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GLOBAL EXISTENCE RESULTS FOR THE ANISOTROPIC BOUSSINESQ SYSTEM IN DIMENSION TWO

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ABSTRACT. We study the global existence issue for the two-dimensional Boussinesq system with *horizontal* viscosity in only one equation. We first examine the case where the Navier-Stokes equation *with no vertical viscosity* is coupled with a transport equation. Second, we consider a coupling between the classical two-dimensional incompressible Euler equation and a transport-diffusion equation with diffusion *in the horizontal direction only.* For the both systems and for arbitrarily large data, we construct global weak solutions *à la Leray.* Next, we state global well-posedness results for more regular data. Our results strongly rely on the fact that the diffusion occurs in a direction perpendicular to the buoyancy force.

1. INTRODUCTION

The Boussinesq system describes the influence of the convection (or convection-diffusion) phenomenon in a viscous or inviscid fluid. It is used as a toy model for geophysical fluids whenever rotation and stratification play an important role (see for example J. Pedlosky's book [23]). In the two-dimensional case, the Boussinesq system reads:

$$(B_{\kappa,\nu}) \quad \begin{cases} \partial_t\theta + u\cdot\nabla\theta - \kappa\Delta\theta = 0 \\ \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla\Pi = \theta\,e_2 \quad \text{with} \quad e_2 = (0,1), \\ \operatorname{div} u = 0. \end{cases}$$

Above, $u = u(t,x)$ denotes the velocity vector-field and $\theta = \theta(t,x)$ is a scalar quantity such as the concentration of a chemical substance or the temperature variation in a gravity field, in which case $\theta\,e_2$ represents the buoyancy force. The nonnegative parameters κ and ν denote respectively the molecular diffusion and the viscosity. In order to simplify the presentation, we restrict ourselves to the whole plan case (that is the space variable x describes the whole $\mathbb{R}^2$) and focus on the evolution for positive times (i.e. $t \in \mathbb{R}_+$).

In the case where both κ and ν are positive, classical methods allow to establish the global existence of regular solutions (see for example [6, 18]). On the other hand, if $\kappa = \nu = 0$ then constructing global unique solutions for some nonconstant θ_0 is a challenging open problem (even in the two-dimensional case) which has many similarities with the global existence problem for the three-dimensional incompressible Euler equations.

The intermediate situation where the diffusion acts only on one of the equations has been investigated in a number of recent papers. Under various regularity assumptions on the initial data, it has been shown that for arbitrarily large initial data, systems $(B_{\kappa,0})$ with $\kappa > 0$ and $(B_{0,\nu})$ with $\nu > 0$ admit a global unique solution (see for example [1, 7, 13, 14, 15, 19, 20]).

In the present paper, we aim at making one more step toward the study of the system with $\kappa = \nu = 0$ by assuming that the diffusion or the viscosity occurs in the horizontal direction and in one of the equations only. More precisely, we want to consider the following two systems:

$$(1) \quad \begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0 \\ \partial_t u + u\cdot\nabla u - \nu\partial_1^2 u + \nabla\Pi = \theta\,e_2 \\ \operatorname{div} u = 0 \end{cases}$$

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and

$$(2) \qquad \begin{cases} \partial_t \theta + u \cdot \nabla \theta - \kappa \partial_1^2 \theta = 0 \\ \partial_t u + u \cdot \nabla u + \nabla \Pi = \theta e_2 \\ \operatorname{div} u = 0. \end{cases}$$

Let us stress that the anisotropic viscosity or diffusion assumptions are consistent with the study of geophysical fluids. It turns out that, in certain regimes and after suitable rescaling, the vertical viscosity (or diffusion) is negligible with respect to the horizontal viscosity (or diffusion) (for more details, one may refer to [11]). For the standard three-dimensional incompressible Navier-Stokes equations, the first mathematical results concerning anisotropic viscosity have been obtained in [10, 22].

On the one hand, it is clear that small variations over the classical methods for solving quasi-linear hyperbolic systems would give local well-posedness for Systems (1) and (2) with initial data in Sobolev spaces with suitably large index. On the other hand, since diffusion occurs in only one direction and one equation, it is not obvious that those solutions are actually global. The present paper is dedicated to the study of global existence for the initial value problem associated to Systems (1) and (2) with (possibly) large initial data.

In order to state our main result pertaining to System (1), let us introduce the set $\sqrt{L}$ of those functions f which belong to every space L^p with $2 \le p < \infty$ and satisfy

$$(3) \qquad \|f\|_{\sqrt{L}} := \sup_{p \ge 2} \frac{\|f\|_{L^p}}{\sqrt{p-1}} < \infty.$$

Theorem 1. *Let $s \in]1/2, 1]$. For all function $\theta_0 \in H^s \cap L^\infty$ and divergence free vector-field $u_0 \in H^1$ with vorticity $\omega_0 := \partial_1 v_0^2 - \partial_2 v_0^1$ in $\sqrt{L}$, System (1) with data (θ_0, u_0) admits a unique global solution (θ, u) such that[1] $\theta \in \mathcal{C}_w(\mathbb{R}_+; L^\infty) \cap \mathcal{C}(\mathbb{R}_+; H^{s-\varepsilon})$ for all $\varepsilon > 0$ and*

$$u \in \mathcal{C}_w(\mathbb{R}_+; H^1), \quad \omega \in L_{loc}^\infty(\mathbb{R}_+; \sqrt{L}) \quad and \quad \nabla u \in L_{loc}^2(\mathbb{R}_+; \sqrt{L}).$$

Remark 1. The assumption that $\theta_0 \in H^s(\mathbb{R}^2)$ for some $s > \frac{1}{2}$ is needed for uniqueness only. It turns out that for less regular initial data one can construct finite energy global weak solutions to System (1), in the spirit of those which have been obtained by J. Leray for the standard Navier-Stokes equation in his pioneering paper [21].

We shall also establish a global well-posedness results for smooth initial data.

Let us now state our main result pertaining to System (2):

Theorem 2. *Let $1 < s < \frac{3}{2}$ and $\theta_0 \in H^1$ such that $|\partial_1|^s \theta_0 \in L^2$. Let u_0 be a divergence free vector-field with coefficients in H^1 and vorticity ω_0 in L^∞. Then System (2) with initial data (θ_0, u_0) admits a global unique solution (θ, u) in $\mathcal{C}_w(\mathbb{R}_+; H^1)$ such that, in addition,*

$$\theta \in L^\infty(\mathbb{R}_+; H^1), \quad |\partial_1|^s \theta \in L^\infty(\mathbb{R}_+; L^2), \quad \omega \in L_{loc}^\infty(\mathbb{R}_+; L^\infty)$$
$$\partial_1 \theta \in L^2(\mathbb{R}_+; H^1), \quad |\partial_1|^{1+s} \theta \in L_{loc}^2(\mathbb{R}_+; L^2).$$

Notation 1. In the above statement, operator $|\partial_1|^\sigma$ is defined as follows:

$$|\partial_1|^\sigma f(x) := \frac{1}{4\pi^2} \int e^{ix \cdot \xi} |\xi_1|^\sigma \widehat{f}(\xi) \, d\xi.$$

Remark 2. Further in the paper, we shall also state the global existence of finite energy weak solutions corresponding to less regular initial data (see Theorem 10).

Remark 3. Like in [15], in all the statements pertaining to System (2), one may replace the assumption that $u_0 \in L^2$ (which is slightly restrictive since in the case $\omega_0 \in L^1$ it implies that $\int_{\mathbb{R}^2} \omega_0 \, dx = 0$) by $(u_0 - \sigma) \in L^2$ for some fixed smooth stationary radial solution σ to Euler equations. For the sake of simplicity however, we here restrict ourselves to the case where u_0 is in L^2.

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[1]In all the paper, we agree that if X is a reflexive Banach space, and $I \subset \mathbb{R}$, an interval then $\mathcal{C}_w(I; X)$ stands for the set of weakly continuous functions on I with values in X.

Let us emphasize that, in contrast with the previous studies devoted to the Boussinesq system, all the results presented here strongly rely on the fact that the buoyancy force is *vertical*. As a matter of fact, it is not clear at all that if the direction of the force is changed then having horizontal diffusion allows to improve significantly the results compared to the case $\kappa = \nu = 0$.

Let us now briefly explain where the direction of the buoyancy force comes into play, and give an insight of the main arguments that we used in the proofs. In the case of System (1), the vorticity $\omega := \partial_1 u^2 - \partial_2 u^1$ satisfies

$$\partial_t \omega + u \cdot \nabla \omega - \nu \partial_1^2 \omega = \partial_1 \theta,$$

from which we easily get (at least formally)

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu\|\partial_1\omega\|_{L^2}^2 = -\int_{\mathbb{R}^2} \theta\,\partial_1\omega\,dx.$$

Taking advantage of the Young inequality and of the fact that $\|\theta(t)\|_{L^2} \leq \|\theta_0\|_{L^2}$, it is thus possible to get a bound for ω in $L_{loc}^\infty(\mathbb{R}_+; L^2)$. In fact, it turns out that similar arguments enable us to bound stronger norms of the solution so that it will be possible to prove the global existence part of Theorem 1. As the velocity field that we have constructed fails to be Lipschitz, proving uniqueness requires our using losing estimates for transport or transport-diffusion equations in the spirit of [3, 12, 13, 14].

If we consider System (2) then the vorticity equation reduces to

$$\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta,$$

so that one may write

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 = \int \partial_1\theta\,\omega\,dx.$$

Now, it turns out that the temperature equation in System (2) provides us with the following bound:

$$\|\theta(t)\|_{L^2}^2 + 2\kappa\int_0^t \|\partial_1\theta(\tau)\|_{L^2(\mathbb{R}^2)}^2 d\tau \leq \|\theta_0\|_{L^2}^2.$$

Therefore, using Young's inequality, it is still possible to get a global control on ω in $L_{loc}^\infty(\mathbb{R}_+; L^2)$. In order to get a global result *with uniqueness* however, we have to consider initial data with much more regularity. Indeed, we have to observe that System (2) contains the Euler system as a particular case (just take $\theta \equiv 0$) so that, according to Yudovich result in [24] one can barely expect to have uniqueness if the vorticity is not in L^∞. Now, from the vorticity equation, it is clear that bounding the vorticity in L^∞ requires that $\partial_1\theta$ is in $L_{loc}^1(\mathbb{R}_+; L^\infty)$. If we assume that $\theta_0 \in H^1$) then we shall be able to prove that the horizontal smoothing effect ensures that $\nabla\theta$ is in $L_{loc}^2(\mathbb{R}_+; H^1)$. As the space H^1 fails to be embedded in L^∞ however, we will have to assume even more regularity in the horizontal direction, and to check that this additional regularity is preserved for all positive time. These plain considerations explain the assumptions made in the statement of Theorem 2. As for uniqueness, it will follow from adaptations of the Yudovich method in [24].

The paper unfolds as follows: section 2 is devoted to the study of System (1) whereas Section 3 deals with System (2). A few technical lemmas have been postponed in the final section of the paper (in particular losing a priori estimates for transport equation with anisotropic diffusion).

As usual, we agree that C denotes a harmless positive constant, the meaning of which is clear from the context.

2. The case of an horizontal viscosity

This part is devoted to the study of the initial value problem for System (1) under various regularity hypotheses. We aim at getting global results for possibly large data.

More precisely, in the first subsection, we prove that for any data $(\theta_0, u_0) \in L^2 \times H^1$ with $\operatorname{div} u_0 = 0$, System (1) admits at least one global solution with finite energy. The next subsection is devoted to a local well-posedness result for smooth data, together with a continuation criterion involving the L^∞ norm of $(\theta, \nabla u)$. In subsection 2.3, we state a sharper continuation criterion involving a *weaker* norm of the velocity which is (formally) controlled for all time by System

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(1). This will enable us to state that the system is globally well-posed in Sobolev spaces with large enough index (see Theorem 5). The last two subsections are devoted to the proof of our global existence and uniqueness result for rough data (namely Theorem 1).

2.1. Global weak solutions.
In order to motivate our statement, let us first write out the "natural" energy estimates associated to System (1).

On the one hand, because $\operatorname{div} u = 0$, we have

$$(4) \qquad \|\theta(t)\|_{L^2} \leq \|\theta_0\|_{L^2}.$$

On the other hand, taking the L^2 inner product of the velocity equation with u, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_1 u\|_{L^2}^2\, d\tau\right) \leq \|\theta\|_{L^2}\|u\|_{L^2}.$$

Using Gronwall lemma and (4), we thus get

$$(5) \qquad \|u(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_1 u\|_{L^2}^2\, d\tau \leq \left(\|u_0\|_{L^2} + t\,\|\theta_0\|_{L^2}\right)^2.$$

Let us stress the fact that the above energy bounds imply that all the components of ∇u except $\partial_2 u^1$ are smoothed out for positive time. Indeed, combining the $L^2_{loc}(\mathbb{R}_+; L^2)$ bound for $\partial_1 u$ which is available from (5) with the fact that $\operatorname{div} u = 0$ ensures that $\partial_1 u^1$ $\partial_1 u^2$ and $\partial_2 u^2$ are in $L^2_{loc}(\mathbb{R}_+; L^2)$. However, as the last component $\partial_2 u^1$ is unlikely to be bounded in $L^2_{loc}(\mathbb{R}_+; L^2)$ if no stronger assumption, it is not clear that one may construct global weak solutions for L^2 data (in contrast with the standard Navier-Stokes equations [21] or with the Boussinesq system with isotropic viscosity [13]).

This induces us to consider initial velocity fields in H^1. Now, in order to get a global bound for the H^1 norm of the velocity, one may consider the vorticity equation

$$(6) \qquad \partial_t \omega + u \cdot \nabla \omega - \nu \partial_1^2 \omega = \partial_1 \theta.$$

Combining an energy method with Young's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu\|\partial_1\omega\|_{L^2}^2 = -\int \theta\,\partial_1\omega\,dx \leq \frac{\nu}{2}\|\partial_1\omega\|_{L^2}^2 + \frac{1}{2\nu}\|\theta\|_{L^2}^2.$$

Therefore,

$$\frac{d}{dt}\|\omega\|_{L^2}^2 + \nu\|\partial_1\omega\|_{L^2}^2 \leq \frac{1}{\nu}\|\theta\|_{L^2}^2,$$

whence, according to (4),

$$(7) \qquad \|\omega(t)\|_{L^2}^2 + \nu\int_0^t \|\partial_1\omega\|_{L^2}^2\, d\tau \leq \|\omega_0\|_{L^2}^2 + \frac{t}{\nu}\|\theta_0\|_{L^2}^2.$$

In short, one may formally bound u in $L^\infty_{loc}(\mathbb{R}_+; H^1)$ and θ in $L^\infty(\mathbb{R}_+; L^2)$ which leads to the following statement.

