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A "MICRO-MACRO" APPROACH FOR STRUCTURAL ANALYSIS ON PARALLEL ARCHITECTURE COMPUTERS

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Abstract. *The multi-level approach proposed herein has two main features. Firstly, it is a mixed domain decomposition method. It introduces a decomposition of the structure into substructures and interfaces; the unknowns of the problem are both the displacement and the efforts on the interfaces. The LArge Time INcrement (*LATIN*) method is then used to build an iterative algorithm designed to solve the problem concurrently. In order to attain the maximum efficiency expected when increasing the number of substructures, a global mechanism, suitable for propagating information among all the substructures, is required.*

The second feature lies in a homogenisation technique for building this mechanism. This is related to the homogenisation between a macro level (in order to represent the long wave length effects) and a micro level, as derived from the previous substructuring. The design of the homogenised operator, along with the information exchange mechanism between the two levels, leads to a new multi-level approach. This naturally takes place within the LATIN *framework, which has formerly been designed for non-linear problems of evolution. An improved performance for this new approach is then to be expected.*

1 INTRODUCTION

The most powerful computers are now based on a parallel architecture[11]. In order to use them efficiently, specific algorithms have been designed, such as domain decomposition methods, which are well-suited to multiprocessor computers, with either shared or distributed memory.

The underlying approach used in this work proceeds with a decomposition of the structure into substructures and interfaces, in a mixed fashion: the unknowns of the problem are both the displacement and the efforts on the interfaces. The LArge Time INcrement (LATIN) method is then used to build an iterative algorithm in order to solve the problem concurrently[8, 2]. When increasing the number of subdomains, the maximum efficiency expected from domain decomposition methods is not attained. Using a global mechanism to propagate information among all of the substructures can overcome this drawback[1]. Such a mechanism has now been implemented in several domain decomposition-like algorithms, such as the FETI method[4] or the Balancing Domain Decomposition method[10].

An initial extension to the LATIN approach, which takes into account two scales arising from the substructuring has been applied in order to improve performance. Its feasibility has been shown for linear elasticity and perfect interfaces[3]; its performance is similar to those obtained from different versions of the FETI method. The resulting algorithm is numerically scalable, i.e. the number of iterations needed to reach convergence is independent of the number of substructures. At each iteration, the algorithm leads to solving a large-scale problem on the whole structure in order to propagate information.

The approach proposed herein is related to the homogenisation techniques between a macro level and a micro level, which have also been derived from the substructuring. Once the macro level has been chosen (in order to represent the large wave effects), the homogenised operator is defined, along with the information exchange mechanism between the two levels. The macro level problem is then chosen to be the global information exchange mechanism. This approach does not require any specific treatment for boundary-located areas, and deals with structures that are not required to be a periodic medium.

In Section 2, we specify the notations for the reference problem and its substructuring. Then, two scales are defined for both the kinematic and static quantities in Section 3. The LATIN method is used to build an iterative algorithm in Section 4, and Section 5 presents an example of parallel computation with a first version of such a multi-level approach.

2 PROBLEM TO BE SOLVED

The reference problem is related to the quasi-static behaviour of a structure denoted by Ω, over a time interval $[0, T]$, for small perturbations and isothermic evolutions. Loadings are a prescribed displacement $\underline{U}_d$ on one part of the boundary $\partial_1\Omega$, a prescribed traction force $\underline{F}_d$ on a complementary part of the boundary $\partial_2\Omega$, and an eventual body force $\underline{f}_d$ onto Ω, see Figure 1.

In order to simplify the presentation, the reference problem is written for the special

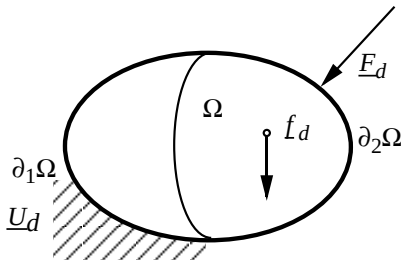

Figure 1. Reference problem

case of linear elasticity (the non-linear case is treated with a full Large Time INcrement approach[7]). Time is no longer taken into account because only the final configuration is of interest herein.

The displacement field at each point $\underline{M}$ of Ω is $\underline{U}(\underline{M})$; the associated space is $\mathcal{U}$ ($\mathcal{U}_0$ will denote the related virtual field space). $\boldsymbol{\varepsilon}$ is the strain field and the current state of the structure is given at each point by the stress field $\boldsymbol{\sigma}$.

