On the Concept of Time-Frequency Distributions Based on Complex-Lag Moments

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### Work context: Analysis of Signals with complex Time-Frequency behaviour

Signals issued from diverse physical phenomena:

- Very challenging to analyse in Time-Frequency (TF) domain
- Signals composed of several TF components characterized by various non-linear contents
  - inner interferences due to the TF non-linearity
  - cross terms due to the multi-component structures
Outline

1. Time-Frequency Distribution Based on Complex-Lag Arguments
2. Complex-Lag Moment for Multi-Component Signals
3. Results
4. Conclusion
1. Time-Frequency Distribution Based on Complex-Lag Arguments

Concept of complex-lag distributions

- A way for inner interferences reduction with respect of Wigner distribution.

\[ s(t) = A e^{j \phi(t)} \]

\[ WVD(t, \omega) = \tilde{\mathcal{F}}_{\tau} \left[ s(t + \frac{\tau}{2}) s^*(t - \frac{\tau}{2}) \right] \]

\[ WVD(t, \omega) = \delta \left( \omega - \phi'(t) \right) *_{\omega} \tilde{\mathcal{F}}_{\tau} \left[ e^{j Q_{wv}(t, \tau)} \right] \]

with Spread Factor \( Q_{wv}(t, \tau) = \phi^{(3)}(t) \frac{\tau^3}{2^2 3!} + \phi^{(5)}(t) \frac{\tau^5}{2^4 5!} + \ldots \)
Consideration of lags on imaginary axis

\[
CTD(t, \omega) = \mathcal{F}_\tau \left[ s(t + \frac{\tau}{4})s^*(t - \frac{\tau}{4})s^{-j}(t + j\frac{\tau}{4})s^{j}(t - j\frac{\tau}{4}) \right]
\]

\[
CTD(t, \omega) = \delta \left( \omega - \phi'(t) \right) \ast \omega \mathcal{F}_\tau \left[ e^{jQ_{ct}(t, \tau)} \right]
\]

with Spread Factor \( Q_{ct}(t, \tau) = \phi^{(5)}(t) \frac{\tau^5}{4^4 \cdot 5!} + \phi^{(9)}(t) \frac{\tau^9}{4^8 \cdot 9!} + \ldots \)

- Cauchy’s integral formula ⇒ computation of the $K^{th}$ order derivative of $\phi$ at instant $t$ as

$$
\phi^{(K)}(t) = \frac{K!}{2\pi j} \oint_{\gamma} \frac{\phi(z)}{(z - t)^{K+1}} \, dz
$$

$$
\phi^{(K)}(t) = \frac{K!}{N \tau^K} \sum_{p=0}^{N-1} \phi \left( t + \tau e^{j \frac{2\pi p}{N}} \right) e^{-j \frac{2\pi pK}{N}} + \varepsilon
$$
\[ GCD^K_N[s](t, \omega) = \mathcal{F}_\tau \left[ GCM^K_N[s](t, \tau) \right] \]
\[ = \delta \left( \omega - \phi^K(t) \right) \ast \omega \mathcal{F}_\tau \left[ e^{jQ(t, \tau)} \right] \]

where \[ GCM^K_N[s](t, \tau) = \prod_{p=0}^{N-1} s^{N-K} \omega_{N,p}^{N-K} \left( t + \omega_{N,p} \sqrt{\tau \frac{K!}{N}} \right) \]

with \( \omega_{N,p} = e^{j2\pi p/N} \)

and \( Q(t, \tau) = N \sum_{p=1}^{+\infty} \phi(Np+K)(t) \frac{Np+1}{(Np+K)!} \left( \frac{K!}{N} \right)^{Np+1} \)