Theorem 3. *Let $u_0 \in H^1$ be a divergence free vector-field and $\theta_0 \in L^2$. System (1) has a unique global solution (θ, u) such that[2]*

$$(8) \qquad \theta \in \mathcal{C}_b(\mathbb{R}_+; L^2), \quad u \in \mathcal{C}_w(\mathbb{R}_+; H^1) \quad and \quad u^2 \in L^2_{loc}(\mathbb{R}_+; H^2).$$

Proof: It is only a matter of making the above computations rigorous. For that, one may for instance use the Friedrichs method: define the spectral cut-off J_n by

$$\widehat{J_n f}(\xi) = 1_{[0,n]}(|\xi|)\widehat{f}(\xi), \qquad n \geq 1,$$

and solve the following ODE in the space $L_n^2 := \{f \in (L^2(\mathbb{R}^2))^3 \,/\, \operatorname{supp}\widehat{f} \subset B(0,n)\}$:

$$(9) \qquad \begin{cases} \partial_t\theta + J_n \operatorname{div}(J_n\theta\, J_n u) = 0, \\ \partial_t u + \mathcal{P}J_n \operatorname{div}(\mathcal{P}J_n u \otimes \mathcal{P}J_n u) - \nu\partial_1^2\mathcal{P}J_n u = \mathcal{P}J_n(\theta e_2), \\ (\theta, u)|_{t=0} = J_n(\theta_0, u_0). \end{cases}$$

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[2] We agree that if X is a Banach space, and $I \subset \mathbb{R}$, an interval then $\mathcal{C}_b(I; X)$ stands for the set of continuous bounded functions on I with values in X.

From the Cauchy-Lipschitz theorem, we get a unique maximal solution (θ_n, u_n) in $\mathcal{C}^1([0, T_n^*[; L_n^2)$. Because $J_n^2 = J_n$, $\mathcal{P}^2 = \mathcal{P}$ and $J_n \mathcal{P} = \mathcal{P} J_n$, we discover that $(\theta_n, \mathcal{P} u_n)$ and $(J_n \theta_n, J_n u_n)$ are also solutions. By uniqueness, we thus have $\mathcal{P} u_n = u_n$ (i. e. $\operatorname{div} u_n = 0$), $J_n u_n = u_n$ and $J_n \theta_n = \theta_n$. Therefore,

$$(10) \qquad \begin{cases} \partial_t \theta_n + J_n \operatorname{div}(\theta_n u_n) = 0, \\ \partial_t u_n + \mathcal{P} J_n \operatorname{div}(u_n \otimes u_n) - \nu \partial_1^2 u_n = \mathcal{P} J_n(\theta_n e_2), \\ \operatorname{div} u_n = 0. \end{cases}$$

As Operators J_n and $\mathcal{P} J_n$ are orthogonal projectors for the L^2 inner product, the above formal calculations remain unchanged. Therefore, we still have as before

$$(11) \qquad \|\theta_n(t)\|_{L^2}^2 = \|J_n \theta_0\|_{L^2}^2 \leq \|\theta_0\|_{L^2}^2,$$

$$(12) \qquad \|u_n(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_1 u_n\|_{L^2}^2 \, d\tau \leq \left(\|u_0\|_{L^2} + t \|\theta_0\|_{L^2} \right)^2.$$

This implies that (θ_n, u_n) remains bounded in L_n^2 for finite time, whence $T_n^* = +\infty$.

Next, applying the curl operator to $(10)_2$, we get

$$\partial_t \omega_n + J_n(u_n \cdot \nabla \omega_n) - \nu \partial_1^2 \omega_n = \partial_1 \theta_n \quad \text{with} \quad \omega_n = \partial_1 u_n^2 - \partial_2 u_n^1.$$

Arguing as for proving (7), we thus get

$$(13) \qquad \|\omega_n(t)\|_{L^2}^2 + \nu \int_0^t \|\partial_1 \omega_n\|_{L^2}^2 \, d\tau \leq \|\omega_0\|_{L^2}^2 + \frac{t}{\nu} \|\theta_0\|_{L^2}^2.$$

This implies that $(\omega_n)_{n \in \mathbb{N}}$ is bounded in $L_{loc}^\infty(\mathbb{R}_+; L^2)$. Now, it is well known that the divergence-free property entails that

$$\|\nabla u_n\|_{L^2} = \|\omega_n\|_{L^2} \quad \text{et} \quad \Delta u_n^2 = \partial_1 \omega_n.$$

So one can conclude that

- $(\theta_n)_{n \in \mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}_+; L^2)$,
- $(u_n)_{n \in \mathbb{N}}$ is bounded in $L_{loc}^\infty(\mathbb{R}_+; H^1)$,
- $(u_n^2)_{n \in \mathbb{N}}$ is bounded in $L_{loc}^2(\mathbb{R}_+; H^2)$.

This is enough to pass to the limit (up to extraction) in (10). Indeed, putting together the continuous embedding $H^1 \hookrightarrow L^4$ and Hölder inequality, we see that the first two properties imply that $(\theta_n u_n)_{n \in \mathbb{N}}$ is bounded in $L_{loc}^4(\mathbb{R}_+; L^{\frac{4}{3}})$ whence, by embedding, in $L_{loc}^4(\mathbb{R}_+; H^{-\frac{1}{2}})$. Therefore $(\partial_t \theta_n)_{n \in \mathbb{N}}$ is bounded in $L_{loc}^4(\mathbb{R}_+; H^{-\frac{3}{2}})$. Likewise, $(\partial_t u_n)_{n \in \mathbb{N}}$ is bounded in $L_{loc}^2(\mathbb{R}_+; H^{-1})$. Since the embeddings $H^{-\frac{3}{2}} \hookrightarrow L^2$ and $H^{-1} \hookrightarrow H^1$ are locally compact, the classical Aubin-Lions argument (see e.g. [2]) allows to conclude that, up to extraction, sequence $(\theta_n, u_n)_{n \in \mathbb{N}}$ has a limit (θ, u) satisfying System (1) and that

$$\theta \in L^\infty(\mathbb{R}_+; L^2), \quad u \in L_{loc}^\infty(\mathbb{R}_+; H^1) \quad \text{and} \quad u^2 \in L_{loc}^2(\mathbb{R}_+; H^2).$$

From standard arguments relying on the time continuity of (θ, u) in low norms, it is easy to prove the weak time continuity result. Finally, since θ is transported by the flow of a divergence free vector-field with coefficients in $L_{loc}^2(\mathbb{R}_+; H^1)$, we get in addition that $\theta \in \mathcal{C}(\mathbb{R}_+; L^2)$ (see e.g. [16]). ∎

2.2. **Local smooth solutions.** Here we aim at proving the local well-posedness for System (1) with initial data (θ_0, u_0) in $H^{s-1} \times H^s$ for some $s > 2$.

The proof will follow from an energy method once the system has been localized in dyadic frequencies. This localization may be done by means of a nonhomogeneous Littlewood-Paley decomposition. In order to define the dyadic blocks Δ_q used in this decomposition, one may proceed as in [8]: starting from a couple (χ, φ) of smooth nonnegative functions such that

$$\operatorname{Supp} \chi \subset \{\xi \in \mathbb{R}^2 \, / \, |\xi| \leq 4/3\}, \qquad \operatorname{Supp} \varphi \subset \{\xi \in \mathbb{R}^2 \, / \, 3/4 \leq |\xi| \leq 8/3\},$$

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1 \quad \text{for all} \quad \xi \in \mathbb{R}^2,$$

we set
$$\Delta_q := 0 \ \text{if} \ q \leq -2, \quad \Delta_{-1} := \chi(D) \quad \text{and} \quad \Delta_q := \varphi(2^{-q}D) \ \text{if} \ q \geq 0.$$
We also introduce the low frequency cut-off
$$S_q := \sum_{p \leq q-1} \Delta_q$$
and (for technical purposes) the *modified* low frequency cut-off $\overline{S}_q$ defined by

(14) $$\overline{S}_{-1} = \Delta_{-1} = S_0 \quad \text{and} \quad \overline{S}_q = S_q \ \text{if} \ q \neq -1.$$

It may be easily checked that
$$u = \sum_{q \geq -1} \Delta_q u \quad \text{for all tempered distribution} \ u$$
and that the set of tempered distributions u satisfying
$$\left(\sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|_{L^2}^2 \right)^{\frac{1}{2}} < \infty$$
coincides with the Sobolev space H^s, the above left-hand side defining a norm equivalent to the usual one.

Let us now state the main result of this subsection:

Proposition 4. *Let (θ_0, u_0) be in $H^{s-1} \times H^s$ with $s > 2$. Assume that $\mathrm{div}\, u_0 = 0$. There exists a positive time T depending only (continuously) on ν and on $\|(\theta_0, \omega_0)\|_{H^{s-1}}$ such that System (1) admits a unique solution (θ, u) in $\mathcal{C}([0,T]; H^{s-1} \times H^s)$. Moreover, $u^2 \in L^2([0,T]; H^{s+1})$.*

Proof: The uniqueness is a straightforward consequence of a more general result (see Proposition 9) the proof of which is postponed to subsection 2.5. So let us focus on the existence part of the above proposition, which is mostly a consequence of the $H^{s-1} \times H^s$ a priori estimates associated to System (1).

1. A priori estimates in $H^{s-1} \times H^s$. Let $(\theta, u) \in \mathcal{C}^1([0,T]; H^\infty)$ satisfy (1). We claim that there exists a constant C depending only on s and such that for all $t \in [0,T]$, we have

(15) $$\|(\theta, \omega)(t)\|_{H^{s-1}}^2 + \nu \int_0^t \|\partial_1 \omega\|_{H^{s-1}}^2 \, d\tau \leq \|(\theta_0, \omega_0)\|_{H^{s-1}}^2 \, e^{\frac{t}{\nu}} \, e^{C \int_0^t \|(\theta, \nabla v)\|_{L^\infty} \, d\tau}.$$

Indeed, applying operator Δ_q to the equation satisfied by θ yields
$$\partial_t \Delta_q \theta + \overline{S}_{q-1} u \cdot \nabla \Delta_q \theta = F_q(u, \theta) \quad \text{with} \quad F_q(u, \theta) := \overline{S}_{q-1} u \cdot \nabla \Delta_q \theta - \Delta_q(u \cdot \nabla \theta).$$
Taking the L^2 inner product of the above equality with $\Delta_q \theta$ and using the divergence free condition, we thus get
$$\frac{1}{2} \frac{d}{dt} \|\Delta_q \theta\|_{L^2}^2 \leq \|F_q(u, \theta)\|_{L^2} \|\Delta_q \theta\|_{L^2}.$$
In the appendix (see Inequality (68)), we state that
$$\|F_q(u, \theta)\|_{L^2} \leq C \Big(\|\nabla u\|_{L^\infty} \sum_{q' \geq q-4} 2^{q-q'} \|\Delta_{q'} \theta\|_{L^2} + \|\theta\|_{L^\infty} \sum_{|q'-q| \leq 1} \|\Delta_{q'} \omega\|_{L^2} \Big).$$

Plugging this inequality in (35) then multiplying both sides by $2^{2q(s-1)}$ and summing up over $q \geq -1$, we get

(16) $$\frac{d}{dt} \|\theta\|_{H^{s-1}}^2 \leq C \Big(\|\nabla u\|_{L^\infty} \|\theta\|_{H^{s-1}}^2 + \|\theta\|_{L^\infty} \|\omega\|_{H^{s-1}} \|\theta\|_{H^{s-1}} \Big).$$

In order to get a H^{s-1} estimate for ω, one may apply Δ_q to the vorticity equation. With the above notation, we get
$$\partial_t \Delta_q \omega + \overline{S}_{q-1} u \cdot \nabla \Delta_q \omega - \nu \partial_1 \Delta_q \omega = \partial_1 \Delta_q \theta + F_q(u, \omega).$$

Taking the L^2 inner product of this inequality with $\Delta_q\omega$ and using once again the divergence free condition, we get after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q\omega\|_{L^2}^2 + \nu\|\partial_1\Delta_q\omega\|_{L^2}^2 = -\int \Delta_q\theta\partial_1\Delta_q\omega\,dx + \int F_q(u,\omega)\Delta_q\omega\,dx.$$

Now, we notice that, by virtue of the Young inequality,

$$-\int \Delta_q\theta\partial_1\Delta_q\omega\,dx \leq \frac{\|\Delta_q\theta\|_{L^2}^2}{2\nu} + \frac{\nu}{2}\|\partial_1\Delta_q\omega\|_{L^2}^2,$$

and, according to (70),

$$\|F_q(u,\omega)\|_{L^2} \leq C\|\nabla u\|_{L^\infty}\sum_{q'\geq q-4} 2^{q-q'}\|\Delta_{q'}\omega\|_{L^2}.$$

Therefore

$$\frac{d}{dt}\|\Delta_q\omega\|_{L^2}^2 + \nu\|\partial_1\Delta_q\omega\|_{L^2}^2 \leq \nu^{-1}\|\Delta_q\theta\|_{L^2}^2 + C\|\Delta_q\omega\|_{L^2}\|\nabla u\|_{L^\infty}\sum_{q'\geq q-4} 2^{q-q'}\|\Delta_{q'}\omega\|_{L^2}.$$

Multiplying both sides by $2^{2q(s-1)}$ then summing up over $q \geq -1$, we end up with

$$\frac{d}{dt}\|\omega\|_{H^{s-1}}^2 + \nu\|\partial_1\omega\|_{H^{s-1}}^2 \leq \nu^{-1}\|\theta\|_{H^{s-1}}^2 + C\|\nabla u\|_{L^\infty}\|\omega\|_{H^{s-1}}^2.$$

It is now clear that adding up this latter inequality to (16) then applying Gronwall lemma completes the proof of Inequality (15).

2. The proof of local existence. One can use again the Friedrichs method introduced in the proof of Theorem 3. As Operators J_n are orthogonal projectors for all Sobolev spaces, they do not modify the energy estimates leading to Inequality (15). Therefore, the approximate solution (θ_n, u_n) to (10) satisfies

$$\|(\theta_n,\omega_n)(t)\|_{H^{s-1}}^2 + \nu\int_0^t \|\partial_1\omega_n\|_{H^{s-1}}^2\,d\tau \leq \|J_n(\theta_0,\omega_0)\|_{H^{s-1}}^2\,e^{\frac{t}{\nu}}\,e^{C\int_0^t \|(\theta_n,\nabla u_n)\|_{L^\infty}\,d\tau}.$$

Of course, the L^2 norm of u_n is controlled by virtue of (12). As $s > 2$, the space H^{s-1} continuously embeds in L^∞. Since $\|\nabla u_n\|_{H^{s-1}} = \|\omega_n\|_{H^{s-1}}$, the previous inequality thus entails that

$$X_n(t) \leq \|(\theta_0,\omega_0)\|_{H^{s-1}}\,e^{\frac{t}{2\nu}}\,e^{C\int_0^t X_n(\tau)\,d\tau} \quad \text{with} \quad X_n^2(t) := \|(\theta_n,\omega_n)(t)\|_{H^{s-1}}^2 + \nu\int_0^t \|\partial_1\omega_n\|_{H^{s-1}}^2\,d\tau.$$

This inequality may be easily integrated into

$$\exp\left(-C\int_0^t X_n(\tau)\,d\tau\right) \geq 1 - 2C\nu X_0 e^{\frac{t}{2\nu}} \quad \text{for all} \quad t \geq 0.$$

Therefore, there exists a $c > 0$ such that if we set

(17) $$T := 2\nu\log\left(\frac{c}{\nu\|(\theta_0,\omega_0)\|_{H^{s-1}}}\right)$$

then

- $(\theta_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T]; H^{s-1})$,
- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty([0,T]; H^s)$,
- $(u_n^2)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]; H^{s+1})$.