The reference problem is to find $\mathbf{s} = (\underline{U}; \boldsymbol{\sigma})$ which satisfies:

- kinematic equations:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\underline{U}), \quad \underline{U}_{|\partial_1\Omega} = \underline{U}_d, \quad \underline{U} \in \mathcal{U} \tag{1}$$

- equilibrium equations:

$$\forall\, \underline{U}^\star \in \mathcal{U}_0, \quad \int_\Omega \mathrm{Tr}[\boldsymbol{\sigma}\boldsymbol{\varepsilon}(\underline{U}^\star)]d\Omega = \int_\Omega \underline{f}_d \cdot \underline{U}^\star d\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{U}^\star dS \tag{2}$$

- constitutive relation ($\mathbf{K}$ is Hooke's tensor):

$$\boldsymbol{\sigma} = \mathbf{K}\boldsymbol{\varepsilon} \tag{3}$$

In order to obtain an accurate solution to the previous problem when several structural effects occur — like high-stress gradient areas — or when a precise model of the material's behaviour is needed, one expects to use a large number of degree of freedom in a finite element approach. This problem will be called the fine-scale problem. A direct solution to the fine-scale problem can be very costly. One way to solve it efficiently is to use a substructuring approach, thereby allowing us to perform the treatment on multi-processor computers[11].

The originality of the substructuring approach used herein is to decompose the former structure into substructures and interfaces, with each of them being a mechanical entity in its own right: each possesses its own behaviour and equation[6]. Such a substructuring

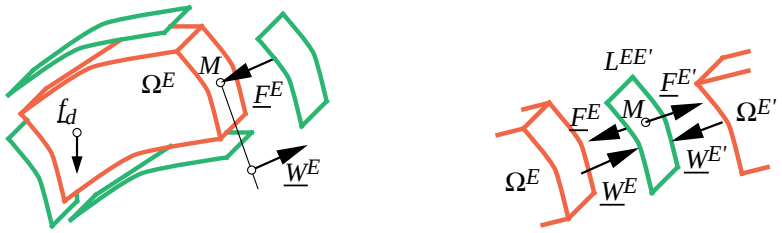

Figure 2. Substructure and interface

leads to introducing additional quantities; the state of the liaison $\Gamma^{EE'}$ between substructures Ω^E and $\Omega^{E'}$ is defined by efforts and displacements on the interface $(\underline{F}^E;\underline{W}^E)$ and $(\underline{F}^{E'};\underline{W}^{E'})$. They represent the action of $\Gamma^{EE'}$ on Ω^E and $\Omega^{E'}$, respectively, such as shown in Figure 2.

As the size of the reference problem increases, domain decomposition methods become more efficient than direct methods. Meanwhile, efficiency is not as high as expected as the number of substructures increases; in particular, numerical scalability is not attained. An additional effort has to be made in order to improve the performance of such algorithms. Classically a global problem has to be used for the whole structure[1]. For instance, the FETI method[4], which is a dual Schur complement problem solved with a projected conjugate gradient, must balance the efforts field on its boundary globally for each subdomain. At each iteration, it has to solve a global problem for which the unknowns are the rigid body movements of the subdomains. When using a primal Schur complement method[9], the Balancing Domain Decomposition method[10] uses a similar feature during the preconditioning step, during which a Neumann problem is solved for each subdomain at each iteration.

In the case of the LATIN method, an initial multi-level extension of the algorithm has been performed[3]. It uses a coarse discretisation of the whole problem at the large scale to propagate globally the information among all the substructures. We are proposing herein another large scale problem generated from a homogenisation-like approach[7] which is expected to be more efficient.

3 USING TWO SCALES

The solution is considered to possess two parts, each related to the micro scale m and to the macro scale M. Such an approach uses homogenisation techniques similar to those which can be developed with multigrid approaches[5]. The state of the structure is then described with two parts, each defined on one scale.

In order to simplify the notations, further substructure-related quantities will no longer be denoted by a superscript E.

3.1 Kinematic splitting

The displacement field is $\underline{U} = \underline{U}^m + \underline{U}^M$. $\underline{U}^M$ is regular on the whole structure except on the eventual discontinuity surfaces (unilateral contact, for instance): $\underline{U}^M \in \mathcal{U}^M$. This is the large variation length displacement. $\underline{U}^m$ is not required to be continuous throughout the interfaces between substructures: $\underline{U}^m \in \mathcal{U}_E^m$. This is the micro correction displacement.

