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ON A LOCAL CHARACTERIZATION OF PSEUDOCONVEX DOMAINS

NIKOLAI NIKOLOV, PETER PFLUG, PASCAL J. THOMAS, WLODZIMIERZ ZWONEK

Abstract. Pseudoconvexity of a domain in \( \mathbb{C}^n \) is described in terms of the existence of a locally defined plurisubharmonic/holomorphic function near any boundary point that is unbounded at the point.

1. Introduction and results

It is well-known that a domain \( D \subset \mathbb{C}^n \) is pseudoconvex if and only if any of the following conditions holds:

(i) there is a smooth strictly plurisubharmonic function \( u \) on \( D \) with \( \lim_{z \to \partial D} u(z) = \infty \);

(ii) for any \( a \in \partial D \) there is a \( u_a \in \mathcal{PSH}(D) \) with \( \lim_{z \to a} u_a(z) = \infty \);

(iii) there is an \( f \in \mathcal{O}(D) \) such that for any \( a \in \partial D \) and any neighborhood \( U_a \) of \( a \) one has that \( \lim \sup_{G \ni z \to a} |f(z)| = \infty \) for any connected component \( G \) of \( D \cap U_a \) with \( a \in \partial G \);

(iv) for any \( a \in \partial D \) there is a neighborhood \( U_a \) of \( a \) and an \( f_a \in \mathcal{O}(D \cap U_a) \) such that for any neighborhood \( V_a \subset U_a \) of \( a \) and any connected component \( G \) of \( D \cap V_a \) with \( a \in \partial G \) one has \( \lim \sup_{G \ni z \to a} |f_a(z)| = \infty \) (see Corollary 4.1.26 in [2]).

If \( D \) is \( C^1 \)-smooth, we may assume that \( D \cap U_a \) is connected in (iii) and (iv).

Our first aim is to see that in (i) in general 'lim' cannot be weakened by 'limsup' even if \( D \) is \( C^1 \)-smooth.

Theorem 1. For any \( \varepsilon \in (0,1) \) there is a non-pseudoconvex bounded domain \( D \subset \mathbb{C}^2 \) with \( C^{1,1-\varepsilon} \)-smooth boundary and a negative function \( u \in \mathcal{PSH}(D) \) with \( \lim \sup_{z \to a} u(z) = 0 \) for any \( a \in \partial D \).

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In particular, \( v := -\log(-u) \in \mathcal{PSH}(D) \) with \( \lim \sup_{z \to a} v(z) = \infty \) for any \( a \in \partial D \).

If we do not require smoothness of \( D \), following the idea presented in the proof, we may just take \( D = \{ z \in \mathbb{C}^n : \min\{||z||, ||z - a||\} < 1 \} \), \( 0 < ||a|| < 2 \), \( n \geq 2 \).

On the other, this cannot happen if \( D \) is \( \mathcal{C}^2 \)-smooth.

**Proposition 2.** Let \( D \subset \mathbb{C}^n \) be a \( \mathcal{C}^2 \)-smooth domain with the following property: for any boundary point \( a \in \partial D \) there is a neighborhood \( U_a \) of \( a \) and a function \( u_a \in \mathcal{PSH}(D \cap U_a) \) such that \( \lim \sup_{z \to a} u_a(z) = \infty \). Then \( D \) is pseudoconvex.

However, if we replace 'limsup' by 'lim', we may remove the hypothesis about smoothness of the boundary.

**Proposition 3.** Let \( D \subset \mathbb{C}^n \) be a domain with the following property: for any boundary point \( a \in \partial D \) there is a neighborhood \( U_a \) of \( a \) and a function \( u_a \in \mathcal{PSH}(D \cap U_a) \) such that \( \lim_{z \to a} u_a(z) = \infty \). Then \( D \) is pseudoconvex.

Note that the assumption in Proposition 3 is formally weaker that to assume that \( D \) is locally pseudoconvex.