Mimicking the compactness argument used for proving Theorem 3, one can now conclude that there exists a solution (θ, u) satisfying $(\theta, u) \in L^\infty([0,T]; H^{s-1} \times H^s)$ and $u^2 \in L^2([0,T]; H^{s+1})$. The time continuity follows from the fact that θ and ω satisfy transport equations with H^{s-1} initial data and a $L^2([0,T]; H^{s-1})$ source term. This completes the proof of Proposition 4. $\blacksquare$

2.3. **Global smooth solutions.** Here we aim at proving that the local smooth solutions which have been constructed so far may be extended to all positive time. Exhibiting a polynomial control of $\|\nabla u(t)\|_{\sqrt{L}}$ (where the space $\sqrt{L}$ has been defined in (3)) is the cornerstone of this extension. More precisely, we shall first prove that the $L^1([0,T]; LL)$ norm of ∇u with

$$(18) \qquad LL := \left\{ f \in \mathcal{S}' \, / \, \|f\|_{LL} := \sup_{q \geq 0} \frac{\|S_q f\|_{L^\infty}}{q+1} < \infty \right\}$$

controls the Sobolev regularity of the solutions to System (1). Next, we shall state that, under the hypotheses of Proposition 4, the norm of ∇u in $L^2([0,T]; \sqrt{L})$ (which is *stronger* to the $L^1([0,T]; LL)$ norm) may be bounded for all time by a fixed polynomial the coefficients of which depend only on low norms of the data, and on ν. Combining this with Proposition 4 will lead to the following global existence statement:

Theorem 5. *Let* (θ_0, u_0) *be in* $H^{s-1} \times H^s$ *for some* $s > 2$. *Assume that* $\operatorname{div} u_0 = 0$. *Then system* (1) *has a unique global solution* (θ, u) *such that*

$$(\theta, u) \in \mathcal{C}(\mathbb{R}_+; H^{s-1} \times H^s) \quad and \quad u^2 \in L^2_{loc}(\mathbb{R}_+; H^{s+1}).$$

As a first step for proving Theorem 5, let us show the following lemma:

Lemma 6. *Let* (θ, u) *be a solution to* (1) *in* $\mathcal{C}([0,T]; H^{s-1} \times H^s)$ *with* $s > 2$. *If*

$$\int_0^T \|\nabla u(t)\|_{LL} \, dt < \infty$$

then (θ, u) *may be continued beyond* T *into a smooth solution of* (1).

Proof: Putting together the lower bound for the lifespan of (θ, u) given by (17) and the uniqueness of smooth solutions, it suffices to state that under the assumptions of the lemma, we have

$$\sup_{0 \leq t < T} \left(\|\theta(t)\|_{H^{s-1}} + \|\omega(t)\|_{H^{s-1}} \right) < \infty.$$

First, as θ is transported by the vector-field u (which is lipschitz for $s > 2$ implies $H^{s-1} \hookrightarrow L^\infty$), we get the following control:

$$\|\theta(t)\|_{L^\infty} = \|\theta_0\|_{L^\infty} \quad \text{for all} \quad t \in [0,T].$$

In consequence, Inequality (15) ensures that

$$(19) \qquad \|(\theta, \omega)(t)\|_{H^{s-1}}^2 + \nu \int_0^t \|\partial_1 \omega\|_{H^{s-1}}^2 \, d\tau \leq \|(\theta_0, \omega_0)\|_{H^{s-1}}^2 \, e^{(\nu^{-1} + C\|\theta_0\|_{L^\infty})t} \, e^{C \int_0^t \|\nabla u\|_{L^\infty} \, d\tau}.$$

On the other hand, in the appendix, it is shown that

$$(20) \qquad \|\nabla u\|_{L^\infty} \leq C\left(1 + \|\nabla u\|_{LL} \log(e + \|\omega\|_{H^{s-1}})\right).$$

Putting together (19) and (20), we deduce that for all $t \in [0,T]$,

$$\log\left(e + \|(\theta, \omega)(t)\|_{H^{s-1}}^2\right) \leq \log\left(e + \|(\theta_0, \omega_0)\|_{H^{s-1}}^2\right) + C\left(1 + \nu^{-1} + \|\theta_0\|_{L^\infty}\right)t$$

$$+ C \int_0^t \|\nabla u\|_{LL} \log\left(e + \|(\theta, \omega)(t)\|_{H^{s-1}}^2\right) d\tau,$$

whence, according to Gronwall Lemma,

$$\log\left(e + \|(\theta, \omega)(t)\|_{H^{s-1}}^2\right) \leq \left(\log\left(e + \|(\theta_0, \omega_0)\|_{H^{s-1}}^2\right) + C\left(1 + \nu^{-1} + \|\theta_0\|_{L^\infty}\right)t\right) e^{C \int_0^t \|\nabla u\|_{LL} \, d\tau}.$$

As the argument of exp is, by assumption, bounded for $t \in [0,T]$, we gather that $(\theta, \omega) \in L^\infty([0,T]; H^{s-1})$, which completes the proof of the lemma. ∎

The next step involves showing that the norm used in the previous lemma is controlled by the system. In fact, we shall state a slightly more accurate result:

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Lemma 7. *Let (θ, u) be a solution to (1) in $\mathcal{C}([0,T]; H^2 \times H^3)$. There exists a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ depending only (continuously) on ν, $\|u_0\|_{L^2}$, $\|\theta_0\|_{L^2 \cap L^\infty}$ and $\|\omega_0\|_{\sqrt{L}}$ such that*

$$\int_0^t \|\nabla u\|_{\sqrt{L}}^2 \, dt \leq f(t) \quad \text{for all } t \in [0,T).$$

Proof: Let us first notice that, because $\operatorname{div} u = 0$ and u is Lipschitz, we have

(21)
$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p} \quad \text{for all} \quad p \in [2, \infty] \quad \text{and} \quad t \in [0,T).$$

In order to get a control of ∇u in $L^2([0,T); \sqrt{L})$, we are going to state that

(22)
$$\|\omega(t)\|_{\sqrt{L}}^2 \leq \|\omega_0\|_{\sqrt{L}}^2 + \frac{t}{2\nu} \|\theta_0\|_{L^2 \cap L^\infty}^2.$$

For showing that, one may multiply the vorticity equation with $|\omega|^{p-2}\omega$ and perform a space integration. As our hypotheses on the solution entail that $\omega \in \mathcal{C}^1([0,T); H^1)$ and thus $\omega \in \mathcal{C}^1([0,T); L^p)$ for all $p \in [2, +\infty)$, we get

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int |\omega|^p \, dx + (p-1)\nu \int |\partial_1\omega|^2|\omega|^{p-2}\,dx \;&\leq\; (p-1)\int |\theta||\partial_1\omega||\omega|^{p-2}\,dx, \\
&\leq\; (p-1)\nu\int |\partial_1\omega|^2|\omega|^{p-2}\,dx + \frac{p-1}{4\nu}\int |\theta|^2|\omega|^{p-2}\,dx \\
&\leq\; (p-1)\nu\int |\partial_1\omega|^2|\omega|^{p-2}\,dx + \frac{p-1}{4\nu}\|\theta\|_{L^p}^2\|\omega\|_{L^p}^{p-2}.
\end{aligned}
$$

Therefore,

$$\frac{d}{dt}\|\omega\|_{L^p}^2 \leq \left(\frac{p-1}{2\nu}\right)\|\theta\|_{L^p}^2$$

hence, by virtue of (21),

$$\|\omega(t)\|_{L^p}^2 \leq \|\omega_0\|_{L^p}^2 + \left(\frac{p-1}{2\nu}\right)\|\theta_0\|_{L^p}^2 \, t,$$

whence (22).

This *does not* imply that $\nabla u \in L^\infty([0,T); \sqrt{L})$ for the classical result on Calderon-Zygmund operators (see e.g. [8], Chap. 3) gives only that

(23)
$$\|\nabla u(t)\|_{L^p} \leq Cp\|\omega(t)\|_{L^p} \quad \text{for all} \quad p \in [2, \infty).$$

However, because $\partial_1\omega = \Delta u^2$ and $\partial_1 u^1 = -\partial_2 u^2$, Inequalities (5) and (7) entail that

(24)
$$\sqrt{\nu}\|\partial_i u^j\|_{L_t^2(H^1)} \leq \|u_0\|_{H^1} + \left(t + \sqrt{\frac{t}{\nu}}\right)\|\theta_0\|_{L^2} \quad \text{for } t \in [0,T) \text{ and } (i,j) \neq (2,1).$$

By virtue of Lemma 13 (see the appendix), we thus get the desired bound for all the components of ∇u except $\partial_2 u^1$. In order to get a suitable bound for $\partial_2 u^1$, one may use the fact that $\partial_2 u^1 = \partial_1 u^2 - \omega$. Putting together Inequalities (22) and (24), it is now easy to conclude ∎

Proof of Theorem 5 : For the sake of simplicity, we restrict ourselves to the case $s \geq 3$ so that one may use Lemma 7. The case $2 < s < 3$ easily follows from the case $s \geq 3$: it is only a matter of smoothing out the initial data then pass to the limit.

So let us assume from now on that $s \geq 3$ and let us denote by (θ, u) the maximal solution supplied by Proposition 4, and by T^* the lifespan of (θ, u). If we assume (by contradiction) that T^* is finite then Proposition 7 ensures that

(25)
$$\nabla u \in L^2([0, T^*); \sqrt{L}).$$

Remark that the space $\sqrt{L}$ is continuously embedded in the space $LL^{\frac{1}{2}}$ defined by

(26)
$$LL^{\frac{1}{2}} := \left\{ f \in \mathcal{S}' \;/\; \|f\|_{LL^{\frac{1}{2}}} := \sup_{q \geq 0} \frac{\|S_q f\|_{L^\infty}}{\sqrt{q+1}} < \infty \right\}.$$

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Indeed, thanks to Bernstein inequality, there exists a constant C such that for all $N \in \mathbb{N}$ and $p \in [2, \infty[$, we have

$$(27) \qquad \|S_N \nabla u\|_{L^\infty} \leq C 2^{\frac{2N}{p}} \|\nabla u\|_{L^p} \leq C 2^{\frac{2N}{p}} \sqrt{p-1} \|\nabla u\|_{\sqrt{L}}.$$

If we choose $p = N + 2$ then we get

$$(28) \qquad \|S_N \nabla u\|_{L^\infty} \leq C \sqrt{N+1} \|\nabla u\|_{\sqrt{L}},$$

whence the desired embedding.

It is obvious that $LL^{\frac{1}{2}} \hookrightarrow LL$. Resorting to (25), we thus get $\nabla u \in L^1([0, T^*); LL)$ and Lemma 6 ensures that the solution (θ, u) may be continued beyond T^*. This contradicts the definition of T^*. ∎

2.4. **Global well-posedness for rough data.** In this section, we want to state global existence with uniqueness for a class of data as large as possible. Having in mind the previous subsection, it seems reasonable to require that $\theta_0 \in L^2 \cap L^\infty$, that $u_0 \in H^1$ and that $\omega_0 \in \sqrt{L}$. As those regularity assumptions are (formally) conserved by the system during the evolution, we thus expect to get a global solution (θ, u) such that $\nabla u \in L^2_{loc}(\mathbb{R}_+; \sqrt{L})$. This would imply that $u \in L^2_{loc}(\mathbb{R}_+; \mathrm{LogLip}^{\frac{1}{2}})$ where $\mathrm{LogLip}^{\frac{1}{2}}$ stands for the set of bounded functions f such that

$$\sup_{\substack{x \neq y \\ |x-y| \leq 1/2}} \frac{|f(y) - f(x)|}{|x - y| \log^{\frac{1}{2}}(|x-y|^{-1})} < \infty.$$

The above inequality is an obvious corollary of (28) and of Proposition 2.107 in [4].

Even though the vector-field u fails to be lipschitz, it has enough regularity so that we have $\theta \in \mathcal{C}(\mathbb{R}_+; H^{s-\varepsilon})$ for all $\varepsilon > 0$ if we start from θ_0 in H^s for some $s \in]-1, 1]$ (this is in fact a consequence of Theorem 3.12 in [12]). These plain observations will lead us to the following statement which obviously contains Theorem 1:

Theorem 8. *Let $\theta_0 \in L^2 \cap L^\infty$, and $u_0 \in H^1$ with $\mathrm{div}\, u_0 = 0$ and $\omega_0 \in \sqrt{L}$. Then System (1) admits a global solution (θ, u) such that*

$$(29) \qquad \begin{aligned} &\theta \in \mathcal{C}_b(\mathbb{R}_+; L^2) \cap \mathcal{C}_w(\mathbb{R}_+; L^\infty) \cap L^\infty(\mathbb{R}_+; L^\infty), \\ &u \in \mathcal{C}_w(\mathbb{R}_+; H^1), \quad u^2 \in L^2_{loc}(\mathbb{R}_+; H^2), \quad \omega \in L^\infty_{loc}(\mathbb{R}_+; \sqrt{L}), \quad \nabla u \in L^2_{loc}(\mathbb{R}_+; \sqrt{L}). \end{aligned}$$

If in addition $\theta_0 \in H^s$ for some real number $s \in (0, 1]$ then $\theta \in \mathcal{C}(\mathbb{R}_+; H^{s-\varepsilon})$ for all $\varepsilon > 0$.

Finally, if $s > 1/2$ then the solution is unique.

Proof: The uniqueness is a consequence of Proposition 9 below so let us focus on the existence part of the statement. To achieve it, one may smooth out the initial data $(\theta_0, u_0)_{n \in \mathbb{N}}$ so as to get a sequence $(\theta_{0,n}, u_{0,n})_{n \in \mathbb{N}}$ of H^∞ functions which tends to (θ_0, u_0) in (say) $L^2 \times H^1$. Resorting to Theorem 5, we get a sequence $(\theta_n, u_n)_{n \in \mathbb{N}}$ of smooth global solutions. Moreover, by virtue of Inequalities (5), (7), (21) and (22), we have

- $(\theta_n)_{n \in \mathbb{N}}$ bounded in $L^\infty(\mathbb{R}_+; L^2 \cap L^\infty)$,
- $(u_n)_{n \in \mathbb{N}}$ bounded in $L^\infty_{loc}(\mathbb{R}_+; H^1)$,
- $(u_n^2)_{n \in \mathbb{N}}$ bounded in $L^2_{loc}(\mathbb{R}_+; H^2)$,
- $(\omega_n^2)_{n \in \mathbb{N}}$ bounded in $L^\infty_{loc}(\mathbb{R}_+; \sqrt{L})$.