Both of these are subjected an orthogonality-like condition (the macro displacement is an averaged displacement):

$$\forall \underline{U}^m \in \mathcal{U}_E^m, \forall \underline{U}^M \in \mathcal{U}^M, \quad \int_\Omega \underline{U}^m \cdot \underline{U}^M d\Omega = \sum_E \int_{\Omega^E} \underline{U}^m \cdot \underline{U}^M d\Omega = 0 \tag{4}$$

The strain field is then:

$$\boldsymbol{\varepsilon}(\underline{U}) = \boldsymbol{\varepsilon}(\underline{U}^m) + \boldsymbol{\varepsilon}(\underline{U}^M) = \boldsymbol{\varepsilon}^m + \boldsymbol{\varepsilon}^M \tag{5}$$

From a kinematic point of view, the displacement on the interfaces is split into two scales: $\underline{W} = \underline{W}^m + \underline{W}^M$, with $\underline{W}^m \in \mathcal{W}_E^m$, and $\underline{W}^M \in \mathcal{W}_E^M$.

3.2 Dual static splitting

Both stress and efforts can be derived from previous representations, with an energy splitting, which is a key point of the proposed approach:

$$(\underline{F}, \underline{W}) = (\underline{F}, \underline{W}^m) + (\underline{F}, \underline{W}^M) = (\underline{F}^m, \underline{W}^m) + (\underline{F}^M, \underline{W}^M) \tag{6}$$

$(.,.)$ is the symmetric form:

$$(\underline{F}, \underline{W}) = \sum_E \int_{\partial\Omega^E} \underline{F}^E \cdot \underline{W}^E d\Omega \tag{7}$$

and:

$$\int_\Omega \mathrm{Tr}[\boldsymbol{\sigma}\boldsymbol{\varepsilon}] d\Omega = \int_\Omega (\mathrm{Tr}[\boldsymbol{\sigma}\boldsymbol{\varepsilon}^m] + \mathrm{Tr}[\boldsymbol{\sigma}\boldsymbol{\varepsilon}^M]) d\Omega = \int_\Omega (\mathrm{Tr}[\boldsymbol{\sigma}^m\boldsymbol{\varepsilon}^m] + \mathrm{Tr}[\boldsymbol{\sigma}^M\boldsymbol{\varepsilon}^M]) d\Omega \tag{8}$$

The last relations allow us to define the micro and macro dual unknowns: $\boldsymbol{\sigma}^M$ and $\underline{F}^M$ are the macro stress and efforts, and $\boldsymbol{\sigma}^m$ and $\underline{F}^m$ are the micro correction stress and correction efforts.

4 A LATIN APPROACH

Within the linear elasticity case, the LATIN framework is transformed since time no longer plays a role. The usual duality is now related to energy, i.e. it occurs between displacements and efforts, and no longer between celerity and efforts, as with the dissipation duality for non-linearities or for dynamic problems.

The problem now consists of finding $\mathbf{s} = (\underline{U}, \underline{W}; \boldsymbol{\sigma}, \underline{F})$ which satisfies:

- an initial group of equations ($\mathbf{\Gamma}$) with possibly non-linear, but local-in-space, equations:
  - for each substructure Ω^E, Hooke's law (3),
  - for each interface $\Gamma^{EE'}$, the corresponding behaviour:
    equilibrium $(\underline{F}+\underline{F}', \underline{W}^\star)=0$ with $\underline{W}^\star=\underline{W}^{m\star}+\underline{W}^{M\star}$; this equilibrium can then be rewritten in the micro-macro form:

$$\forall \underline{W}^{m\star}\in\mathcal{W}^m_{E,0},\quad (\underline{F}^m+\underline{F}'^m,\underline{W}^{m\star})=0 \tag{9}$$
$$\forall \underline{W}^{M\star}\in\mathcal{W}^M_{E,0},\quad (\underline{F}^M+\underline{F}'^M,\underline{W}^{M\star})=0 \tag{10}$$

    and for a perfect liaison, displacement continuity: $\underline{W}^m=\underline{W}'^m$ and $\underline{W}^M=\underline{W}'^M$