**Remark.** The three propositions above have real analogues replacing (non)pseudoconvex domains by (non)convex domains and plurisubharmonic functions by convex functions (for the analogue of Proposition 3 use e.g. Theorem 2.1.27 in [2] which implies that if \( D \) is a nonconvex domain in \( \mathbb{R}^n \), then there exists a segment \( [a, b] \) such that \( c = \frac{a+b}{2} \in \partial D \) but \( [a, b] \setminus \{c\} \subset D \). The details are left to the reader.

Recall now that a domain \( D \subset \mathbb{C}^n \) is called *locally weakly linearly convex* if for any boundary point \( a \in \partial D \) there is a complex hyperplane \( H_a \) through \( a \) and a neighborhood \( U_a \) of \( a \) such that \( H_a \cap D \cap U_a = \emptyset \). D. Jacquet asked whether a locally weakly linearly convex domain is already pseudoconvex (see [3], page 58). The answer to this question is affirmative by Proposition 3. The next proposition shows that such a domain has to be even taut\footnote{This means that \( \mathcal{O}(D, D) \) is a normal family, where \( D \subset \mathbb{C} \) is the open unit disc. Note that any taut domain is pseudoconvex and any bounded pseudoconvex domain with \( \mathcal{C}^1 \)-smooth boundary is taut.} if it is bounded.

**Proposition 4.** Let \( D \subset \mathbb{C}^n \) be a bounded domain with the following property: for any boundary point \( a \in \partial D \) there is a neighborhood \( U_a \) of \( a \) and a function \( f_a \in \mathcal{O}(D \cap U_a) \) such that \( \lim_{z \to a} |f_a(z)| = \infty \). Then \( D \) is taut.
Let $D \subset \mathbb{C}^n$ be a domain and let $K_D(z)$ denote the Bergman kernel of the diagonal. It is well-known that $\log K_D \in \mathcal{PSH}(D)$. Recall that (v) if $D$ is bounded and pseudoconvex, and $\limsup_{z \to a} K_D(z) = \infty$ for any $a \in \partial D$, then $D$ is an $L^2_h$-domain of holomorphy ($L^2_h(D) := L^2(D) \cap \mathcal{O}(D)$) (see [6]).

We show that the assumption of pseudoconvexity is essential.

**Proposition 5.** There is a non-pseudoconvex bounded domain $D \subset \mathbb{C}^2$ such that $\limsup_{z \to a} K_D(z) = \infty$ for any $a \in \partial D$.

Note that the domain $D$ with $u = \log K_D$ presents a similar kind of example as that in Proposition 1 (however, the domain has weaker regularity properties).

The example given in Proposition 5 is a domain with non-schlicht envelope of holomorphy. This is not accidental as the following result shows.

**Proposition 6.** Let $D \subset \mathbb{C}^n$ be a domain such that $\limsup_{z \to a} K_D(z) = \infty$ for any $a \in \partial D$. Assume that one of the following conditions is satisfied:

- the envelope of holomorphy $\hat{D}$ of $D$ is a domain in $\mathbb{C}^n$,
- for any $a \in \partial D$ and for any neighborhood $U_a$ of $a$ there is a neighborhood $V_a \subset U_a$ of $a$ such that $V_a \cap D$ is connected (this is the case when e.g. $D$ is a $C^1$-smooth domain).

Then $D$ is pseudoconvex.

**Remark.** Note that the domain in the example is not fat. We do not know what will happen if $D$ is assumed to be fat.

Making use of the reasoning in [6] we shall see how Proposition 5 implies that the domain from this proposition admits a function $f \in L^2_h(D)$ satisfying the property $\limsup_{z \to a} |f(z)| = \infty$ for any $a \in \partial D$.

**Theorem 7.** Let $D$ be the domain from Proposition 5. Then there is a function $f \in L^2_h(D)$ such that $\limsup_{z \to a} |f(z)| = \infty$ for any $a \in \partial D$.