As explained in the proof of Lemma (7), these proprieties imply that $(\nabla u_n)_{n \in \mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R}_+; \sqrt{L})$.

Now, taking advantage of the losing estimates proved in Proposition 16, we deduce that if, in addition, $\theta_0 \in H^s$ for some $s \in (0, 1)$ then, for all $\varepsilon > 0$, $(\theta_n)_{n \in \mathbb{N}}$ is bounded in $L^\infty_{loc}(\mathbb{R}_+; H^{s-\varepsilon})$. In order to conclude to the existence part of the statement, one may use again a compactness argument *à la Aubin-Lions* as in the proof of Theorem 3. ∎

2.5. **Uniqueness for rough data.** The difference $(\delta\!\theta, \delta u, \delta\Pi)$ between two solutions (θ_1, u_1, Π_1) and (θ_2, u_2, Π_2) satisfies:

(30)
$$\begin{cases} \partial_t \delta\!\theta + \operatorname{div}(u_2\,\delta\!\theta) = -\operatorname{div}(\delta u\,\theta_1), \\ \partial_t \delta u + \operatorname{div}(u_2 \otimes \delta u) + \operatorname{div}(\delta u \otimes u_1) - \nu\partial_1^2 \delta u + \nabla\delta\Pi = \delta\!\theta\, e_2. \end{cases}$$

First of all, let us notice that if $\theta_1 \in L^\infty([0,T]; H^\gamma)$ then the right-hand side of the first equation is at most in $L^\infty([0,T]; H^{\gamma-1})$. As, under the hypotheses of Theorem 5, u_2 fails to be lipschitz but has gradient in $L^2_{loc}(\mathbb{R}_+; \sqrt{L})$, Proposition 16 will enable us to bound $\delta\!\theta$ in $H^{\gamma-1-\varepsilon}$ provided that $\operatorname{div}(\delta u\,\theta_1)$ be bounded in $L^1([0,T]; H^{\gamma-1})$, a condition which requires a control over the $L^1([0,T]; L^\infty)$ norm of δu. If the velocity equation had a full Laplace operator then the resulting smoothing effect would be strong enough so as to provide us with a control over this norm. We shall see that in the framework of anisotropic viscosity, one can still get an appropriate bound for δu in $L^1([0,T]; L^\infty)$ provided $\gamma > 1/2$. In short, we expect to be able to control $\delta\!\theta$ in $L^\infty([0,T]; H^{\beta-1})$ for some $\beta \in]1/2, \gamma[$, δu in $L^\infty([0,T]; H^\alpha)$ and $\partial_1 \delta u$ in $L^2([0,T]; H^\alpha)$ for some $\alpha \in]1/2, \beta[$. This motivates the following statement which implies the uniqueness part of Theorem 8:

Proposition 9. *Let (θ_1, u_1) and (θ_2, u_2) be two solutions of (1) on $[0,T] \times \mathbb{R}^2$ with the same initial data. Assume that $\nabla u_i \in L^1([0,T]; LL^{\frac{1}{2}})$ and that there exists some $\gamma \in]1/2, 1[$ such that*

$$\theta_i \in L^\infty([0,T]; L^\infty \cap H^\gamma) \quad \text{and} \quad u_i \in L^\infty([0,T]; H^\gamma) \quad \text{for } i = 1, 2.$$

Then the two solutions coincide.

Proof: According to the above heuristics, we have to bound

$$\delta\!\theta \text{ in } L^\infty([0,T]; H^{\beta-1}), \quad \delta u \text{ in } L^\infty([0,T]; H^\alpha) \text{ and } \partial_1\delta u \text{ in } L^2([0,T]; H^\alpha)$$

for some fixed (α, β) such that $1/2 < \alpha < \beta < \gamma$.

In order to bound $\delta\!\theta$, one may use Proposition 16 with the vector-field u_2. Because $\delta\!\theta(0) = 0$, we have

$$\|\delta\!\theta(t)\|_{H^{\beta-1}} \leq C \int_0^t \|\operatorname{div}(\delta u\,\theta_1)\|_{H^{\gamma-1}} \quad \text{for all} \quad t \in [0,T].$$

In order to bound the right-hand side, one may resort to the following *Bony's decomposition* [5]:

(31)
$$\operatorname{div}(\delta u\,\theta_1) = \operatorname{div}\left(T_{\delta u}\theta_1 + R(\delta u, \theta_1)\right) + \sum_{i=1}^2 T_{\partial_i\theta_1}\delta u^i,$$

where the paraproduct operator T (resp. reminder operator R) is defined by

$$T_f g := \sum_q S_{q-1}f\Delta_q g \quad \left(\text{resp.} \quad R(f,g) := \sum_q \Delta_q f\left(\Delta_{q-1}g + \Delta_q g + \Delta_{q-1}g\right)\right).$$

Let us stress that the condition $\operatorname{div}\delta u = 0$ has been used in order to have the derivative act on the left for the first two terms of (31).

From standard continuity results for operators T and R (see e.g. [4]) we have

$$\|T_{\delta u}\theta_1 + R(\delta u, \theta_1)\|_{H^\gamma} \leq C\|\delta u\|_{L^\infty}\|\theta_1\|_{H^\gamma}.$$

As for the last term, given that $\gamma - 1 < 0$, one can write

$$\|T_{\partial_i\theta_1}\delta u^i\|_{H^{\gamma-1}} \leq C\|\nabla\theta_1\|_{H^{\gamma-1}}\|\delta u\|_{L^\infty}.$$

We eventually get

(32)
$$\|\delta\!\theta\|_{L_t^\infty(H^{\beta-1})} \leq C\|\theta_1\|_{L_t^\infty(H^\gamma)}\|\delta u\|_{L_t^1(L^\infty)}.$$

In order to bound δu, one may use Proposition 17. We get for all $t \in [0,T]$,

$$\|\delta u\|_{L_t^\infty(H^\alpha)} + \|\partial_1\delta u\|_{L_t^2(H^\alpha)} \leq C\left(\|\delta\!\theta\|_{L_t^2(H^{\beta-1})} + \|\delta u \cdot \nabla u_1\|_{L_t^2(H^{\beta-1})}\right)$$

for some constant C depending only on α, β, ν and u_2. Using again the Bony decomposition

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and arguing exactly as for proving (32), we get

$$\|\delta u \cdot \nabla u_1\|_{H^{\beta-1}} \le C\|\delta u\|_{L^\infty}\|u_1\|_{H^\beta}.$$

Therefore, given that $u_1 \in L^\infty([0,T]; H^\beta)$,

(33) $$\|\delta u\|_{L_t^\infty(H^\alpha)} + \|\partial_1 \delta u\|_{L_t^2(H^\alpha)} \le C\big(\|\delta\theta\|_{L_t^2(H^{\beta-1})} + \|\delta u\|_{L_t^2(L^\infty)}\big).$$

In order to complete the proof of the proposition, it is only a matter of showing that $\|\delta u\|_{L_t^1(L^\infty)}$ may be bounded in terms of $\|\delta u\|_{L_t^\infty(H^\alpha)}$ and of $\|\partial_1 \delta u\|_{L_t^2(H^\alpha)}$. This is the only point where the assumption $\alpha > 1/2$ (and thus $\gamma > 1/2$) is going to play a role. First of all, thanks to the trace theorem, one may write (with obvious notation)

$$H^\alpha(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}_{x_2}; H^{\alpha-\frac{1}{2}}(\mathbb{R}_{x_1})).$$

Therefore, one may write

(34) $$\|\delta u\|_{L_{x_2}^\infty(H_{x_1}^{\alpha-\frac{1}{2}})} \le C\|\delta u\|_{H^\alpha} \quad \text{and} \quad \|\partial_1 \delta u\|_{L_{x_2}^\infty(H_{x_1}^{\alpha-\frac{1}{2}})} \le C\|\partial_1 \delta u\|_{H^\alpha}.$$

As for all $\alpha \in\,]0,1[$, Gagliardo-Nirenberg inequality implies that

$$\|\delta u(\cdot, x_2)\|_{L^\infty(\mathbb{R})} \le C\|\delta u(\cdot, x_2)\|_{H^{\alpha-\frac{1}{2}}(\mathbb{R})}^\alpha \|\partial_1 \delta u(\cdot, x_2)\|_{H^{\alpha-\frac{1}{2}}(\mathbb{R})}^{1-\alpha} \quad \text{for all} \quad x_2 \in \mathbb{R},$$

we have, by combination with (34),

$$\|\delta u\|_{L^\infty(\mathbb{R}^2)} \le C\|\delta u\|_{H^\alpha(\mathbb{R}^2)}^\alpha \|\partial_1 \delta u\|_{H^\alpha(\mathbb{R}^2)}^{1-\alpha}.$$

Coming back to (32) and (33), we deduce that for some constant C depending only on ν, T and on the norms of (θ_1, u_1) and (θ_2, u_2), we have

$$\|\delta\theta\|_{L_t^\infty(H^{\beta-1})} \le C t^{\frac{1}{2}+\frac{\alpha}{2}} \delta U(t)$$
$$\delta U(t) \le C\Big(t^{\frac{1}{2}}\|\delta\theta\|_{L_t^\infty(H^{\beta-1})} + t^{\frac{\alpha}{2}}\delta U(t)\Big)$$

with

$$\delta U(t) := \|\delta u\|_{L_t^\infty(H^\alpha)} + \|\partial_1 \delta u\|_{L_t^2(H^\alpha)}.$$

Inserting the first inequality in the second one, one may conclude that $\delta u \equiv 0$ (and thus $\delta\theta \equiv 0$) on a suitably small time interval. Finally, let us notice that our assumptions on the solutions ensure that $\delta\theta \in \mathcal{C}([0,T]; H^{\beta-1})$ and $\delta u \in \mathcal{C}([0,T]; H^\alpha)$. Using a classical connectivity argument, it is now easy to get the uniqueness on the whole interval $[0,T]$. ■

3. The case of an horizontal diffusivity

This section is devoted to the study of System (2). In other words, in contrast with the previous section, we now assume that the velocity satisfies the incompressible Euler equation with buoyancy force whereas the temperature experiences diffusion *in the horizontal variable only*.

We aim at stating various global existence results for arbitrarily large data. More precisely, we first prove that any data $\theta_0 \in L^2$ and $u_0 \in H^1$ with $\operatorname{div} u_0 = 0$ generates a global weak solution with finite energy. The rest of this section is mainly devoted to the proof of Theorem 2. As a first step, in subsection 3.2, we state H^1 a priori estimates for the la temperature. In the next subsection, we prove a uniqueness result for a large class of solutions. As this uniqueness result requires in particular that $\nabla\theta \in L_{loc}^1(\mathbb{R}_+; L^\infty)$ and that $\nabla u \in L_{loc}^1([0,T]; L)$, our next task amounts to finding additional regularity conditions on the data which may be propagated globally by the system. It turns out that it is possible to propagate some *anisotropic* Sobolev regularity over the temperature, and thus to complete the proof of Theorem 2.

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3.1. Global weak solutions: the case $\theta_0 \in L^2$ and $u_0 \in H^1$. Let us first derive the formal energy estimates for System (2) in the case $\theta_0 \in L^2$ and $u_0 \in H^1$. First, multiply $(2)_1$ by θ and integrate over $[0,t] \times \mathbb{R}^2$. We get

$$(35) \qquad \|\theta(t)\|_{L^2}^2 + 2\kappa \int_0^t \|\partial_1 \theta(s)\|_{L^2}^2 \, ds \leq \|\theta_0\|_{L^2}^2.$$

Combining this with the standard energy estimate for u yields

$$(36) \qquad \|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}.$$

In order to get a H^1 bound for the velocity, one may consider the vorticity equation:

$$\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta.$$

Multiplying by ω then integrating with respect to the space variable, we find that

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^2 \leq \|\omega\|_{L^2} \|\partial_1 \theta\|_{L^2}$$

whence,

$$\|\omega(t)\|_{L^2} \ \leq \ \|\omega_0\|_{L^2} + \int_0^t \|\partial_1 \theta\|_{L^2} \, ds,$$

$$(37) \qquad\qquad\qquad \leq \ \|\omega_0\|_{L^2} + \sqrt{\frac{t}{2\kappa}} \|\theta_0\|_{L^2}.$$

Now, using a Friedrichs method quite similar to that of the proof of Theorem 3, we easily get the following statement:

Theorem 10. *Let $\theta_0 \in L^2$ and $u_0 \in H^1$ with $\mathrm{div}\, u_0 = 0$. Then System (2) with data (θ_0, u_0) has a global solution (θ, u) such that*

$$\theta \in \mathcal{C}_w(\mathbb{R}_+; L^2), \quad \partial_1 \theta \in L^2(\mathbb{R}_+; L^2) \quad and \quad u \in \mathcal{C}(\mathbb{R}_+; H^1).$$

3.2. H^1 a priori estimates for the temperature. In the present paragraph, we show that one may get (at least formally) a global control over both the H^1 norm of θ and of u.