- a second group ($\mathbf{A_d}$) with possibly global, but linear equations, in which the splitting into the two levels had been previously performed:
  - kinematic admissibility:

$$\boldsymbol{\varepsilon}^m=\boldsymbol{\varepsilon}(\underline{U}^m),\quad \underline{U}^m{}_{|\partial\Omega^E}=\underline{W}^m,\quad \underline{U}^m\in\mathcal{U}^m_E,\quad \underline{W}^m\in\mathcal{W}^m_E \tag{11}$$
$$\boldsymbol{\varepsilon}^M=\boldsymbol{\varepsilon}(\underline{U}^M),\quad \underline{U}^M{}_{|\partial\Omega^E}=\underline{W}^M,\quad \underline{U}^M\in\mathcal{U}^M,\quad \underline{W}^M\in\mathcal{W}^M_E \tag{12}$$

  - static admissibility:

$$\forall \underline{U}^{m\star}\in\mathcal{U}^m_{E,0},\quad \int_{\Omega^E}\mathrm{Tr}[\boldsymbol{\sigma}^m\boldsymbol{\varepsilon}(\underline{U}^{m\star})]d\Omega=\int_{\Omega^E}\underline{f}_d\cdot\underline{U}^{m\star}d\Omega+\int_{\partial\Omega^E}\underline{F}^m\cdot\underline{W}^{m\star}dS \tag{13}$$

$$\forall \underline{U}^{M\star}\in\mathcal{U}^M_0,\quad \sum_E\int_{\Omega^E}\mathrm{Tr}[\boldsymbol{\sigma}^M\boldsymbol{\varepsilon}(\underline{U}^{M\star})]d\Omega=$$
$$=\sum_E\int_{\Omega^E}\underline{f}_d\cdot\underline{U}^{M\star}d\Omega+\sum_E\int_{\partial\Omega^E}\underline{F}^M\cdot\underline{W}^{M\star}dS \tag{14}$$

A two-stage iterative algorithm can now be built[6], see Figure 3. Its main feature is to successively produce an element of $\mathbf{\Gamma}$ and an element of $\mathbf{A_d}$ at each iteration. Each stage involves a search direction, $\mathbf{E}^+$ and $\mathbf{E}^-$. These are the parameters of the method.

4.1 Local stage

Once $\mathbf{s}$ is known, the problem is then to find $\hat{\mathbf{s}}\in\mathbf{\Gamma}$ such that $\hat{\mathbf{s}}-\mathbf{s}$ belongs to a search direction $\mathbf{E}^+$, i.e. for linear elasticity:

$$(\hat{\boldsymbol{\sigma}}-\boldsymbol{\sigma})+\mathbf{K}(\hat{\boldsymbol{\varepsilon}}-\boldsymbol{\varepsilon})=0 \tag{15}$$

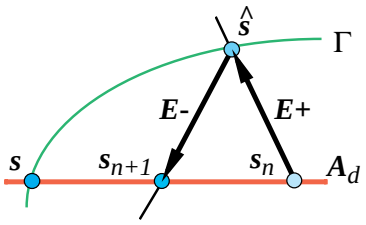

Figure 3. LATIN iterative scheme

for each substructure, and:

$$\begin{aligned} \forall \underline{W}^{m\star} \in \mathcal{W}_E^m, \quad (\underline{\hat{F}}^m - \underline{F}^m, \underline{W}^{m\star}) - (\mathbf{k}^m(\underline{\hat{W}}^m - \underline{W}^m), \underline{W}^{m\star}) = 0 \\ \forall \underline{W}^{M\star} \in \mathcal{W}_E^M, \quad (\underline{\hat{F}}^M - \underline{F}^M, \underline{W}^{M\star}) - (\mathbf{k}^M(\underline{\hat{W}}^M - \underline{W}^M), \underline{W}^{M\star}) = 0 \end{aligned} \tag{16}$$

for each interface. Of course, $\hat{\mathbf{s}}$ also has to satisfy Hooke's law for each substructure and the behaviour of each interface.

It can can be noticed that micro and macro quantities are not separated in the search direction (15) in order to ensure coupling between the two scales. For the boundary fields, the search direction (16) is expressed separately on each level. One characteristic of the parameter $\mathbf{k}^m$, when using a multi-level approach, is its relationship with an interface characteristic[3] and no longer with the structure characteristic, as that was the case for the formerly mono-level approach[8].