### 2. Proof of Proposition 1

First, we shall prove two lemmas.

**Lemma 8.** For any $\varepsilon \in (0,1)$ and $C_1, C_2 > 0$, there exists an $F \in C^{1,1-\varepsilon}(\mathbb{R})$ such that:

(i) $\text{supp} \ F \subset [-1,1], \ 0 \leq F(x) \leq C_1$ for all $x \in \mathbb{R};$

(ii) there is a dense open set $\mathcal{U} \subset [-1,1]$ such that $F''(x)$ exists and $F''(x) \leq -C_2 < 0$ for all $x \in \mathcal{U};$

(iii) $F$ vanishes on a Cantor subset of $[-1,1]$. 

Proof. An elementary construction yields an even non-negative smooth function $b$ supported on $[-3/4, +3/4]$, decreasing on $[0, 3/4]$, such that $b(x) = 1 - 4x^2$ for $|x| \leq 1/4$, $|b'(x)| \leq C_3$, $-8 \leq b''(x) \leq C_4$ for all $x \in \mathbb{R}$, where $C_3, C_4 > 0$.

For any $a, p > 0$, we set $b_{a,p}(x) := ab(x/p)$, $x \in \mathbb{R}$.

We shall construct two decreasing sequences of positive numbers $(a_n)_{n \geq 0}$ and $(p_n)_{n \geq 0}$, and intervals $\{I_{n,i}, J_{n,i}, n \geq 0, 1 \leq i \leq 2^n\}$.

Set $I_{0,1} := (-1, +1)$ and $J_{0,1} := [-p_0/4, p_0/4]$, where $p_0 < 1$. Then $I_{1,1} := (-1, -p_0)$ and $I_{1,2} := (p_0, 1)$.

In general, if the intervals of the n-th "generation" $I_{n,i}$ are known, we require

(1) \[ p_n < \frac{|I_{n,i}|}{2}, \]

where $|J|$ denotes the length of an interval $J$. Denote by $c_{n,i}$ the center of $I_{n,i}$ and put $J_{n,i} := [c_{n,i} - p_n/4, c_{n,i} + p_n/4]$. Denote respectively by $I_{n+1,2i-1}$ and $I_{n+1,2i}$ the first and second component of $I_{n,i} \setminus J_{n,i}$.

Now we write

$$f_n(x) := \sum_{i=1}^{2^n} b_{a_n,p_n}(x - c_{n,i}), \quad x \in \mathbb{R}, \quad F_n := \sum_{m=0}^{n} f_m.$$ 

Note that the terms in the sum defining $f_n$ have disjoint supports contained in $[c_{n,i} - 3p_n/4, c_{n,i} + 3p_n/4] \subset I_{n,i}$, $(J_{n,i}$ does not contain the support of the corresponding term in $f_n$; it is only a place, where that term coincides with a quadratical polynomial) so that $|f_n'(x)| \leq C_3 a_n/p_n$. The function $F = \lim_{n \to \infty} F_n$ will be of class $C^1$ if

(2) \[ \sum_{n=0}^{\infty} \frac{a_n}{p_n} < \infty. \]

Also, note that

$$|F_n'(x)| \leq |F_{n-1}'(x)| + C_4 \frac{a_n}{p_n^2}, \text{ so } \sup |F_n'| \leq C_4 \sum_{m=1}^{n} \frac{a_m}{p_m^2}.$$ 

From now on we choose

(3) \[ \frac{a_n}{p_n^2} = BA^n, \text{ for some } A > 1, B > 0 \text{ to be determined.} \]

We then have $\sup |F_n''| \leq C_4 BA^{n+1}/(A - 1)$.