To start with, let us point out that Inequalities (35), (36) and (37) provide us with a bound for θ in $L^\infty(\mathbb{R}_+; L^2)$, for $\partial_1 \theta$ in $L^2(\mathbb{R}_+; L^2)$ and for u in $L^\infty_{loc}(\mathbb{R}_+; H^1)$. We claim that if we assume in addition that $\nabla \theta_0 \in L^2$ then one may bound $\nabla \theta$ in $L^\infty_{loc}(\mathbb{R}_+; L^2)$. Indeed, applying operator ∂_i $(i = 1, 2)$ to the equation satisfied by θ yields

$$\partial_t \partial_i \theta + u \cdot \nabla \partial_i \theta + \partial_i u \cdot \nabla \theta - \kappa \partial_1^2 \partial_i \theta = 0.$$

Let us multiply this equality by $\partial_i \theta$, integrate over $\mathbb{R}^2$ then add up the equalities for $i = 1, 2$. Integrating by parts where needed and using the fact that $\mathrm{div}\, u = 0$, we easily find that

$$(38) \qquad \frac{1}{2} \frac{d}{dt} \|\nabla \theta\|_{L^2}^2 + \kappa \|\partial_1 \nabla \theta\|_{L^2}^2 + \sum_{1 \leq i,j \leq 2} \int \partial_i \theta \, \partial_j u^i \, \partial_j \theta \, dx = 0.$$

For $(i, j) \neq (2, 2)$, the terms in the above summation are easy to handle. Indeed, taking advantage of the anisotropic Hölder inequality, one can write

$$\left| \int \partial_i \theta \, \partial_j u^i \, \partial_j \theta \, dx \right| \leq \|\nabla u\|_{L^2} \|\partial_1 \theta\|_{L^2_{x_1}(L^\infty_{x_2})} \|\nabla \theta\|_{L^\infty_{x_1}(L^2_{x_2})}.$$

Let us admit the following two inequalities (the proof of which is postponed in the appendix):

$$(39) \qquad \|f\|_{L^2_{x_1}(L^\infty_{x_2})} \leq C\|f\|_{L^2}^{\frac{1}{2}} \|\partial_2 f\|_{L^2}^{\frac{1}{2}} \quad \text{and} \quad \|f\|_{L^\infty_{x_1}(L^2_{x_2})} \leq C\|f\|_{L^2}^{\frac{1}{2}} \|\partial_1 f\|_{L^2}^{\frac{1}{2}}.$$

Applying these inequalities to $\partial_1 \theta$ and to $\nabla \theta$, and using the fact that $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$, we deduce that

$$\left| \int \partial_i \theta \, \partial_j u^i \, \partial_j \theta \, dx \right| \leq C\|\omega\|_{L^2} \|\nabla \theta\|_{L^2} \|\partial_1 \nabla \theta\|_{L^2} \quad \text{if} \quad (i, j) \neq (2, 2).$$

In order to bound the term corresponding to $(i,j) = (2,2)$, one may use the fact that $\partial_2 u^2 = -\partial_1 u^1$ and integrate by parts. We get

$$\int \partial_2 u^2 (\partial_2 \theta)^2 \, dx = - \int \partial_1 u^1 (\partial_2 \theta)^2 \, dx = 2 \int u^1 \partial_2 \theta \, \partial_1 \partial_2 \theta \, dx.$$

Therefore, thanks to the anisotropic Hölder inequalities and to (39),

$$
\begin{aligned}
\left| \int \partial_2 u^2 (\partial_2 \theta)^2 \, dx \right| & \leq \quad 2 \|\partial_1 \partial_2 \theta\|_{L^2} \|u^1\|_{L^2_{x_1}(L^\infty_{x_2})} \|\partial_2 \theta\|_{L^\infty_{x_1}(L^2_{x_2})}, \\
& \leq \quad C \|\partial_1 \partial_2 \theta\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} \|\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \theta\|_{L^2}^{\frac{1}{2}}, \\
& \leq \quad C \|u\|_{L^2}^{\frac{1}{2}} \|\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \theta\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \theta\|_{L^2}^{\frac{3}{2}}.
\end{aligned}
$$

So finally, Young inequality leads to

$$\left| \sum_{1 \leq i,j \leq 2} \int \partial_i \theta \, \partial_j u^i \, \partial_j \theta \, dx \right| \leq \frac{\kappa}{2} \|\partial_1 \nabla \theta\|_{L^2}^2 + \frac{C}{\kappa} \|\omega\|_{L^2}^2 \left(1 + \frac{\|u\|_{L^2}^2}{\kappa^2} \right) \|\nabla \theta\|_{L^2}^2.$$

Plugging this inequality in (38) and using Gronwall lemma, we end up with

$$\|\nabla \theta(t)\|_{L^2}^2 + \kappa \int_0^t \|\partial_1 \nabla \theta(s)\|_{L^2}^2 \, ds \leq \|\nabla \theta_0\|_{L^2}^2 \exp \left\{ \frac{C}{\kappa} \int_0^t \|\omega\|_{L^2}^2 \left(1 + \frac{\|u\|_{L^2}^2}{\kappa^2} \right) d\tau \right\}.$$

Putting together (35), (36) and (37), we conclude that

$$(40) \qquad \|\theta(t)\|_{H^1}^2 + \kappa \int_0^t \|\partial_1 \theta(s)\|_{H^1}^2 \, ds \leq C(t, \kappa, \theta_0, u_0)$$

with $C(t, \kappa, \theta_0, u_0) := \|\theta_0\|_{H^1}^2 \exp \left\{ \dfrac{Ct}{\kappa} \left(\|\omega_0\|_{L^2}^2 + \dfrac{t}{\kappa} \|\theta_0\|_{L^2}^2 \right) \left(1 + \dfrac{\|u_0\|_{L^2}^2 + t^2 \|\theta_0\|_{L^2}^2}{\kappa^2} \right) \right\}.$

3.3. **A uniqueness result.** In this section, we establish a uniqueness result for System (2) under "minimal" assumptions. In order to motivate those assumptions, let us remind that in the isotropic case (that is with a full Laplacian in the temperature equation) which has been investigated in [15], uniqueness is true in the class of $C^{0,1}(\mathbb{R}_+; L^2)$ solutions which satisfy in addition

$$(41) \qquad \nabla \theta \in L^1_{loc}(\mathbb{R}_+; L^\infty) \quad \text{and} \quad \nabla u \in L^1_{loc}(\mathbb{R}_+; L).$$

As in the case that we now consider the smoothing effect over the temperature is obviously *weaker*, we expect the conditions leading to uniqueness to be *stronger* than (41). We shall prove the following result:

Proposition 11. *Let (θ_1, u_1) and (θ_2, u_2) be two solutions of (2) with the same data. Assume that both solutions belong to $L^\infty([0,T]; H^1) \cap C^{0,1}([0,T]; L^2)$ and that, in addition, $\partial_1 \theta_2 \in L^2([0,T]; H^1)$ and $\nabla u_2 \in L^1([0,T]; L)$. Then $(\theta_1, u_1) \equiv (\theta_2, u_2)$ on $[0,T] \times \mathbb{R}^2$.*

Proof: With the usual notation, $(\delta\!\theta, \delta u)$ satisfies:

$$
\begin{cases}
\partial_t \delta\!\theta + u_1 \cdot \nabla \delta\!\theta + \operatorname{div}(\delta u \, \theta_2) - \kappa \partial_1^2 \delta\!\theta = 0 \\
\partial_t \delta u + u_1 \cdot \nabla \delta u + \delta u \cdot \nabla u_2 = -\nabla \delta\!\Pi + \delta\!\theta \, e_2.
\end{cases}
$$

From a standard energy method, we get

$$(42) \qquad \frac{1}{2} \frac{d}{dt} \|\delta\!\theta\|_{L^2}^2 + \kappa \|\partial_1 \delta\!\theta\|_{L^2}^2 \leq \left| \int \operatorname{div}(\theta_2 \delta u) \delta\!\theta \, dx \right|,$$

$$(43) \qquad \frac{1}{2} \frac{d}{dt} \|\delta u\|_{L^2}^2 \leq \left| \int \delta u \cdot \nabla u_2 \, \delta\!\theta \, dx \right| + \left| \int \delta\!\theta \delta u^2 \, dx \right|.$$

In order to bound the right-hand side of (42), one may write

$$(44) \qquad \int \operatorname{div}(\theta_2 \delta u) \delta\!\theta = - \int \theta_2 \delta u^1 \partial_1 \delta \theta \, dx - \int \theta_2 \delta u^2 \partial_2 \delta\!\theta \, dx.$$

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The first term is easy to deal with: using Cauchy-Schwarz inequality, we get

$$(45) \qquad \left| \int \theta_2 \delta u_1 \partial_1 \vartheta \, dx \right| \le \|\delta u\|_{L^2} \|\theta_2\|_{L^\infty} \|\partial_1 \vartheta\|_{L^2}.$$

Next, applying the following inequality (see the proof in the appendix)

$$(46) \qquad \|\theta_2\|_{L^\infty} \le C\|\theta_2\|_{L^2}^{1/4} \|\partial_1\theta_2\|_{L^2}^{1/4} \|\partial_2\theta_2\|_{L^2}^{1/4} \|\partial_1\partial_2\theta_2\|_{L^2}^{1/4},$$

and using Young inequality, we find that

$$(47) \qquad \left| \int \theta_2 \delta u_1 \partial_1 \vartheta \, dx \right| \le \frac{C}{\kappa} \|\theta_2\|_{H^1}^{3/2} \|\partial_1\partial_2\theta_2\|_{L^2}^{1/2} \|\delta u\|_{L^2}^2 + \frac{\kappa}{6} \|\partial_1\vartheta\|_{L^2}^2.$$

The second term of (44) is more intricate. If we integrate by parts and use the fact that $\operatorname{div} \delta u = 0$, we get

$$
\begin{aligned}
-\int \theta_2 \delta u^2 \partial_2 \vartheta \, dx &= \int \partial_2 \theta_2 \delta u^2 \, \vartheta \, dx + \int \theta_2 \partial_2 \delta u^2 \vartheta \, dx, \\
&= \int \partial_2 \theta_2 \delta u^2 \, \vartheta \, dx - \int \theta_2 \partial_1 \delta u^1 \vartheta \, dx, \\
&= A_1 + A_2 + A_3
\end{aligned}
$$

with

$$A_1 := \int \partial_2 \theta_2 \, \delta u^2 \, \vartheta \, dx, \quad A_2 := \int \theta_2 \delta u^1 \partial_1 \vartheta \, dx \ \text{ and } \ A_3 := \int \partial_1 \theta_2 \delta u^1 \vartheta \, dx.$$

The term A_2 may be bounded according to (47). In order to bound A_3, we use the anisotropic Hölder inequality and (39). This leads to

$$
\begin{aligned}
|A_3| &\le \|\vartheta\|_{L^\infty_{x_1}(L^2_{x_2})} \|\partial_1\theta_2\|_{L^2_{x_1}(L^\infty_{x_2})} \|\delta u\|_{L^2}, \\
&\le C\|\vartheta\|_{L^2}^{1/2} \|\partial_1\vartheta\|_{L^2}^{1/2} \|\partial_1\theta_2\|_{L^2}^{1/2} \|\partial_1\partial_2\theta_2\|_{L^2}^{1/2} \|\delta u\|_{L^2},
\end{aligned}
$$

whence, resorting again to Young inequality,

$$(48) \qquad |A_3| \le \frac{C}{\kappa} \|\delta\theta\|_{L^2}^2 \|\partial_1\theta_2\|_{L^2}^2 \|\partial_1\partial_2\theta_2\|_{L^2}^2 + \|\delta u\|_{L^2}^2 + \frac{\kappa}{6} \|\partial_1\vartheta\|_{L^2}^2.$$

The term A_1 is the most difficult to deal with. To get an appropriate bound, let us first notice that, as $\operatorname{div} \delta u = 0$, we may write

$$\delta u^2 = (1 - \partial_2^2)^{-1}(\delta u^2) + (1 - \partial_2^2)^{-1}\partial_2\partial_1\delta u^1.$$

Therefore, integrating by parts, we get $A_1 = A_1^1 + A_1^2 + A_1^3$ with

$$
\begin{aligned}
A_1^1 &:= \int (1 - \partial_2^2)^{-1}(\delta u^2) \, \partial_2\theta_2 \, \vartheta \, dx, \\
A_1^2 &:= -\int \partial_2(1 - \partial_2^2)^{-1}(\delta u^1) \, \partial_1\partial_2\theta_2 \, \vartheta \, dx, \\
A_1^3 &:= -\int \partial_2(1 - \partial_2^2)^{-1}(\delta u^1) \, \partial_2\theta_2 \, \partial_1\vartheta \, dx.
\end{aligned}
$$

First of all, we have

$$|A_1^3| \le \|\partial_2(1 - \partial_2^2)^{-1}\delta u\|_{L^2_{x_1}(L^\infty_{x_2})} \|\partial_2\theta_2\|_{L^\infty_{x_1}(L^2_{x_2})} \|\partial_1\vartheta\|_{L^2}.$$

Taking advantage of (39), we get

$$\|\partial_2(1 - \partial_2^2)^{-1}\delta u^1\|_{L^2_{x_1}(L^\infty_{x_2})} \le C\|\partial_2(1 - \partial_2^2)^{-1}(\delta u^1)\|_{L^2}^{\frac{1}{2}} \|\partial_2^2(1 - \partial_2^2)^{-1}\delta u^1\|_{L^2}^{\frac{1}{2}} \le C\|\delta u\|_{L^2}$$

$$\|\partial_2\theta_2\|_{L^\infty_{x_1}(L^2_{x_2})} \le C\|\partial_2\theta\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2\theta\|_{L^2}^{\frac{1}{2}}.$$

In consequence, thanks to Young inequality, we have

$$|A_1^3| \le \frac{C}{\kappa} \|\partial_2\theta\|_{L^2} \|\partial_1\partial_2\theta\|_{L^2} \|\delta u\|_{L^2}^2 + \frac{\kappa}{6} \|\partial_1\delta\theta\|_{L^2}^2.$$

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To deal with A_1^2, one may write that, by virtue of (39) and of Young inequality

$$
\begin{aligned}
|A_1^2| &\leq \|\partial_2(1-\partial_2^2)^{-1}\delta u^1\|_{L_{x_1}^2(L_{x_2}^\infty)}\|\partial_1\partial_2\theta_2\|_{L^2}\|\delta\vartheta\|_{L_{x_1}^\infty(L_{x_2}^2)},\\
&\leq C\|\delta u\|_{L^2}\|\partial_1\partial_2\theta_2\|_{L^2}\|\delta\vartheta\|_{L^2}^{\frac{1}{2}}\|\partial_1\delta\vartheta\|_{L^2}^{\frac{1}{2}},\\
&\leq \frac{C}{\kappa}\|\delta u\|_{L^2}^2\|\partial_1\partial_2\theta_2\|_{L^2}^2 + \frac{3\kappa}{2}\|\delta\vartheta\|_{L^2}^2 + \frac{\kappa}{6}\|\partial_1\delta\vartheta\|_{L^2}^2.
\end{aligned}
$$

Finally, for A_1^1 we have

$$
\begin{aligned}
|A_1^1| &\leq \|(1-\partial_2^2)^{-1}(\delta u_2)\|_{L_{x_1}^2(L_{x_2}^\infty)}\|\partial_2\theta_2\|_{L^2}\|\delta\vartheta\|_{L_{x_1}^\infty(L_{x_2}^2)},\\
&\leq C\|\delta u\|_{L^2}\|\partial_2\theta_2\|_{L^2}\|\delta\vartheta\|_{L^2}^{\frac{1}{2}}\|\partial_1\delta\vartheta\|_{L^2}^{\frac{1}{2}},\\
&\leq \frac{C}{\kappa}\|\delta u\|_{L^2}^2\|\partial_1\theta_2\|_{L^2}^2 + \frac{3\kappa}{2}\|\delta\vartheta\|_{L^2}^2 + \frac{\kappa}{6}\|\partial_1\delta\vartheta\|_{L^2}^2.
\end{aligned}
$$

Putting together all the previous inequalities, we conclude that

$$
\text{(49)} \qquad |A_1| \leq \frac{\kappa}{2}\|\partial_1\delta\vartheta\|_{L^2}^2 + 3\kappa\|\delta\vartheta\|_{L^2}^2 + \frac{C}{\kappa}\|\partial_1\theta_2\|_{H^1}^2\|\delta u\|_{L^2}^2.
$$

