Q) \) is a continuous map \( \mathcal{H}^{m-|\beta|} \to \mathcal{H}^{m-|\alpha-\beta|} \). If \( \langle f, L f \rangle \geq \mu \| f \|_{\mathcal{H}^m}^2 - \nu \| f \|_{\mathcal{H}^{-m}}^2 \) for some \( \mu, \nu > 0 \) then \( L \) is a closed bounded from below form on \( \mathcal{H}^m \) and the self-adjoint operator \( H_0 \) associated to it verifies Proposition 8.

**Note added July 2014:** The theory can be extended to metric spaces by using the \( C^* \)-algebra introduced and studied in my paper “On the structure of the essential spectrum of elliptic operators on metric spaces”, J. Funct. Analysis 260, 1734–1765 (2011) and arXiv:1003.3454.

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**References**


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