All the successive terms $f_m, m > n$, are supported on intervals of the form $I_{m,j}$, thus vanish on the interval $J_{n,i}$, so on those intervals $F$ is a
smooth function and
\[ F'' = F'' = F''_{n-1} - \frac{a_n}{p_n^2} \leq C_4 \frac{BA^n}{A - 1} - 8BA^n, \]
therefore, if we choose
\[ A > 1 + \frac{C_4}{4}, \]
we have \( F''(x) \leq -4BA^n \) for all \( x \in J_{n,i} \), and \( 1 \leq i \leq 2^n \).

Set \( U := \bigcup_{n,i} J_{n,i}^0 \). We have seen that \( |I_{n+1,i}| < \frac{|I_{n,j}|}{2} \) (and those quantities do not depend on \( i \) or \( j \)), so that the complement of \( U \) has empty interior. This proves claim (ii), by choosing \( B = C_2/4 \). The other claims are clear from the form of the function \( F \), once we provide the sequences \((a_n)\) and \((p_n)\) satisfying (3), (4), (2), and (1).

Let \( a_n := a_0 \gamma^n \), \( p_n = p_0 \delta^n \). Then (3) is satisfied by construction and \( a_0 = Bp_0^2 \). Fix \( \delta, p_0 \in (0, 1/2) \). It follows that \( p_n < |J_{n,i}|/4 \) for all \( n \) (by an easy induction). Hence, (4) holds.

By our explicit form, (4) means that \( \gamma \delta^{-2} > 1 + \frac{C_4}{4} \), while (2) means \( \gamma \delta^{-1} < 1 \), so with \( \delta^{-1} > 1 + \frac{C_4}{4} \), it is easy to choose \( \gamma \). Finally, \( \|F\|_{\infty} \leq a_0(1 - \gamma)^{-1} < C_1 \) for \( a_0 \) small enough, which can be achieved by decreasing \( p_0 \) further.

Given any \( \varepsilon > 0 \), we can modify the choices of \( \delta \) and \( \gamma \) to obtain that \( F' \in C^1_{1-\varepsilon} \) (the Hölder class of order \( 1 - \varepsilon \)). Given any two points \( x, y \in [-1, +1] \) and any integer \( n \geq 1 \),
\[
|F'(x) - F'(y)| \leq |x - y|\|F''\|_{\infty} + 2 \sum_{m \geq n} \|f_m'\|_{\infty}
\leq C \left((\gamma \delta^{-2})^n |x - y| + (\gamma \delta^{-1})^n\right),
\]
where \( C > 0 \) is a positive constant depending on the parameters we have chosen. Take \( n \) such that \( \delta |x - y| \leq \delta^n \leq |x - y| \). Then
\[
\frac{|F'(x) - F'(y)|}{|x - y|^{1-\varepsilon}} \leq C' (\gamma \delta^{-2+\varepsilon})^n,
\]
and it will be enough to choose \( \delta \) and \( \gamma \) so that \( \gamma \delta^{-2+\varepsilon} \leq 1 \) and \( \gamma \delta^{-2} > 1 + \frac{C_4}{4} \), which can be achieved once we pick \( \delta \) small enough. The rest of the parameters are then chosen as above.

Remark. It is clear that \( F \) cannot be of class \( C^2(\mathbb{R}) \). We do not know if our argument can be pushed to get \( F \in C^{1,1}_{1}(\mathbb{R}) \).

Lemma 9. For any \( \varepsilon \in (0, 1) \) there exists a non-pseudoconvex bounded \( C^{1,1-\varepsilon} \)-smooth domain \( D \subset \mathbb{C}^2 \) boundary such that \( \partial D \) contains a dense subset of points of strict pseudoconvexity.
Proof. We start with the unit ball and cave it in somewhat at the North Pole to get an open set of points of strict pseudoconcavity on the boundary. Let $r_0 < 1/3$ and for $x \in [0, 1)$,

$$\psi_0(x) = \min\{\log(1 - x^2), x^2 - r_0^2\}$$

We take $\psi$ a $C^\infty$ regularization of $\psi_0$ such that $\psi = \psi_0$ outside of $(r_0/2, r_0)$. Consider the Hartogs domain

$$D_0 := \left\{(z, w) \in \mathbb{C}^2 : |z| < 1, \log |w| < \frac{1}{2} \psi(|z|) \right\}.$$

Notice that $D_0 \setminus \{|z| \leq r_0\} = \mathbb{B}_2 \setminus \{|z| \leq r_0\}$, so that $\partial D$ is smooth near $|z| = 1$.