Since the search direction $\mathbf{E}^+$ is local and linear, the resulting local stage problem is also local in terms of space variable. It can be easily parallelised once the interfaces have been mapped onto the available processors.

4.2 Linear stage

Once $\hat{\mathbf{s}}$ is known, the problem is then to find $\mathbf{s} \in \mathbf{A_d}$, in such a way that $\mathbf{s} - \hat{\mathbf{s}}$ belongs to the search direction $\mathbf{E}^-$, i.e. for each substructure:

$$(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) - \mathbf{K}(\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}) = 0 \tag{17}$$

and for each interface:

$$\begin{aligned} \forall \underline{W}^{m\star} \in \mathcal{W}_E^m, \quad (\underline{F}^m - \underline{\hat{F}}^m, \underline{W}^{m\star}) + (\mathbf{k}^m(\underline{W}^m - \underline{\hat{W}}^m), \underline{W}^{m\star}) = 0 \\ \forall \underline{W}^{M\star} \in \mathcal{W}_E^M, \quad (\underline{F}^M - \underline{\hat{F}}^M, \underline{W}^{M\star}) + (\mathbf{k}^M(\underline{W}^M - \underline{\hat{W}}^M), \underline{W}^{M\star}) = 0 \end{aligned} \tag{18}$$

As previously noted, $\mathbf{K}$ does not separate micro and macro quantities in (17), although $\mathbf{k}^m$ and $\mathbf{k}^M$ do in (18).

Since $\mathbf{s} \in \mathbf{A_d}$, it is said to be admissible. It is subjected to both kinematic (11 and 12) and static (13 and 14) admissibilities. Such a problem is linear and global per each

substructure only. It can also be parallelised when the substructures also get distributed among the processors.

We are now going to describe more precisely the kind of problems to be solved during the linear stage. Taking into account the properties that $\hat{\mathbf{s}}$ has to satisfy, the search direction (17) leads to $\boldsymbol{\sigma} = \mathbf{K}\boldsymbol{\varepsilon}$, i.e.:

$$\forall \boldsymbol{\varepsilon}^\star = \boldsymbol{\varepsilon}^{m\star} + \boldsymbol{\varepsilon}^{M\star}, \quad \int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\sigma}\boldsymbol{\varepsilon}^\star] d\Omega = \int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\varepsilon}\mathbf{K}\boldsymbol{\varepsilon}^\star] d\Omega \tag{19}$$

4.2.1 Micro scale problem

Let's look more closely at the micro scale problem, in which all macro quantities are supposed to be known. One must find $\mathbf{s}^m = (\underline{U}^m, \underline{W}^m; \boldsymbol{\sigma}^m, \underline{F}^m)$ which satisfies for each substructure the kinematic and static admissibilities (11) and (13), as well as the search direction (17). The later then leads to:

$$\int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\sigma}^m \boldsymbol{\varepsilon}^{m\star}] d\Omega = \int_{\Omega^E} \mathrm{Tr}[(\mathbf{K}\boldsymbol{\varepsilon}^m + \mathbf{K}\boldsymbol{\varepsilon}^M)\boldsymbol{\varepsilon}^{m\star}] d\Omega \tag{20}$$

Using the static admissibility and the search direction, the resulting displacement-oriented formulation consists of finding on each substructure $\underline{U}^m$ and $\underline{W}^m$ such that:

$$\boldsymbol{\varepsilon}^m = \boldsymbol{\varepsilon}(\underline{U}^m), \quad \underline{U}^m{}_{|\partial\Omega^E} = \underline{W}^m, \quad \underline{U}^m \in \mathcal{U}_E^m, \quad \underline{W}^m \in \mathcal{W}_E^m \tag{21}$$

and:

$$\begin{aligned} \forall \underline{U}^{m\star} \in \mathcal{U}_{E,0}^m, \quad & \int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}^m)\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{m\star})] + \int_{\partial\Omega^E} \underline{U}^m \cdot \mathbf{k}^m \underline{U}^{m\star} dS = \\ & = \int_{\Omega^E} \underline{f}_d \cdot \underline{U}^{m\star} d\Omega + \int_{\partial\Omega^E} (\hat{\underline{F}}^m + \mathbf{k}^m \hat{\underline{W}}^m) \cdot \underline{U}^{m\star} dS - \int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\varepsilon}^M \mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{m\star})] d\Omega \end{aligned} \tag{22}$$

