Some properties of (C,E,P)-algebras : Overgneration and 0-order estimates.

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Abstract

We give a new definition of the so-called overgenerated rings, which are the usual tool used to define the asymptotic structure of a $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra, written as a factor space $\mathcal{M}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$. With this new definition and in the particular case of $\mathcal{E} = \mathrm{C}^\infty$, we show that a moderate element i.e. in $\mathcal{M}_{(A,\mathcal{E},\mathcal{P})}$ is negligible if and only if it satisfies the C^0-order estimate for the ideal $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$.

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1 Introduction

More than ten years ago, J.-A. Marti has introduced the structure of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras [8] which is a refinement of the Colombeau simplified algebras of new generalized functions [2, 6, 10]. These $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras are constructed on a base (pre)sheaf $\mathcal{E}$, which is usually a (pre)sheaf of algebras equipped with a topology $\mathcal{P}$ and on an asymptotic structure given by a ring $\mathcal{C}$ of generalized constants. Except in a few cases (for example [1]), the sheaf $\mathcal{E}$ is chosen to be the sheaf of smooth functions. Roughly speaking a presheaf of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras is a presheaf of factor algebras $\mathcal{M}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$, where $\mathcal{M}_{(A,\mathcal{E},\mathcal{P})}$ (resp $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$) is the (pre)sheaf of algebras of moderate elements (resp the (pre)sheaf of ideals of negligible elements) over the factor ring $\mathcal{C} = A/I_A$. (The moderateness and the negligibility are defined by the asymptotic structure given by the ring $\mathcal{C}$.)

Although these $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-type algebras have proved their efficiency to give a meaning and to solve singular differential problems, the investigations on their intrinsic properties have not yet been developed, expect first attempts concerning topology in [4] and asymptotic analysis in [3]. In this paper, after recalling for sake of self contentedness the basic notions on $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-type algebras (Section 2), we present a new definition of the so-called overgenerated rings in Section 3. This concept of overgeneration is one essential part of the theory for it allows to adapt the algebraic structure to the singularities of the problems. Taking advantage of this new definition and analogously to the theorem 1.2.3. of [6] for Colombeau simplified algebras, we show in Section 4 that a moderate element (i.e. in $\mathcal{M}_{(A,\mathcal{E},\mathcal{P})}$) is negligible (i.e. in $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$) if and only if it satisfies the C^0-order estimate of the ideal, for the case of $\mathcal{E} = \mathrm{C}^\infty$ endowed with its classical topology. This property allows radical simplification of many proofs of existence and uniqueness for differential problems.

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2 The presheaf of $(\mathcal{C},\mathcal{E},\mathcal{P})$-type algebras

We begin by recalling the algebraical construction of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras [8, 9] as improved in [3]. Let:
(1) Λ be a directed set with partial order relation $\preceq$;
(2) A be a solid subring of the ring $\mathbb{K}^\Lambda$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$): whenever $(|s_\lambda|)_\lambda \leq (|r_\lambda|)_\lambda$ for some $((s_\lambda)_\lambda,(r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times A$, that is, $|s_\lambda| \leq |r_\lambda|$ for all λ, it follows that $(s_\lambda)_\lambda \in A$;
(3) I_A be a solid ideal of A;
(4) $\mathcal{E}$ be a sheaf of $\mathbb{K}$-topological algebras over a topological space X.

Suppose that for any open set Ω in X, the topology of the algebra $\mathcal{E}(\Omega)$ is defined by a family $\mathcal{P}(\Omega)$ of seminorms such that:
(5) Whenever Ω_1, Ω_2 are two open subsets of X with $\Omega_2 \subset \Omega_1$ and ρ_2^1 is the restriction operator $\mathcal{E}(\Omega_1) \to \mathcal{E}(\Omega_2)$, then, for each $p_2 \in \mathcal{P}(\Omega_2)$, the seminorm $p_1 = p_2 \circ \rho_2^1$ extends p_2 to $\mathcal{P}(\Omega_1)$;
(6) Whenever $\Theta = (\Omega_h)_{h\in H}$ is a family of open sets in X with $\Omega = \cup_{h\in H}\Omega_h$, then, for each $p \in \mathcal{P}(\Omega)$, there exist a finite subfamily $(\Omega_i)_{1\leq i\leq n}$ of Θ and corresponding seminorms $p_i \in \mathcal{P}(\Omega_i)$, $1 \leq i \leq n$, such that

$$\forall u \in \mathcal{E}(\Omega), \quad p\,(u) \leq p_1\,(u\,|_{\Omega_1}) + \ldots + p_n(u|_{\Omega_n}).$$