Now define $\Phi(z) = \Phi(x + iy) = F(x/r_0)\chi(y/r_0)$, where $F$ is the function obtained in Lemma $\S$ and $\chi$ is a smooth, even cut-off function on $\mathbb{R}$ such that $0 \leq \chi \leq 1$, $\text{supp} \chi \subset (-2, 2)$, and $\chi \equiv 1$ on $[-1, 1]$. We define

$$D := \left\{(z, w) \in \mathbb{C}^2 : |z| < 1, \log |w| < \frac{1}{2} \psi(|z|) + \Phi(z) \right\}.$$

Recall that for a Hartogs domain $\{\log |w| < \varphi(z), |z| < 1\}$, if $\varphi$ is of class $C^2$ at $z_0$, a boundary point $(z_0, w_0)$ with $|z_0| < 1$ is strictly pseudo-convex (respectively, strictly pseudoconcave) if and only if $\Delta \varphi(z_0) < 0$ (respectively, $\Delta \varphi(z_0) > 0$). Choosing an appropriate regularization (convolution by a smooth positive kernel of small enough support), we may get that:

- $\Delta \psi(|z|) \leq -4$ for $|z| \geq r_0$,
- $\Delta \psi(|z|) = 4$ for $|z| \leq r_0/2$, and is always $\leq 4$.

We consider points $z_0 = x + iy$. If $|x| > r_0$, $\Phi(z_0) = 0$ and we have pseudoconvex points (the boundary is a portion of the boundary of the ball).

On the other hand, when $x \in r_0\mathcal{U}$ (where $\mathcal{U}$ is the dense open set defined in Lemma $\S$),

$$\Delta \Phi(z_0) = \frac{1}{r_0^2} \left( F''(x/r_0)\chi(y/r_0) + F(x/r_0)\chi''(y/r_0) \right).$$

The only values of $z_0$ for which $F(x/r_0)\chi''(y/r_0) \neq 0$ or $\chi(y/r_0) < 1$ verify $|z_0| > r_0$, and at those points we have, using the fact that $F''(x/r_0) < 0$,

$$\frac{1}{2} \Delta \psi(|z_0|) + \Delta \Phi(z_0) \leq -4 + \frac{1}{r_0^2} C_1 \|\chi''\|_\infty \leq -1 \quad \text{for } x \leq r_0^2/2.$$

Note that the graphs of both functions cut inside the interval $(r_0/2, r_0)$. Indeed, $x^2 - r_0^2 > \log(1 - x^2)$ for $x \geq r_0^2$ and $x^2 - r_0^2 < \log(1 - x^2)$ for $x \leq r_0^2/2$. 
if we choose $C_1$ small enough. Hence we have strict pseudoconvexity again.

So we may restrict attention to $|y| \leq r_0$ and $\Delta \Phi(z_0) = F''(x/r_0)/r_0^2$. Therefore

$$\frac{1}{2} \Delta \psi(|z_0|) + \Delta \Phi(z_0) \leq 2 - C_2/r_0^2 < -2$$

for a $C_2$ chosen large enough.

Finally, notice that points $(z_0, w_0)$ with $|z_0| < r_0/2$ and $F(x) = 0$ verify $(z_0, w_0) \in \partial D_0 \cap \partial D$, $D_0 \subset D$, and $D_0$ is strictly pseudoconcave at $(z_0, w_0)$, so $D$ is as well. □

Proof of Proposition 2. We may assume that $D$ has a global defining function $r : U \to \mathbb{R}$ with $U = U(\partial D)$, $r \in C^2(U)$, and grad $r \neq 0$ on $U$, such that $D \cap U = \{z \in U : r(z) < 0\}$.