Now, inserting Inequalities (47), (48) and (49) in (44), we deduce that there exists an integrable function f_2 over $[0,T]$ depending only on (θ_2, u_2) and on κ such that

$$
\text{(50)} \qquad \frac{1}{2}\frac{d}{dt}\|\delta\vartheta\|_{L^2}^2 \leq f_2(t)\,\|(\delta\vartheta, \delta u)\|_{L^2}^2.
$$

Adapting the well-known Yudovich's argument (see [17] and [24]), it is now easy to complete the proof of uniqueness. Indeed, from Inequality (43), we get for all $p \in [2, \infty[$,

$$
\text{(51)} \qquad \frac{1}{2}\frac{d}{dt}\|\delta u\|_{L^2}^2 \leq \|\nabla u_2\|_{L^p}\|\delta u\|_{L^\infty}^{\frac{2}{p}}\|\delta u\|_{L^2}^{2-\frac{2}{p}} + \|\delta\vartheta\|_{L^2}\|\delta u\|_{L^2}.
$$

Setting $X_\varepsilon(t) := \sqrt{\|(\delta\vartheta, \delta u)(t)\|_{L^2}^2 + \varepsilon^2}$ for $\varepsilon > 0$, and using (50) and (51), we obtain

$$
\frac{d}{dt}X_\varepsilon \leq p\|\nabla u_2\|_L\|\delta u\|_{L^\infty}^{\frac{2}{p}}X_\varepsilon^{1-\frac{2}{p}} + (\frac{1}{2} + f_2)X_\varepsilon.
$$

Now, if we set $Y_\varepsilon = X_\varepsilon \exp\left(-\int_0^t(\frac{1}{2} + f_2(\tau))\,d\tau\right)$, we have

$$
\frac{2}{p}Y_\varepsilon^{\frac{2}{p}-1}\frac{d}{dt}Y_\varepsilon \leq 2\|\nabla u_2\|_L\|\delta u\|_{L^\infty}^{\frac{2}{p}},
$$

whence

$$
Y_\varepsilon(t) \leq \left(\varepsilon^{\frac{2}{p}} + 2\int_0^t \|\nabla u_2\|_L\|\delta u\|_{L^\infty}^{\frac{2}{p}}\,d\tau\right)^{\frac{p}{2}}.
$$

Having ε tend to 0, we discover that for all $t \in \mathbb{R}_+$,

$$
\|(\delta\vartheta, \delta u)(t)\|_{L^2}^2 \leq \|\delta u\|_{L_t^\infty(L^\infty)}^2\left(2\int_0^t \|\nabla u_2\|_L\,d\tau\right)^p.
$$

By Sobolev embedding and thanks to (23) with $p = 4$, we have

$$
\text{(52)} \qquad \|\delta u\|_{L^\infty} \leq C\big(\|\delta u\|_{L^2} + \|\delta\omega\|_{L^4}\big).
$$

As the assumptions made in the proposition ensure that $\omega_i \in L^\infty([0,T]; W^{1,4})$ and that $u_i \in L^\infty([0,T]; L^2)$, we deduce that $\delta u \in L^\infty([0,T] \times \mathbb{R}^2)$. Therefore, there exists some $T_0 > 0$ such that the right-hand side of the above inequality tend to 0 when p goes to infinity. This yields uniqueness on $[0, T_0]$. From a standard connectivity argument, it is now easy to conclude to uniqueness on the whole interval $[0, T]$. ∎

3.4. **Anisotropic a priori estimates.** If in addition to the H^1 hypothesis on (θ_0, u_0), we assume that $\omega_0 \in L^p$ for some p in $[2, \infty[$, then the vorticity equation

$$(53) \qquad \partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta$$

implies that

$$(54) \qquad \|\omega(t)\|_{L^p} \leq \|\omega_0\|_{L^p} + \int_0^t \|\partial_1 \theta\|_{L^p}.$$

Now, remind that as $\theta_0 \in H^1$, a bound for $\partial_1 \theta$ in $L^2_{loc}(\mathbb{R}_+; H^1)$ is available, whence also in $L^2_{loc}(\mathbb{R}_+; L^p)$ by Sobolev embedding. In fact, we even have a more accurate information if $\omega_0 \in \sqrt{L}$. Indeed, Lemma 13 ensures that H^1 is continuously embedded in $\sqrt{L}$ so that, according to (54),

$$(55) \qquad \|\omega(t)\|_{\sqrt{L}} \leq \|\omega_0\|_{\sqrt{L}} + C\sqrt{t}\,\|\partial_1 \theta\|_{L^2_t(H^1)}.$$

However, this bound *does not* imply that $\nabla u \in L^1_{loc}(\mathbb{R}_+; L)$ so that one cannot get uniqueness by a direct application of Proposition 11. In fact, thanks to (23), it is obvious that $\nabla u \in L^1_{loc}(\mathbb{R}_+; L)$ provided $\omega \in L^1_{loc}(\mathbb{R}_+; L^2 \cap L^\infty)$. According to (54), having $\partial_1 \theta$ in $L^1(\mathbb{R}_+; L^\infty)$ will entail that the vorticity is bounded.

In order to get this, we shall first show that one may propagate some additional horizontal Sobolev regularity for θ. By virtue of Lemma 14 (see the appendix), this will enable us to estimate $\partial_1 \theta$ in $L^1(\mathbb{R}_+; L^\infty)$ (and even in $L^2_{loc}(\mathbb{R}_+; L^\infty)$ actually).

More precisely, we assume from now on that $(\theta_0, u_0) \in H^1(\mathbb{R}^2)$ and $\omega_0 \in \sqrt{L}$, and that, in addition, $|\partial_1|^{1+s}\theta_0 \in L^2$ for some $s \in (0, \frac{1}{2}]$. In order to propagate the additional regularity, one may apply operator $|\partial_1|^{1+s}$ to the equation

$$\partial_t \theta + u \cdot \nabla \theta - \kappa \partial_1^2 \theta = 0,$$

and take the $L^2(\mathbb{R}^2)$ inner product with $|\partial_1|^{1+s}\theta$. After integrating by parts, we find that

$$(56) \qquad \frac{1}{2}\frac{d}{dt}\||\partial_1|^{1+s}\theta\|_{L^2}^2 + \kappa\||\partial_1|^{2+s}\theta\|_{L^2}^2 \leq \left|\left(|\partial_1|^{1+s}(u \cdot \nabla \theta), |\partial_1|^{1+s}\theta\right)_{L^2}\right|.$$

Bounding the right-hand side is the main difficulty. First of all, let us notice that $i|\partial_1| = \partial_1 R_1$ where R_1 stands for the Riesz operator with respect to the first variable. As $|\partial_1|^s$ is a symmetric operator, one may write

$$\left|\left(|\partial_1|^{1+s}(u \cdot \nabla \theta), |\partial_1|^{1+s}\theta\right)_{L^2}\right| = \left|\left(\partial_1(u \cdot \nabla \theta), R_1|\partial_1|^{1+2s}\theta\right)_{L^2}\right| \leq I_1 + I_2$$

with

$$I_1 := \left|\left(\partial_1 u \cdot \nabla \theta, R_1|\partial_1|^{1+2s}\theta\right)_{L^2}\right| \quad \text{and} \quad I_2 := \left|\left(u \cdot \nabla \partial_1 \theta, R_1|\partial_1|^{1+2s}\theta\right)_{L^2}\right|.$$

The term I_2 is easy to deal with. Indeed, for $s \in (0, 1/2]$, we have, according to Hölder and Parseval inequalities,

$$\begin{aligned} I_2 &\leq \|u\|_{L^\infty}\|\nabla \partial_1 \theta\|_{L^2}\||\partial_1|^{1+2s}\theta\|_{L^2}, \\ &\leq \|u\|_{L^\infty}\|\partial_1 \theta\|_{H^1}^2. \end{aligned}$$

Thanks to (52) and by virtue of Inequalities (36), (40) and (55), we thus have

$$(57) \qquad \int_0^t I_2(\tau)\,d\tau \leq C(t, \kappa, \theta_0, u_0)$$

where, from now on, $C(t, \kappa, \theta_0, u_0)$ denotes a positive continuous function depending only on t, κ and on the norm of (θ_0, u_0) in $H^1 \times (H^1 \cap W^{1,4})$.

In order to bound the term I_1 one may write $I_1 \leq I_1^1 + I_1^2$ with

$$\begin{aligned} I_1^1 &:= \left|\left(\partial_1 u^1 \partial_1 \theta, R_1|\partial_1|^{1+2s}\theta\right)_{L^2}\right|, \\ I_1^2 &:= \left|\left(\partial_1 u^2 \partial_2 \theta, R_1|\partial_1|^{1+2s}\theta\right)_{L^2}\right|. \end{aligned}$$

For I_1^1, as $\partial_1 u_1 = -\partial_2 u_2$, integrating by parts yields

$$\begin{aligned} I_1^1 &\leq \left|\left(u^2 \partial_2 \partial_1 \theta, R_1|\partial_1|^{1+2s}\theta\right)_{L^2}\right| + \left|\left(u^2 \partial_1 \theta, R_1|\partial_1|^{1+2s}\partial_2 \theta\right)_{L^2}\right|, \\ &\leq \widetilde{I}_1^1 + \widehat{I}_1^1 \end{aligned}$$

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with
$$\widetilde{I}_1^1 := \left| \left(u^2 \partial_2 \partial_1 \theta, R_1 |\partial_1|^{1+2s} \theta \right)_{L^2} \right| \quad \text{and} \quad \widehat{I}_1^1 := \left| \left(|\partial_1|^{2s} (u^2 \partial_1 \theta), R_1 |\partial_1| \partial_2 \theta \right) \right|.$$

In order to bound the term $\widetilde{I}_1^1$, one may combine Hölder Inequality and (52). As $0 < s \le 1/2$, we get
$$\begin{aligned} \widetilde{I}_1^1 &\le \|u\|_{L^\infty} \|\partial_1 \theta\|_{H^1} \||\partial_1|^{1+2s} \theta\|_{L^2}, \\ &\le C \left(\|u\|_{L^2} + \|\omega\|_{L^4} \right) \|\partial_1 \theta\|_{H^1}^2. \end{aligned}$$

In consequence, by virtue of (36), (40) and (55), we have
$$\tag{58} \int_0^t \widetilde{I}_1^1(\tau) d\tau \le C(t, \kappa, \theta_0, u_0).$$

As for $\widehat{I}_1^1$, we use the fact that
$$\tag{59} \widehat{I}_1^1 \le \||\partial_1|^{2s} (u^2 \partial_1 \theta)\|_{L^2} \|\partial_1 \partial_2 \theta\|_{L^2}.$$

Because $s \in (0, 1/2]$, we have
$$\||\partial_1|^{2s} (u^2 \partial_1 \theta)\|_{L^2} \le \|u^2 \partial_1 \theta\|_{H^1},$$

whence
$$\begin{aligned} \||\partial_1|^{2s} (u^2 \partial_1 \theta)\|_{L^2} &\le \|u^2 \partial_1 \theta\|_{L^2} + \|\partial_1 \theta \nabla u^2\|_{L^2} + \|u^2 \partial_1 \nabla \theta\|_{L^2}, \\ &\le \|u\|_{L^\infty} \|\partial_1 \theta\|_{L^2} + \|\nabla u^2\|_{L^4} \|\partial_1 \theta\|_{L^4} + \|u\|_{L^\infty} \|\nabla \partial_1 \theta\|_{L^2}. \end{aligned}$$

Thanks to the Sobolev embedding $H^1 \hookrightarrow L^4$ and to (23), (52) we get
$$\||\partial_1|^{2s} (u^2 \partial_1 \theta)\|_{L^2} \le C \left(\|u\|_{L^\infty} + \|\omega\|_{L^4} \right) \|\partial_1 \theta\|_{H^1}.$$

Coming back to (59) and using (52), one can now conclude that
$$\tag{60} \int_0^t \widehat{I}_1^1(\tau) d\tau \le C(t, \kappa, \theta_0, u_0).$$

The term I_1^2 is more intricate to deal with. To start with, we integrate by parts to rewrite this term as follows:
$$I_1^2 \le \left| \int u_2 \, \partial_1 \partial_2 \theta \, R_1 |\partial_1|^{1+2s} \theta \, dx \right| + \left| \int |\partial_1|^s (u_2 \partial_2 \theta) |\partial_1|^{2+s} \theta \, dx \right|,$$

from which we get the following bound:
$$I_1^2 \le \|u_2\|_{L^\infty} \|\partial_1 \partial_2 \theta\|_{L^2} \||\partial_1|^{1+2s} \theta\|_{L^2} + \||\partial_1|^s (u_2 \partial_2 \theta)\|_{L^2} \||\partial_1|^{2+s} \theta\|_{L^2}.$$

As $s \in (0, 1/2]$, Young inequality enables us to write
$$\tag{61} I_1^2 \le \|u\|_{L^\infty} \|\partial_1 \theta\|_{H^1}^2 + \frac{\kappa}{2} \||\partial_1|^{2+s} \theta\|_{L^2}^2 + \frac{1}{2\kappa} \||\partial_1|^s (u_2 \partial_2 \theta)\|_{L^2}^2.$$

Let us admit (see the proof in appendix) that there exists a constant C such that for all $s \in (0, 1/2]$ we have
$$\tag{62} \||\partial_1|^s (u_2 \partial_2 \theta)\|_{L^2} \le C \|u\|_{H^1} \left(\|\partial_2 \theta\|_{L^2} + \|\partial_1 \partial_2 \theta\|_{L^2} \right).$$

Using (52) and plugging (36), (40) and (55) in (61), we get
$$\tag{63} \int_0^t I_1^2(\tau) \, d\tau \le C(t, \kappa, \theta_0, u_0) + \frac{\kappa}{2} \int_0^t \||\partial_1|^{2+s} \theta(\tau)\|_{L^2}^2 \, d\tau.$$

It is now suitable to integrate (56) with respect to time and to plug (57), (58), (60) in (63). We eventually get for all $s \in (0, 1/2]$,
$$\tag{64} \||\partial_1|^{1+s} \theta(t)\|_{L^2}^2 + \kappa \int_0^t \||\partial_1|^{2+s} \theta\| \, d\tau \le \||\partial_1|^{1+s} \theta_0\|_{L^2}^2 + C(t, \kappa, \theta_0, u_0).$$

Resorting to Lemma (14) with $s_1 = 1 + s$ and $s_2 = 1$, we find that
$$\int_0^t \|\partial_1 \theta\|_{L^\infty}^2 \, d\tau \le C \int_0^t \left(\|\partial_1 \theta\|_{L^2}^2 + \||\partial_1|^{2+s} \theta\|_{L^2}^2 + \|\partial_1 \partial_2 \theta\|_{L^2}^2 \right) d\tau.$$

Therefore, by virtue of Inequalities (40) and (64), we get a bound for $\partial_1\theta$ in $L^2([0,t]; L^\infty)$ in terms of t and of the norms of the initial data. As explained before, this supplies the desired bound for the vorticity in $L^\infty_{loc}(\mathbb{R}_+; L^\infty)$.