Define $\mathcal{C} = A/I_A$ and $|B| = \{(|r_\lambda|)_\lambda\,,\ (r_\lambda)_\lambda \in B\}$ ($B = A$ or I_A). From (2), it follows that $|A|$ is a subset of A and that $A_+ = \{(b_\lambda)_\lambda \in A,\ \forall\lambda\in\Lambda,\ b_\lambda \geq 0\} = |A|$. The same holds for I_A. Furthermore, (2) implies also that A is a $\mathbb{K}$-algebra [3]. With these notations, set

$$\mathcal{M}(\Omega) = \mathcal{M}_{(A,\mathcal{E},\mathcal{P})}(\Omega) = \left\{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall p \in \mathcal{P}(\Omega),\ ((p(u_\lambda))_\lambda \in |A|\right\},$$
$$\mathcal{N}(\Omega) = \mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) = \left\{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall p \in \mathcal{P}(\Omega),\ (p(u_\lambda))_\lambda \in |I_A|\right\}.$$

Proposition-Definition 1 *[3, 8]*
(i) $\mathcal{M}_{(A,\mathcal{E},\mathcal{P})}$ (resp. $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$) is a sheaf of $\mathbb{K}$-subalgebras (resp. of ideals) of the sheaf $\mathcal{E}^\Lambda$ (resp. of $\mathcal{M}_{(A,\mathcal{E},\mathcal{P})}$).
(ii) The factor $\mathcal{M}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$ is a presheaf of algebras over the factor ring $\mathcal{C} = A/I_A$, with localization principle, called presheaf of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras.

Remark that, with (2), the constant sheaf $\mathcal{M}_{(A,\mathbb{K},|.|)}/\mathcal{N}_{(I_A,\mathbb{K},|.|)}$ is exactly equal to $\mathcal{C} = A/I_A$.

Notation 1 *We denote by $[(u_\lambda)_\lambda]_\mathcal{A} = [u_\lambda]_\mathcal{A}$ or $[u_\lambda]$, when no confusion may arise, the class of $(u_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{A}(\Omega)$.*

Remark 1 *We suppose in addition that $\{(a_\lambda)_\lambda \in A \mid \lim_\Lambda a_\lambda = 0\} \neq \emptyset$ and that I_A satisfies*

$$(7) \qquad I_A \subset \{(a_\lambda)_\lambda \in A \mid \lim_\Lambda a_\lambda = 0\},$$

Then there exists a canonical sheaf embedding of $\mathcal{E}$ into $\mathcal{A}$ through the morphism of algebra

$$\sigma_\Omega : \mathcal{E}\,(\Omega) \to \mathcal{A}\,(\Omega)\,, \quad f \mapsto [(f)_\lambda]\,.$$

Indeed, if $[(f)_\lambda] = 0$, we have: $\forall p \in \mathcal{P}(\Omega)$, $(p(f))_\lambda \in |I_A|$. From (7), it follows that $\forall p \in \mathcal{P}(\Omega)$, $p(f) = 0$. Thus $f = 0$.

Remark 2 *For the above algebraic considerations of this section (and specially Proposition-Definition 1), we don't need Λ to be a directed set. However, the previous remark shows the importance of this assumption in order to get non trivial extensions.*

3 Overgenerated rings

In almost all works using $(\mathcal{C},\mathcal{E},\mathcal{P})$-type algebras (see, for examples, [5, 8, 9]), the ring A and the ideal I_A are constructed as polynomially overgenerated rings and satisfy the assumption (7) of Remark 1. For $(a_\lambda)_\lambda, (b_\lambda)_\lambda \in \mathbb{R}^\Lambda$, we shall use the following notation

$$a_\lambda \ll b_\lambda \Leftrightarrow \exists \lambda_0 \in \Lambda,\ \forall \lambda \preceq \lambda_0 : a_\lambda \leq b_\lambda.$$

We first give an improved definition of the overgeneration.

Proposition-Definition 2 ***(Polynomially overgenerated rings)*** *Consider $\mathcal{B}_0$ a family of nets in $(\mathbb{R}_+^*)^\Lambda$ and $\mathcal{B}$ the subset of elements in $(\mathbb{R}_+^*)^\Lambda$ obtained as rational functions with coefficients in $\mathbb{R}_+^*$ of elements in $\mathcal{B}_0$ as variables. Set*

$$A_\mathcal{B} = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \exists (b_\lambda)_\lambda \in \mathcal{B} : |a_\lambda| \ll b_\lambda \right\}.$$

The set $A_\mathcal{B}$ is a solid subring of $\mathbb{K}^\Lambda$, called the ring (polynomially) overgenerated *by $\mathcal{B}_0$ (or by $\mathcal{B}$).*

Usually, the set $\mathcal{B}_0$ is finite and given by the problem itself. (See [5, 9].) The term polynomially refers to the fact that the growth of elements of $A_\mathcal{B}$ is at most polynomial with respect to the elements of $\mathcal{B}_0$. This polynomial overgeneration is sufficient for the non linearities considered in the quoted references, but, for example, does not permit to obtain $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras stable by exponential.