In order to express the macro scale problem in the next section, the structure of the micro scale problem must be detailed. Since the problem is linear, the solution can be expressed as:

$$\underline{U}^m = \hat{\underline{U}}_d + \tilde{\underline{U}} \tag{23}$$

$\tilde{\underline{U}}$ is the part of the solution for when $\boldsymbol{\varepsilon}^M$ is the only loading term on the right hand side of (22). Let $\tilde{\tilde{\underline{U}}} \in \mathcal{U}^M$ be such that:

$$\forall \underline{U}^{M\star} \in \mathcal{U}^M, \quad \int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\tilde{\tilde{\underline{U}}})\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})] d\Omega = \int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\tilde{\underline{U}})\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})] d\Omega \tag{24}$$

Since $\tilde{\underline{U}} \in \mathcal{U}_E^m$ and $\tilde{\tilde{\underline{U}}} \in \mathcal{U}^M$, the later is energetically equivalent to the response in the micro scale to a macro scale loading. Finally, $\tilde{\tilde{\underline{U}}}$ is derived from $\underline{U}^M$ using $\tilde{\underline{U}}$. It can be formally written as:

$$\mathbf{K}\boldsymbol{\varepsilon}(\tilde{\tilde{\underline{U}}}) = \mathbf{L}\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^M) \tag{25}$$

4.2.2 Macro scale problem

This problem is similarly treated with the search direction (19) which then leads to:

$$\int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\sigma}^M \boldsymbol{\varepsilon}^{M\star}] d\Omega = \int_{\Omega^E} \mathrm{Tr}[(\mathbf{K}\boldsymbol{\varepsilon}^M + \mathbf{K}\boldsymbol{\varepsilon}^m)\boldsymbol{\varepsilon}^{M\star}] d\Omega \tag{26}$$

Using expression (25) in the macro scale equilibrium, the displacement-oriented formulation now consists of finding $\underline{U}^M$ and $\underline{W}^M$ such that:

$$\boldsymbol{\varepsilon}^M = \boldsymbol{\varepsilon}(\underline{U}^M), \quad \underline{U}^M{}_{|\partial\Omega^E} = \underline{W}^M, \quad \underline{U}^M \in \mathcal{U}^M, \quad \underline{W}^M \in \mathcal{W}_E^M \tag{27}$$

and:

$$\begin{aligned} &\forall \underline{U}^{M\star} \in \mathcal{U}_0^M, \quad \sum_E \int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\underline{U}^M)(\mathbf{1}+\mathbf{L})\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})] d\Omega = \\ &= \sum_E \int_{\Omega^E} \underline{f}_d \cdot \underline{U}^{M\star} d\Omega + \sum_E \int_{\partial\Omega^E} \underline{F}^M \cdot \underline{U}^{M\star} dS - \sum_E \int_{\Omega^E} \mathrm{Tr}[\boldsymbol{\varepsilon}(\hat{\underline{U}}_d)\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^{M\star})] d\Omega \end{aligned} \tag{28}$$

Regularity constraints for $\underline{U}^M$ lead to a global problem on the whole structure for the macro scale. The solution $\underline{U}^M$ now allows us to solve the micro scale problem (22).

$(\mathbf{1}+\mathbf{L})\mathbf{K}\boldsymbol{\varepsilon}(\underline{U}^M)$ is the homogenised stress and $(\mathbf{1}+\mathbf{L})\mathbf{K}$ is the homogenised Hooke's tensor.

5 EXAMPLE

The reported results have been obtained with a first version of the micro-macro approach[3]. In particular, the macro scale operator of the macro-scale problem (28) is different from the previous one.

With this version and for the examples presented herein, the finite element meshes of the two scales are embedded. Information transfer between grids is then performed with a hierarchical finite element projection[14]. Moreover, the orthogonality condition (4) is replaced by the hierarchical splitting of finite element shape functions[13]. When using the homogenised operator $(\mathbf{1}+\mathbf{L})\mathbf{K}$, performances are expected to be improved.

Nevertheless, the general layout of both multi-level approaches is similar and has been described in Table 1. The approaches have been implemented in the industrial-type code CASTEM 2000[12] developed at the CEA in Saclay. All of the computations have been performed on an SGI ORIGIN 2000 with 32 processors and 8 Gb central shared memory.