3.5. A global existence result.
This paragraph is devoted to proving the following result (which obviously implies Theorem (2)):

Theorem 12. *Let $(\theta_0, u_0) \in H^1$ with $\mathrm{div}\, u_0$. System (2) has a global solution (θ, u) such that*

$$(\theta, u) \in \mathcal{C}_w(\mathbb{R}_+; H^1) \quad and \quad \partial_1\theta \in L^2_{loc}(\mathbb{R}_+; H^1).$$

If in addition $\omega_0 \in \sqrt{L}$ then one may construct a global solution which also satisfies

$$\omega \in L^\infty_{loc}(\mathbb{R}_+; \sqrt{L}).$$

If in addition $\omega_0 \in L^\infty$ and there exists $s \in (0, 1/2]$ such that $|\partial_1|^{1+s}\theta_0 \in L^2$ then the above solution is unique, strongly continuous in time with values in H^1, and satisfies

$$|\partial_1|^{1+s}\theta \in \mathcal{C}(\mathbb{R}_+; L^2) \quad and \quad |\partial_1|^{2+s}\theta \in L^2_{loc}(\mathbb{R}_+; L^2).$$

Proof: The result may be obtained by means of the Friedrichs method. With the notation of the previous section, we solve the following ODE in L^2_n:

$$\begin{cases} \partial_t\theta + J_n \,\mathrm{div}\,(J_n\theta\, J_n u) - \kappa\partial_1^2 \mathcal{P} J_n\theta = 0, \\ \partial_t u + \mathcal{P} J_n \,\mathrm{div}\,(\mathcal{P} J_n u \otimes \mathcal{P} J_n u) = \mathcal{P} J_n(\theta e_2), \\ (\theta, u)|_{t=0} = J_n(\theta_0, u_0). \end{cases}$$

Cauchy-Lipschitz theorem gives a unique maximal solution (θ_n, u_n) in the space $\mathcal{C}^1([0, T_n^*); L^2_n)$. As $J_n^2 = J_n$, $\mathcal{P}^2 = \mathcal{P}$ et $J_n\mathcal{P} = \mathcal{P} J_n$, we deduce that $\mathcal{P} u_n = u_n$, $J_n u_n = u_n$ and $J_n\theta_n = \theta_n$. Therefore (θ_n, u_n) satisfies

(65)
$$\begin{cases} \partial_t\theta_n + J_n \,\mathrm{div}\,(\theta_n u_n) - \kappa\partial_1^2\theta_n = 0, \\ \partial_t u_n + \mathcal{P} J_n \,\mathrm{div}\,(u_n \otimes u_n) = \mathcal{P} J_n(\theta_n e_2). \end{cases}$$

As usual, because operators J_n and $\mathcal{P} J_n$ are orthogonal projectors in all the Sobolev spaces, all the previous formal a priori estimates pertaining to Sobolev norms remind true. More precisely, we still have (36), (37) and (40) so that

- $(\theta_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty_{loc}(\mathbb{R}_+; H^1)$,
- $(\partial_1\theta_n)_{n\in\mathbb{N}}$ is bounded in $L^2_{loc}(\mathbb{R}+; H^1)$,
- $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty_{loc}(\mathbb{R}_+; H^1)$.

This is fully enough to pass to the limit (up to extraction) in System (65) and to get the first part of the theorem.

In order to construct weak solutions preserving the $\sqrt{L}$ and the anisotropic regularities, one may smooth out System (2) by means of an artificial viscosity. More precisely, we first solve the following system for $\varepsilon > 0$:

$$\begin{cases} \partial_t\theta + u \cdot \nabla\theta - \kappa\partial_1^2\theta - \varepsilon\Delta\theta = 0 \\ \partial_t u + u \cdot \nabla u + \nabla\Pi - \varepsilon\Delta u = \theta e_2 \\ \mathrm{div}\, u = 0 \end{cases}$$

supplemented with smoothed out initial data $(\theta_0^\varepsilon, u_0^\varepsilon)$.

Resorting again to the Friedrichs method that has been used in the case $\varepsilon = 0$, and noticing that the cut-off operator J_n does not modify the Sobolev estimates, we get a global solution $(\theta^\varepsilon, u^\varepsilon)$ in

$$\mathcal{C}(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^2)$$

satisfying Inequalities (37) and (40) uniformly with respect to ε.

Actually, using standard methods, one can check that the H^2 regularity controls higher Sobolev norms. As the initial data are in H^∞, the solution $(\theta^\varepsilon, u^\varepsilon)$ thus belongs to all the Sobolev spaces, which will enable us to make the following computations rigorous.

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The L^p estimate over the vorticity may be proved by multiplying the vorticity equation

$$\partial_t \omega^\varepsilon + u^\varepsilon \cdot \nabla \omega^\varepsilon - \varepsilon \Delta \omega^\varepsilon = \partial_1 \theta^\varepsilon$$

by $|\omega^\varepsilon|^{p-2}\omega^\varepsilon$, and performing an integration over $\mathbb{R}^2$. This gives again

$$\|\omega^\varepsilon(t)\|_{L^p} \leq \|\omega_0\|_{L^p} + \int_0^t \|\partial_1 \theta^\varepsilon\|_{L^p} \leq \|\omega_0\|_{L^p} + C\sqrt{pt}\,\|\partial_1 \theta^\varepsilon\|_{L_t^2(H^1)}.$$

It is also clear that all the anisotropic Sobolev estimates remain the same, uniformly with respect to ε. Therefore, having ε tend to 0 yields the end of the existence part of Theorem 10.

Finally, the uniqueness result is a mere consequence of Proposition 11. ∎

4. Appendix

4.1. **A few inequalities.** Here we prove a few inequalities which have been used throughout the paper.

Proof of Inequality (20) *:* For proving (20), one may split ∇u into low and high frequencies according to the Littlewood-Paley decomposition. More precisely, for any $N \in \mathbb{N}$ one may write

$$\nabla u = S_N \nabla u + \sum_{q \geq N} \Delta_q \nabla u.$$

We thus have

$$\|\nabla u\|_{L^\infty} \leq \|S_N \nabla u\|_{L^\infty} + \sum_{q \geq N} \|\Delta_q \nabla u\|_{L^\infty},$$

whence, using the definition of $\|\cdot\|_{LL}$ and Bernstein inequalities,

$$\|\nabla u\|_{L^\infty} \leq (N+1)\|\nabla u\|_{LL} + C\sum_{q \geq N} 2^q \|\Delta_q \nabla u\|_{L^2}.$$

Given that $\|\Delta_q \nabla u\|_{L^2} = \|\Delta_q \omega\|_{L^2}$ and that $2 - s < 0$, we readily get

$$\|\nabla u\|_{L^\infty} \leq (N+1)\|\nabla u\|_{LL} + C2^{N(2-s)}\|\omega\|_{H^{s-1}}.$$

Now, if $C\|\omega\|_{H^{s-1}} \leq \|\nabla u\|_{LL}$ then taking $N = 0$ obviously yields the desired inequality. Else, one may choose for N the integer part of

$$\frac{1}{s-2}\log_2\left(\frac{C\|\omega\|_{H^{s-1}}}{\|\nabla v\|_{LL}}\right)$$

and we still get the desired result. ∎

Lemma 13. *In dimension two, the Sobolev space H^1 continuously embeds in the space $\sqrt{L}$.*

Proof: For any $p \in [2, \infty[$ and $v \in H^1$, using the Littlewood-Paley decomposition and a Bernstein inequality enables us to write

$$
\begin{aligned}
\|v\|_{L^p} &\leq \sum_{q \geq -1} \|\Delta_q v\|_{L^p}, \\
&\leq C\sum_{q \geq -1} 2^{-\frac{2q}{p}} 2^q \|\Delta_q v\|_{L^2}, \\
&\leq C\left(\sum_{q \geq -1} 2^{-\frac{4q}{p}}\right)^{\frac{1}{2}} \|v\|_{H^1}, \\
&\leq C\sqrt{p-1}\,\|v\|_{H^1},
\end{aligned}
$$

whence the desired result. ∎

Proof of Inequalities (39)*:* For stating the first inequality, the starting point is the following classical one-dimensional Gagliardo-Nirenberg inequality:

$$(66) \qquad \|f(x_1, \cdot)\|_{L_{x_2}^\infty} \leq \|f(x_1, \cdot)\|_{L_{x_2}^2}^{1/2}\|\partial_2 f(x_1, \cdot)\|_{L_{x_2}^2}^{1/2}.$$

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Taking the $L^2_{x_1}$ norm of both sides and using Cauchy-Schwarz inequality, we get

$$\|f\|_{L^2_{x_1}(L^\infty_{x_2})} \leq C\|f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\partial_2 f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}.$$

For proving the second inequality, it is only a matter of swapping the roles of variables x_1 and x_2, and using Minkowski's inequality. ∎

Proof of Inequality (46): From (66), we deduce that

$$\|f\|_{L^\infty} \leq C\|f\|^{1/2}_{L^\infty_{x_1}(L^2_{x_2})}\|\partial_2 f\|^{1/2}_{L^\infty_{x_1}(L^2_{x_2})}.$$

Applying the second inequality of (39) to f and $\partial_2 f$, it is now easy to complete the proof. ∎

Proof of Inequality (62): Obviously, it suffices to state that

$$\|fg\|_{L^2_{x_2}(H^{1/2}_{x_1})} \leq C\|f\|_{H^1}\big(\|g\|_{L^2} + \|\partial_1 g\|_{L^2}\big).$$

For proving the above inequality, we first notice that the standard product laws for one-dimensional Sobolev spaces ensure that for all fixed x_2, we have

$$\|(fg)(\cdot, x_2)\|_{H^{1/2}(\mathbb{R})} \leq C\|f(\cdot, x_2)\|_{H^{1/2}(\mathbb{R})}\|g(\cdot, x_2)\|_{H^1(\mathbb{R})}.$$

Therefore

$$\|fg\|_{L^2_{x_2}(H^{1/2}_{x_1})} \leq C\|f\|_{L^\infty_{x_2}(H^{1/2}_{x_1})}\|g\|_{L^2_{x_2}(H^1_{x_1})}.$$

Because the trace operator on $x_2 = cste$ is continuous from $H^1(\mathbb{R}^2)$ to $H^{1/2}(\mathbb{R})$, we get the desired inequality. ∎

In the last part of the paper, anisotropic Sobolev norms have been used several times. Below, we state a sufficient condition under which anisotropic Sobolev spaces are embedded in the set of bounded functions.

Lemma 14. *For any couple (s_1, s_2) of positive real numbers satisfying $1/s_1 + 1/s_2 < 2$ there exists a constant C such that*

$$\|u\|_{L^\infty} \leq C\big(\|u\|_{L^2} + \||\partial_1|^{s_1} u\|_{L^2} + \||\partial_2|^{s_2} u\|_{L^2}\big).$$

Proof: Using Fourier variables, we see that

$$\|\widehat{u}(\xi)\|^2_{L^1(\mathbb{R}^2)} \leq \left(\int (1 + |\xi_1|^{2s_1} + |\xi_2|^{2s_2})|\widehat{u}(\xi)|^2 \, d\xi\right) \times \left(\int (1 + |\xi_1|^{2s_1} + |\xi_2|^{2s_2})^{-1} \, d\xi\right).$$

Therefore, it suffices to show that

$$\int (1 + |\xi_1|^{2s_1} + |\xi_2|^{2s_2})^{-1} \, d\xi < \infty.$$

If we make the change of variable

$$\xi_1 = (1 + |\xi_2|^{2s_2})^{\frac{1}{2s_1}} \zeta_1$$

we get

$$\int (1 + |\xi_1|^{2s_1} + |\xi_2|^{2s_2})^{-1} \, d\xi = \int (1 + |\xi_2|^{2s_2})^{-1 + \frac{1}{2s_1}} (1 + \zeta_1^{2s_1})^{-1} \, d\zeta_1 \, d\xi_2.$$

This integral is finite whenever $s_1 > \frac{1}{2}$ and $s_2(1 - \frac{1}{2s_1}) > \frac{1}{2}$, a condition which is equivalent to $1/s_1 + 1/s_2 < 2$. ∎

4.2. **Losing a priori estimates.** The second part of the appendix is mainly devoted to the proof of losing a priori estimates for the following anisotropic Stokes system with convection

$$(67) \qquad \begin{cases} \partial_t w + v \cdot \nabla w - \nu \partial_1^2 w + \nabla \Pi = f + g\, e_2, \\ \operatorname{div} w = 0 \end{cases}$$

in the case where the gradient of the divergence free vector field is only in $L^1([0,T]; LL^{\frac{1}{2}})$ (where $LL^{\frac{1}{2}}$ has been defined in (26)). Remind that those estimates are the key to the proof of uniqueness in Theorem 1. Albeit similar results have been proved before in [12], we also prove losing a priori estimates for ordinary transport equations for the reader convenience.

The key to the proof of all those losing a priori estimates is the following commutator estimate (which is also used in the proof of Inequality (15)).