Remark 3 *With this definition $\mathcal{B}$ is stable by inverse. In many practical cases and, for example, in the case of Colombeau simplified algebras, which are a particular case of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras, $\mathcal{B}$ is exactly the set of invertible elements of the ring of generalized constants.*

As a "canonical" ideal of $A_\mathcal{B}$, one usually choose

$$I_\mathcal{B} = \left\{ (a_\lambda)_\lambda \in \mathbb{K}^\Lambda \mid \forall (b_\lambda)_\lambda \in \mathcal{B} : |a_\lambda| \ll b_\lambda \right\}. \tag{8}$$

A routine checking shows that $I_\mathcal{B}$ is a solid ideal of $A_\mathcal{B}$. We shall always assume the existence of $(r_\lambda)_\lambda \in \mathcal{B}$ such that $\lim_\Lambda r_\lambda = 0$, in order to have (7) and, thus, the canonical embedding of $\mathcal{E}(\Omega)$ into $\mathcal{A}(\Omega)$. (This assumption is satisfied in all practical applications.) We denote by $\mathcal{C}_\mathcal{B} = A_\mathcal{B}/I_\mathcal{B}$ the corresponding ring of generalized numbers.

4 An Austrian Lemma in $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras

We take here $\mathcal{E} = \mathrm{C}^\infty$ with $X = \mathbb{R}^d$, $\mathcal{P}(\mathbb{R}^d)$ being the usual family of seminorms $(P_{K,l})_{K,l}$ defined by

$$P_{K,l}(u) = \sup_{|\alpha|\leq l} P_{K,\alpha}(u) \ \text{ with } P_{K,\alpha}(u) = \sup_{x\in K} |\partial^\alpha u(x)|,\ \ K \subset\subset \Omega,\ \ l = 0 \text{ or } l = 1.$$

and $\partial^\alpha = \dfrac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial z_1^{\alpha_1}\ldots\partial z_d^{\alpha_d}}$ for $z = (z_1,\ldots,z_d) \in \Omega$, $l \in \mathbb{N}$, $\alpha = (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d$. We consider a ring of generalized constants $\mathcal{C} = A_\mathcal{B}/I_\mathcal{B}$ overgenerated as stated in Proposition-Definition 2. The ideal $I_\mathcal{B}$ is defined by (8) and the set of indices Λ is assumed to be left filtering. Recall that

$$\mathcal{M}(\mathbb{R}^d) = \mathcal{M}_{(A_\mathcal{B},\mathrm{C}^\infty,\mathcal{P})}(\mathbb{R}^d) = \{(u_\lambda)_\lambda \in \mathrm{C}^\infty(\Omega)^\Lambda : \forall p \in \mathcal{P}(\mathbb{R}^d),\ (p(u_\lambda))_\lambda \in |A_\mathcal{B}|\},$$
$$\mathcal{N}(\mathbb{R}^d) = \mathcal{N}_{(I_\mathcal{B},\mathrm{C}^\infty,\mathcal{P})}(\mathbb{R}^d) = \{(u_\lambda)_\lambda \in \mathrm{C}^\infty(\Omega)^\Lambda : \forall p \in \mathcal{P}(\mathbb{R}^d),\ (p(u_\lambda))_\lambda \in |I_\mathcal{B}|\}.$$

Proposition 3 *Assume that there exists* $(a_\lambda)_\lambda \in \mathcal{B}$ *with* $\lim_\Lambda a_\lambda = 0$. *Then* $(u_\lambda)_\lambda \in \mathcal{M}_{(A_\mathcal{B},\mathrm{C}^\infty,\mathcal{P})}(\mathbb{R}^d)$ *is in* $\mathcal{N}_{(I_\mathcal{B},\mathrm{C}^\infty,\mathcal{P})}(\mathbb{R}^d)$ *if, and only if,*

$$\forall K \subset\subset \mathbb{R}^2, \quad P_{K,0}(u_\lambda) \in |I_\mathcal{B}|.$$

Remark 4 *We recall that the set* $\mathcal{B}$ *is stable by inverse, which could be assumed for all the* $(\mathcal{C},\mathcal{E},\mathcal{P})$*-algebras considered up to now in the literature. Notice also that one has the existence of* $(a_\lambda)_\lambda \in \mathcal{B}$ *such that* $\lim_\Lambda a_\lambda = 0$ *in all practical cases.*