5.1 Convergence rate

In order to check the performance of the proposed approach, let us first consider the model problem of a slendered bidimensional structure submitted to a parabolic bending

Macro scale dedicated to 1 processor		Micro scale dedicated to n processors
Initialisation		Initialisation
		compute loading contribution
receiving loading contributions	$\longleftarrow$	sending it
assembling them		initialisation of $\hat{\mathbf{s}}$
factorisation of macro scale problem		**factorisation** of micro scale problem
forward-backward: global problem		
sending $\underline{U}^M$	$\longrightarrow$	receiving $\underline{U}^M$
		compute coupling term
		forward-backward to get $\underline{U}^m$
Loop over iterations		Loop over iterations
		compute coupling term
receiving coupling terms	$\longleftarrow$	sending it
assembling them		**local stage**,
forward-backward: global problem		convergence check $\longleftrightarrow$
sending $\underline{U}^M$	$\longrightarrow$	receiving $\underline{U}^M$
		compute coupling term
		forward-backward to get $\underline{U}^m$

Table 1. Micro-macro algorithm

loading (see Figure 4). The reference $(\underline{U};\boldsymbol{\sigma})_{\text{ref}}$ here is the direct finite element solution without decomposition. It allows us to define the convergence rate in the energy norm:

$$\tau = -\log\frac{e_{n+1}}{e_n} \quad \text{where:} \quad e_n^2 = \frac{\frac{1}{2}\displaystyle\int_\Omega \text{Tr}[(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{\text{ref}})\mathbf{K}^{-1}(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{\text{ref}})]d\Omega}{\frac{1}{2}\displaystyle\int_\Omega \text{Tr}[\boldsymbol{\sigma}_{\text{ref}}\mathbf{K}^{-1}\boldsymbol{\sigma}_{\text{ref}}]d\Omega} \tag{29}$$

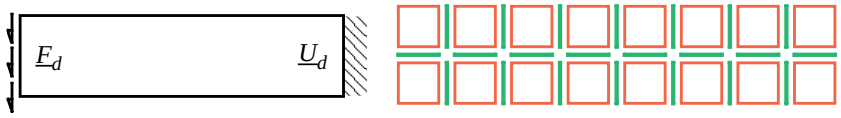

Figure 4. Model problem and example of decomposition into 16 substructures and interfaces

Since only perfect interfaces are used in this example, $\mathcal{U}^M$ denotes regular fields on the whole structure Ω. The interface loading in the global macro scale equilibrium (28) is

then:

$$\sum_E \int_{\partial\Omega^E} \underline{F}^M \cdot \underline{U}^{M\star} dS = \sum_{\Gamma^{EE'}} \int_{\Gamma^{EE'}} (\underline{F}^M + \underline{F}'^M) \cdot \underline{U}^{M\star} dS = 0 \tag{30}$$

Table 2 shows the various characteristics of the solved problems.

substructuring (total)			direct computation		
nb sstr	nb dof	memory requirement factorised matrix	nb dof	nb elements	memory requirement factorised matrix
4	33 800	25 Mb	33 410	8 192	33 Mb
16	135 200	99 Mb	132 354	32 768	255 Mb
64	540 800	398 Mb	526 850	131 072	2 000 Mb*

* estimated

Table 2: Characteristics of the involved problems (micro scale is meshed with 6-node triangles, macro scale with 3-node triangles)

Figure 5 presents the averaged convergence rate (up to a high level of convergence: $e_n \leq 0.01\%$) versus the number of substructures. It illustrates a well-known behavioural characteristic of domain decomposition methods: slowing the convergence rate when increasing the number of subdomains[1].

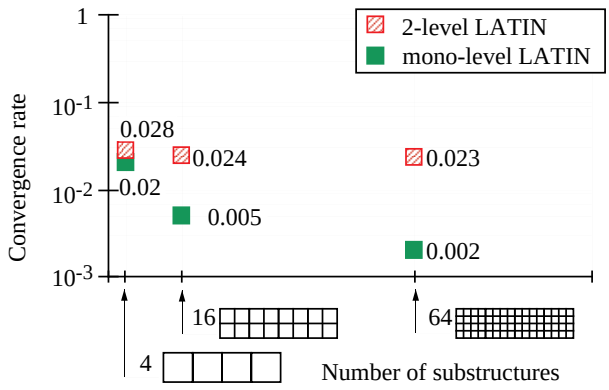

Figure 5. Convergence rate versus number of substructures

The evolution in the stress and displacement fields along iterations are compared in Figure 6 for the mono-level and micro-macro approaches, at iterations 1, 2 and 15. For the later approach, both the micro and macro scale parts of the solution are shown in Figure 7.