Lemma 15. *Let v be a divergence free vector-field over $\mathbb{R}^2$. Let $\omega := \partial_1 v^2 - \partial_2 v^1$. There exists a positive constant C such that for all $q \geq -1$, the term $F_q(v, \rho) := \overline{S}_{q-1} v \cdot \nabla \Delta_q \rho - \Delta_q(v \cdot \nabla \rho)$ (with $\overline{S}_{q-1}$ defined in (14)) satisfies the following estimates :*

$$(68) \qquad \|F_q(v,\rho)\|_{L^2} \leq C \|\nabla v\|_{L^\infty} \sum_{q' \geq q-4} 2^{q-q'} \|\Delta_{q'} \rho\|_{L^2} + \|\rho\|_{L^\infty} \sum_{|q-q'| \leq 4} \|\Delta_{q'} \omega\|_{L^2},$$

$$(69) \qquad \|F_q(v,\rho)\|_{L^2} \leq C \sqrt{q+2}\, \|\nabla v\|_{LL^{\frac{1}{2}}} \sum_{q'} 2^{-|q-q'|} \|\Delta_{q'} \rho\|_{L^2}.$$

In the case $\rho = \omega$, we have in addition

$$(70) \qquad \|F_q(v,\omega)\|_{L^2} \leq C \|\nabla v\|_{L^\infty} \sum_{q' \geq q-4} 2^{q-q'} \|\Delta_{q'} \omega\|_{L^2}.$$

Proof: Decompose $F_q(v,\rho)$ into $F_q^1(v,\rho) + F_q^2(v,\rho) + F_q^3(v,\rho) + F_q^4(v,\rho)$ with

$$F_q^1(v,\rho) := \sum_{q' \geq -1} [\overline{S}_{q'-1} v, \Delta_q] \cdot \nabla \Delta_{q'} \rho, \qquad F_q^2(v,\rho) := \sum_{q' \geq -1} (\overline{S}_{q-1} - \overline{S}_{q'-1}) v \cdot \nabla \Delta_q \Delta_{q'} \rho,$$

$$F_q^3(v,\rho) := -\Delta_q \Big(\sum_{q' \geq 1} S_{q'-1} \partial_i \rho\, \Delta_{q'} v^i \Big), \qquad F_q^4(v,\rho) := -\sum_{q' \geq 0} \partial_i \Delta_q \Big(\Delta_{q'} v^i \Big(\sum_{|\alpha| \leq 1} \Delta_{q'+\alpha} \Big) \rho \Big).$$

Let us emphasize that only the term F_q^1 involves low frequencies of v. Taking advantage of the support properties of the function φ defined at the beginning of Subsection 2.2, we notice that the summation in the definition of F_q^1 may be restricted to those indices q' such that $|q'-q| \leq 4$. Therefore, a standard commutator inequality (see e.g. [4], Chap. 2) ensures that

$$(71) \qquad \|F_q^1(v,\rho)\|_{L^2} \leq C \sum_{|q'-q| \leq 4} \|\nabla \overline{S}_{q'-1} v\|_{L^\infty} \|\Delta_{q'} \rho\|_{L^2}.$$

For $F_q^2(v,\rho)$, we obtain, according to Hölder and Bernstein inequalities, and to the localization properties of the Littlewood-Paley decomposition,

$$(72) \qquad \|F_q^2(v,\rho)\|_{L^2} \leq C \sum_{|q'-q| \leq 1} \|\nabla \check{\Delta}_q v\|_{L^\infty} \|\Delta_q \rho\|_{L^2} \quad \text{with} \quad \check{\Delta}_q := \sum_{|\alpha| \leq 4} \Delta_{q+\alpha}.$$

From the definition of operator $S_{q'-1}$, the localization properties of operators Δ_q and Bernstein inequalities, we get

$$(73) \qquad \|F_q^3(v,\rho)\|_{L^2} \leq \sum_{q' \leq q+3} 2^{q''-q} \|\Delta_{q''} \rho\|_{L^2} \|\check{\Delta}_q \nabla v\|_{L^\infty}.$$

Notice that one can alternately get the following inequality :

$$(74) \qquad \|F_q^3(v,\rho)\|_{L^2} \leq C \|\rho\|_{L^\infty} \sum_{|q'-q| \leq 4} \|\Delta_{q'} \omega\|_{L^2}.$$

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Indeed, it is only a matter of using that the sum defining $F_q^3(v, \rho)$ may be restricted to $q' \geq 1$ and thus, according to Bernstein inequalities and to $\|\nabla \Delta_{q'} v\|_{L^2} = \|\Delta_{q'} \omega\|_{L^2}$, one may write

$$
\begin{aligned}
\|S_{q'-1} \partial_i \rho \, \Delta_{q'} v^i\|_{L^2} &\leq C \|S_{q'-1} \partial_i \rho\|_{L^\infty} 2^{-q'} \|\nabla \Delta_{q'} v^i\|_{L^2}, \\
&\leq C \|\rho\|_{L^\infty} \|\Delta_{q'} \omega\|_{L^2}.
\end{aligned}
$$

Finally the term $F_q^4(v, \rho)$ may be bounded as follows:

$$
(75) \qquad \|F_q^4\|_{L^2} \leq \sum_{q' \geq q-3} 2^{q-q'} \|\Delta_{q'} \rho\|_{L^2} \|\nabla \check{\Delta}_{q'} v\|_{L^\infty}.
$$

Because

$$
\|\nabla \check{\Delta}_q v\|_{L^\infty} \leq C \|\nabla v\|_{L^\infty} \quad \text{and} \quad \|\nabla \check{\Delta}_q v\|_{L^\infty} \leq C \sqrt{q+2} \, \|\nabla v\|_{LL^{\frac{1}{2}}},
$$

Inequalities (71) to (75) enable us to get (68) and (69). Inequality (70) stems from (68). ∎

One can turn to the statement of losing a priori estimates. For technical reasons, we adopt the framework of Besov spaces $B_{2,\infty}^\sigma$. As we have $H^\sigma \hookrightarrow B_{2,\infty}^\sigma$ and $B_{2,\infty}^\sigma \hookrightarrow H^{\sigma'}$ for all $\sigma > \sigma'$, it is of course not difficult to rewrite all those estimates in terms of Sobolev norms.

For the transport equation, we shall prove the following result (in the spirit of [3, 12]).

Proposition 16. *Let ρ satisfy the transport equation*

$$
(76) \qquad \partial_t \rho + v \cdot \nabla \rho = f
$$

with initial data $\rho_0 \in B_{2,\infty}^s$ and source term $f \in L^1([0, T]; B_{2,\infty}^s)$. Assume in addition that $\mathrm{div}\, v = 0$ and that, for some $V \in L^1([0, T])$ we have

$$
(77) \qquad \sup_{N \geq 0} \frac{\|\nabla S_N v(t)\|_{L^\infty}}{\sqrt{1+N}} \leq V(t).
$$

For all $s \in (-1, 1)$, there exists a constant C depending only on s such that for all $\varepsilon \in \,]0, (s+1)/2[$ and $t \in [0, T]$, we have

$$
\|\rho(t)\|_{B_{2,\infty}^{s-\varepsilon}} \leq C \exp\left(\frac{C}{\varepsilon} \left(\int_0^t V(\tau) \, d\tau \right)^2 \right) \left(\|\rho_0\|_{B_{2,\infty}^s} + \|f\|_{L_t^1(B_{2,\infty}^s)} \right).
$$

Proof: Applying Δ_q to Equation (76), one may write

$$
\partial_t \Delta_q \rho + \overline{S}_{q-1} v \cdot \nabla \Delta_q \rho = \Delta_q f + F_q(v, \rho) \quad \text{with} \quad F_q(v, \rho) := \overline{S}_{q-1} v \cdot \nabla \Delta_q \rho - \Delta_q(v \cdot \nabla \rho).
$$

Taking the L^2 inner product of this inequality with $\Delta_q \rho$ and observing that $\mathrm{div}\, \overline{S}_{q-1} v = 0$, we thus get

$$
(78) \qquad \|\Delta_q \rho(t)\|_{L^2} \leq \|\Delta_q \rho_0\|_{L^2} + \int_0^t \|\Delta_q f\|_{L^2} \, d\tau + \int_0^t \|F_q(v, \rho)\|_{L^2} \, d\tau.
$$

From Inequality (69), we readily get for all $\varepsilon \in \,]0, (s+1)/2[$, $q \geq -1$ and $t \in [0, T]$,

$$
(79) \qquad 2^{q(s-\varepsilon)} \|F_q(v(t), \rho(t))\|_{L^2} \leq C \sqrt{q+2} \, V(t) \|\rho(t)\|_{B_{2,\infty}^{s-\varepsilon}}
$$

for some constant C depending only on s.

Set $\eta = \varepsilon / \int_0^T V(\tau) \, d\tau$ and $s_t := s - \eta \int_0^t V(\tau) \, d\tau$ for $t \in [0, T]$. Putting (78) and (79) together yields

$$
\begin{aligned}
2^{(2+q)s_t} \|\Delta_q \rho(t)\|_{L^2} \leq\ & 2^{(2+q)s} \|\Delta_q \rho_0\|_{L^2} 2^{-\eta(2+q) \int_0^t V(\tau') \, d\tau'} \\
& + \int_0^t 2^{(2+q)s_\tau} \|\Delta_q f(\tau)\|_{L^2} 2^{-\eta(2+q) \int_\tau^t V(\tau') \, d\tau'} \, d\tau \\
(80) \qquad & + C \int_0^t \sqrt{2+q} \, V(\tau) 2^{-\eta(2+q) \int_\tau^t V(\tau') \, d\tau'} \|\rho(\tau)\|_{B_{2,\infty}^{s_\tau}} \, d\tau.
\end{aligned}
$$

Notice that if q satisfies

$$
(81) \qquad 2 + q \geq \frac{4C^2}{\eta^2 \log 4}
$$

then the last term may be bounded by

$$\frac{1}{2} \sup_{\tau \in [0,t]} \|\rho(\tau)\|_{B_{2,\infty}^{s\tau}}$$

whereas if q does not satisfy (81) then it may be bounded by

$$\frac{2C^2}{\eta \log 2} \int_0^t V(\tau) \|\rho(\tau)\|_{B_{2,\infty}^{s\tau}} \, d\tau.$$

So finally, taking the supremum over $q \geq -1$ in (80) and using the above two inequalities, we get

$$\sup_{\tau \in [0,t]} \|\rho(\tau)\|_{B_{2,\infty}^{s\tau}} \leq 2\|\rho_0\|_{B_{2,\infty}^s} + 2\int_0^t \|f(\tau)\|_{B_{2,\infty}^{s\tau}} \, d\tau + \frac{C}{\eta} \int_0^t V(\tau) \|\rho(\tau)\|_{B_{2,\infty}^{s\tau}} \, d\tau.$$

Thanks to Gronwall lemma, we end up with

$$\sup_{t \in [0,T]} \|\rho(t)\|_{B_{2,\infty}^{s_t}} \leq 2e^{\frac{C}{\eta} \int_0^T V(t) \, dt} \left(\|\rho_0\|_{B_{2,\infty}^s} + \int_0^T \|f(t)\|_{B_{2,\infty}^{s_t}} \, dt \right),$$

which entails the desired inequality given that $s \geq s_t \geq s - \varepsilon$ for all $t \in [0,T]$. ∎

A similar result turns to be true for System 67. In addition, owing to the anisotropic viscosity, we get an extra horizontal smoothing (which was the key to the proof of Proposition 9. More precisely, we have:

Proposition 17. *Let v and s be as in Proposition* 16. *Then we have*

$$\|w(t)\|_{B_{2,\infty}^{s-\varepsilon}} + \nu^{\frac{1}{2}} \|\partial_1 w\|_{L_t^2(B_{2,\infty}^{s-\varepsilon})}$$

$$\leq C(1 + \sqrt{\nu t}) \exp\left(\frac{C}{\varepsilon} \left(\int_0^t V(\tau) \, d\tau \right)^2 \right) \left(\|\rho_0\|_{B_{2,\infty}^s} + \|f\|_{L_t^1(B_{2,\infty}^s)} + \nu^{-\frac{1}{2}} \|g\|_{L_t^2(B_{2,\infty}^{s-1})} \right).$$

Proof: Let us first apply operator Δ_q to System (67). With the notation introduced in the proof of Proposition 16, we have

$$\partial_t \Delta_q w + \overline{S}_{q-1} v \cdot \nabla \Delta_q w - \nu \partial_1^2 \Delta_q w + \nabla \Delta_q \Pi = \Delta_q f + \Delta_q g \, e_2 + F_q(v,w)$$

with $F_q(v,w)$ satisfying (79).

Taking the L^2 inner product and using the fact that $\operatorname{div} v = \operatorname{div} w = 0$, we see that

$$(82) \quad \frac{1}{2} \frac{d}{dt} \|\Delta_q w\|_{L^2}^2 + \nu \|\partial_1 \Delta_q w\|_{L^2}^2 = \int \Delta_q f \cdot \Delta_q w \, dx + \int F_q(v,w) \cdot \Delta_q w \, dx + \int \Delta_q g \Delta_q w^2 \, dx.$$

Assume that $q \geq 0$. Taking advantage of Parseval equality, one may write

$$\int \Delta_q g \Delta_q w^2 \, dx = -\int (-\Delta)^{-1} \Delta_q g \, \Delta \Delta_q w^2 \, dx,$$

$$= -\int (-\Delta)^{-1} \Delta_q g \, \Delta_q \partial_1^2 w^2 \, dx - \int (-\Delta)^{-1} \Delta_q g \, \Delta_q \partial_2^2 w^2 \, dx.$$

As $\operatorname{div} w = 0$, integrating by parts yields

$$\int \Delta_q g \Delta_q w^2 \, dx = -\int (-\Delta)^{-1} \Delta_q g \, \Delta_q \partial_1^2 w^2 \, dx + \int (-\Delta)^{-1} \Delta_q g \, \Delta_q \partial_1 \partial_2 w^1 \, dx,$$

$$= \int \partial_1 (-\Delta)^{-1} \Delta_q g \, \Delta_q \partial_1 w^2 \, dx - \int \partial_2 (-\Delta)^{-1} \Delta_q g \, \Delta_q \partial_1 w^1 \, dx.$$

Next, applying Bernstein and Young inequalities, we deduce that

$$\int \Delta_q g \Delta_q w^2 \, dx \leq C 2^{-q} \|\Delta_q g\|_{L^2} \|\partial_1 \Delta_q w\|_{L^2} \leq \frac{\nu}{2} \|\partial_1 \Delta_q w\|_{L^2}^2 + \frac{C}{2\nu} 2^{-2q} \|\Delta_q w\|_{L^2}^2.$$

Then coming back to (82) and integrating, we thus get for all $q \geq 0$,

$$\|\Delta_q w\|_{L_t^\infty(L^2)}^2 + \nu \|\partial_1 \Delta_q w\|_{L_t^2(L^2)}^2 \leq \|\Delta_q w_0\|_{L^2}^2$$

$$+ 2\|\Delta_q f\|_{L_t^1(L^2)}^2 + 2\|F_q(v,w)\|_{L_t^1(L^2)}^2 + \frac{C}{\nu} 2^{-2q} \|\Delta_q g\|_{L_t^2(L^2)}^2.$$

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For $q = -1$, we merely have

$$\|\Delta_{-1}w(t)\|_{L^2} \le \|\Delta_{-1}w_0\|_{L^2} + \int_0^t \left(\|\Delta_{-1}f\|_{L^2} + \|\Delta_{-1}g\|_{L^2} + \|F_{-1}(v,w)\|_{L^2} \right) d\tau.$$

Of course $\|\partial_1 \Delta_{-1}w\|_{L_t^2(L^2)} \le C t^{\frac{1}{2}} \|\Delta_{-1}w\|_{L_t^\infty(L^2)}$. So finally, for all $q \ge -1$, we have

$$\|\Delta_q w\|_{L_t^\infty(L^2)} + \nu^{\frac{1}{2}} \|\partial_1 \Delta_q w\|_{L_t^2(L^2)}$$
$$\le 2(1 + \sqrt{\nu t}) \left(\|\Delta_q w_0\|_{L^2} + \|\Delta_q f\|_{L_t^1(L^2)} + \|F_q(v,w)\|_{L_t^1(L^2)} + \frac{C}{\nu^{\frac{1}{2}}} 2^{-q} \|\Delta_q g\|_{L_t^2(L^2)} \right).$$

With Inequality (79) at our disposal, it is now easy to conclude the proof of the proposition. It is just a matter of arguing exactly as in Proposition 16. ∎

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