Proof. Take $K \subset\subset \Omega$. We have to prove that $\forall l \in \mathbb{N}$, $P_{K,l}(u_\lambda) \in |I_\mathcal{B}|$. By induction, it suffices to prove that $P_{K,0}(u_\lambda) \in |I_\mathcal{B}|$ implies $P_{K,1}(u_\lambda) \in |I_\mathcal{B}|$. In fact, this amounts to show that $P_{K,0}(u_\lambda) \in |I_\mathcal{B}|$ implies $P_{K,0}((\partial/\partial x_i)\, u_\lambda) \in |I_\mathcal{B}|$ for $i \in \{1,\ldots,d\}$. Set $\delta = \min(1, \mathrm{dist}(K,\partial\Omega))$ and $L = K + \overline{B}(0,\delta/2)$. We have $K \subset\subset L \subset\subset \Omega$. Since $(u_\lambda)_\lambda \in \mathcal{M}(\mathbb{R}^d)$, there exists $(\beta_\lambda)_\lambda \in \mathcal{B}$ such that

$$\exists \lambda_0 \in \Lambda, \forall \lambda \preceq \lambda_0, \quad P_{L,2}(u_\lambda) \leq \beta_\lambda.$$

We may assume that $\lim_\Lambda \beta_\lambda = +\infty$. Indeed, for any $(\beta_\lambda)_\lambda \in \mathcal{B}$, we set $\beta'_\lambda = a_\lambda^{-1} + \beta_\lambda$ where $(a_\lambda)_\lambda \in \mathcal{B}$ is such that $\lim_\Lambda a_\lambda = 0$. Thus, $\lim_\Lambda \max \beta'_\lambda = +\infty$. Take any $(c_\lambda)_\lambda \in \mathcal{B}$ and define $b_\lambda = a_\lambda c_\lambda / (a_\lambda + c_\lambda)$. Clearly we have $b_\lambda \in |A_\mathcal{B}|$, $b_\lambda \leq c_\lambda$ and $b_\lambda \leq a_\lambda$. Thus $\lim_\Lambda b_\lambda = 0$. Let $(e_i)_{1\leq i\leq d}$ be the canonical base of $\mathbb{R}^d$. There exists λ_1 such that, for all $x \in K$, $x + b_\lambda \beta_\lambda^{-1} e_i \in L$ when $\lambda \preceq \lambda_1$, since $\lim_\Lambda \beta_\lambda^{-1} = 0$. By the Taylor theorem we have, for $x \in K$,

$$u_\lambda\left(x + b_\lambda\beta_\lambda^{-1}e_i\right) = u_\lambda(x) + b_\lambda\beta_\lambda^{-1}\frac{\partial}{\partial x_i}u_\lambda(x) + \frac{1}{2}\left(b_\lambda\beta_\lambda^{-1}\right)^2\frac{\partial^2}{\partial x_i^2}u_\lambda\left(x + \theta b_\lambda\beta_\lambda^{-1}e_i\right)$$

with $0 \leq \theta \leq 1$. It follows that

$$\frac{\partial}{\partial x_i}u_\lambda(x) = b_\lambda^{-1}\beta_\lambda\left(u_\lambda\left(x + b_\lambda\beta_\lambda^{-1}e_i\right) - u_\lambda(x)\right) - \frac{1}{2}\left(b_\lambda\beta_\lambda^{-1}\right)\frac{\partial^2}{\partial x_i^2}u_\lambda\left(x + \theta b_\lambda\beta_\lambda^{-1}e_i\right).$$

Thus

$$\left|\frac{\partial}{\partial x_i}u_\lambda(x)\right| \leq 2b_\lambda^{-1}\beta_\lambda P_{L,0}(u_\lambda) + \frac{1}{2}b_\lambda\beta_\lambda^{-1}P_{L,2}(u_\lambda) \leq 2b_\lambda^{-1}\beta_\lambda P_{L,0}(u_\lambda) + \frac{1}{2}b_\lambda$$

for $\lambda \preceq \lambda_2$ with $\lambda_2 \preceq \lambda_j$, $0 \leq j \leq 1$. As $P_{K,0}(u_\lambda) \in |I_\mathcal{B}|$, we have $P_{L,0}(u_\lambda) \leq (1/4)b_\lambda^2\beta_\lambda^{-1} \in \mathcal{B}$ for $\lambda \preceq \lambda_3$ for some λ_3. Thus

$$\left|\frac{\partial}{\partial x_i}u_\lambda(x)\right| \leq b_\lambda \quad \text{for } \lambda \preceq \lambda_4 \text{ with } \lambda_4 \preceq \lambda_j,\ 3 \leq j \leq 4.$$

Finally, $P_{K,0}((\partial/\partial x_i)\, u_\lambda) \in |I_\mathcal{B}|$ as expected. ■

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