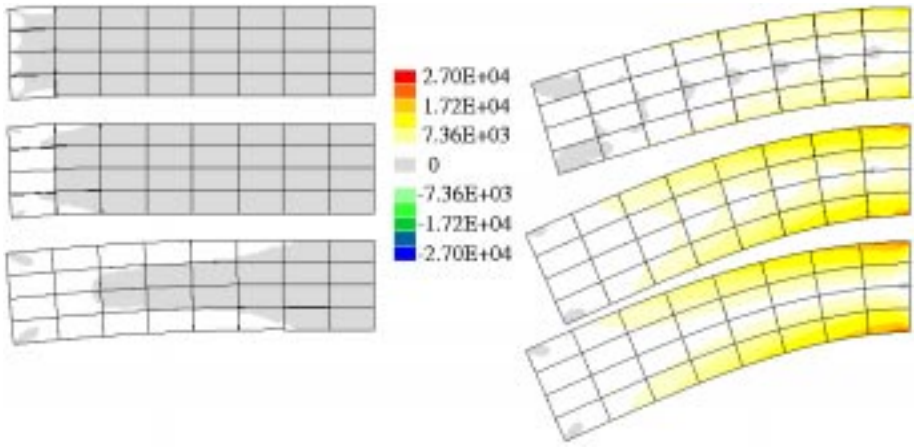

Figure 6: Comparison of solutions at iterations 1,2 and 15 for: a) on the left: the mono-level approach; b) on the right: the micro-macro approach.

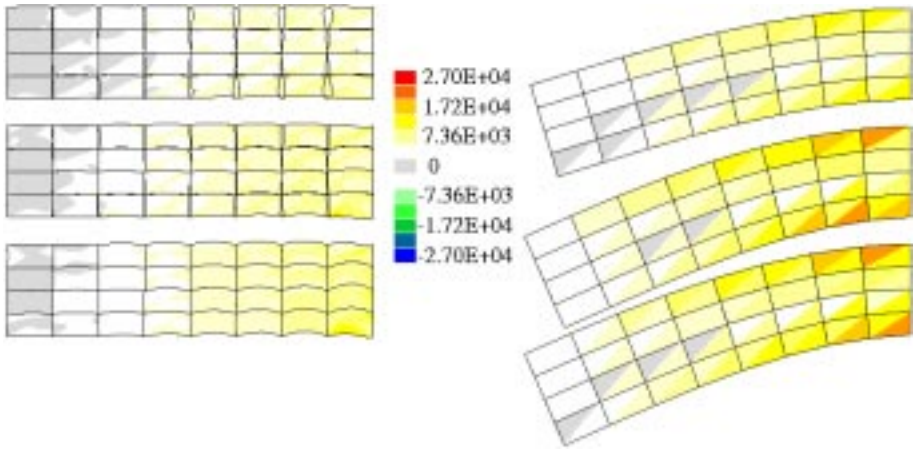

Figure 7: a) on the left: the micro scale solution (with a 10-magnification for displacement); b) on the right: the macro scale solution.

Finally, it should be pointed out that for the mono-level approach, the search direction is $\mathbf{k} = E/L_0$, where E is Young's modulus and L_0 has an optimum value of twice the length of the structure[8]. When using the multi-level or micro-macro approach, the value of $\mathbf{k}^m = E/L_0$ has been set equal to 0.25 times the length of an interface. It is no longer characterised by the global behaviour of the structure.

With a global macro scale problem, the algorithm also becomes numerically scalable: the convergence rate is quasi-independent of the number of substructures.

6 CONCLUSION

The proposed technique belongs to the field of structural analysis with homogenisation. It combines a micro and macro representations of the problem into the LATIN framework.

The second feature herein is to build a mechanical and parallel approach that is related to domain decomposition methods, in order to use parallel architecture computers for problems with a large number of degrees of freedom. It deals with a mixed substructuring method as the unknowns are both the displacement and the efforts into the interfaces.

As the macro scale problem is kept global onto the whole structure, it leads to an information exchange mechanism for all the substructures. The resulting algorithms are numerically scalable, and the convergence rate is improved.

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