“Inner interferences reduction” property of GCD

Increasing parameter \( N \) (\( N \leftrightarrow \) number of complex lags) leads to a reduction of Spread Factor and so attenuation of inner interferences.
Spread Factors in some Time-Frequency Distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Spread factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$STFT \Leftrightarrow GCD_1^1$</td>
<td>$Q(t, \tau) = \phi^{(2)}(t)\frac{\tau^2}{2!} + \phi^{(3)}(t)\frac{\tau^3}{3!} + \ldots$</td>
</tr>
<tr>
<td>$WVD \Leftrightarrow GCD_2^1$</td>
<td>$Q(t, \tau) = \phi^{(3)}(t)\frac{\tau^3}{2^2 \cdot 3!} + \phi^{(5)}(t)\frac{\tau^5}{2^4 \cdot 5!} + \ldots$</td>
</tr>
<tr>
<td>$CTD \Leftrightarrow GCD_4^1$</td>
<td>$Q(t, \tau) = \phi^{(5)}(t)\frac{\tau^5}{4^4 \cdot 5!} + \phi^{(9)}(t)\frac{\tau^9}{4^8 \cdot 9!} + \ldots$</td>
</tr>
<tr>
<td>$GCD_6^1$</td>
<td>$Q(t, \tau) = \phi^{(7)}(t)\frac{\tau^7}{6^6 \cdot 7!} + \phi^{(13)}(t)\frac{\tau^{13}}{6^{12} \cdot 13!} + \ldots$</td>
</tr>
</tbody>
</table>

Example for a test signal $s(t) = e^{j(6 \cos(\pi t) + \frac{2}{3} \cos(3\pi t) + \frac{4}{3} \cos(5\pi t))}$

Inner interferences for $WVD$, $CTD$ and $GCD_6^1$
GCD ↔ Time-Phase Derivative Representation

- GCD offers representations for derivatives of any order $K$ of the instantaneous phase law of the signal.
- Example for a $4^{th}$ order polynomial phase signal

The theoretical derivatives (top) are correctly represented by GCDs of corresponding order (bottom)
Consideration of Cross Terms

- The GCM has a complicate non-linear form
- Multi-component case ⇒ high level of cross terms very difficult to calculate, as follows:

\[ s(t) = s_1(t) + s_2(t) = e^{j\phi_1(t)} + e^{j\phi_2(t)} \]

\[
GCM^K_N [s_1 + s_2] (t, \tau) = \prod_{p=0}^{N-1} \left[ s_1 \left( t + \omega_N, p \sqrt{\frac{K!}{N}} \right) + s_2 \left( t + \omega_N, p \sqrt{\frac{K!}{N}} \right) \right] \omega_N^{N-K, p}
\]

\[
= e^{j\phi_1^K(t) \tau + jQ_1(t, \tau)} + e^{j\phi_2^K(t) \tau + jQ_2(t, \tau)} + CT_{\phi_1, \phi_2}(N, K)
\]
Case of a two-component signal: cross terms have strong energetic level and corrupt the visibility of auto-terms.
Reduction of Cross Terms: “Multi-lag sets” principle

- Auto-terms don’t depend on number of lags $N$
First step: Compute GCMs for several lags sets $N_i$
$\Rightarrow$ auto terms keep the same structures
$\Rightarrow$ cross terms depend of $N$ and are consequently differently located

Second step: Sum the different GCMs

\[
mlsGCM_{\{N_i\}}^K [s] (t, \tau)
\]

\[
= \sum_{i=1}^{P} GCM_{N_i}^K [s] (t, \tau)
\]

\[
= P \sum_{l} e^{j \phi_l^{(K)}(t)\tau + jQ_l(t,\tau)} + \sum_{i=1}^{P} CT_{\phi_l}(N_i, K)
\]

Highlighting of auto-terms / Decreasing of cross-terms
Case of a two-component signal: cross terms level reduction using $mlsGCD$
3. Results

GCD and mlsGCD can perform in many applications dealing with complex TF modulations.

Ex.1: Removal of quasi-stationary corrupting components

Separation of a Sinusoidal Frequency Modulation corrupted by chirp, tone and noise.
Ex.2: Application to Transient signals

- Infinite derivability of transient signals phase law
- Extraction, via derivability property, of transient impulses corrupted by both noise and coherent perturbations

Noise reduction by \textit{mlsGCD}

Transient impulses extraction
4. Conclusion

We achieved...

Distribution able to:
1. focus on arbitrary derivative of phase laws
2. in multi-component context
3. with highly concentrated and cross-terms reduced representations

Prospects
- Signal-dependent choice of lag sets
- 2D extension of complex lag concept
Thank you for your Attention!!!