Now assume the contrary. Then we may find a point $z^0 \in \partial D$ such that the Levi form of $r$ at $z^0$ is not positive semidefinite on the complex tangent hyperplane to $\partial D$ at $z_0$. Therefore, there is a complex tangent vector $a$ with $Lr(z_0, a) \leq -2c < 0$, where $Lr(z_0, a)$ denotes its Levi form at $z^0$ in direction of $a$. Moreover, we may assume that $|\partial r/\partial z_1(z_0)| \geq 2c$.

Now fix an arbitrary boundary point $z \in \partial D \cap B(z_0, \epsilon_0)$. Define $a(z) := a + (\sum_{j=1}^n a_j \frac{\partial r}{\partial z_j}(z), 0, \ldots, 0)$.

3. Proofs of Propositions 2, 3 and 4

Proof of Proposition 2. We may assume that $D$ has a global defining function $r : U \to \mathbb{R}$ with $U = U(\partial D)$, $r \in C^2(U)$, and grad $r \neq 0$ on $U$, such that $D \cap U = \{z \in U : r(z) < 0\}$.

Now assume the contrary. Then we may find a point $z^0 \in \partial D$ such that the Levi form of $r$ at $z^0$ is not positive semidefinite on the complex tangent hyperplane to $\partial D$ at $z_0$. Therefore, there is a complex tangent vector $a$ with $Lr(z_0, a) \leq -2c < 0$, where $Lr(z_0, a)$ denotes its Levi form at $z^0$ in direction of $a$. Moreover, we may assume that $|\partial r/\partial z_1(z_0)| \geq 2c$.

Now choose $V = V(z^0) \subset U$ and $u \in PSH(D \cap V)$ with

$$\limsup_{D \cap V \ni z \to z_0} u(z) = \infty;$$

in particular, there is a sequence of points $D \cap V \ni b^i \to z_0$ such that $u(b^i) \to \infty$.

By the $C^2$-smooth assumption, there is an $\epsilon_0 > 0$ such that for all $z \in B(z_0, \epsilon_0) \subset V$ and all $\bar{a} \in B(a, \epsilon_0)$ we have

$$Lr(z, \bar{a}) \leq -c, \quad |\frac{\partial r}{\partial z_1}(z)| \geq c.$$

Now fix an arbitrary boundary point $z \in \partial D \cap B(z_0, \epsilon_0)$. Define

$$a(z) := a + (\sum_{j=1}^n a_j \frac{\partial r}{\partial z_j}(z), 0, \ldots, 0).$$
Observe that this vector is a complex tangent vector at \( z \) and \( a(z) \in \mathbb{B}(a, \varepsilon_0) \) if \( z \in \mathbb{B}(z_0, \varepsilon_1) \) for a sufficiently small \( \varepsilon_1 < \varepsilon_0 \).

Now, let \( z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1) \). Put
\[
b_1(z) := \frac{Lr(z, a(z))}{2 \partial z_1(z)}
\]
and
\[
\varphi_z(\lambda) = z + \lambda \cdot (\lambda_1 a(z)) + \lambda^2 b_1(z), 0, \ldots, 0), \quad \lambda \in \mathbb{C}.
\]

Moreover, if \( \varepsilon_1 \) is sufficiently small, we may find \( \delta, \rho_0 > 0 \) such that for all \( z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1) \) we have
\[
\overline{D} \cap \mathbb{B}(z, \delta) - t \nu(z) \subset D, \quad 0 < t \leq t_0,
\]
where \( \nu(z) \) denotes the outer unit normal vector of \( D \) at \( z \).

Next using the Taylor expansion of \( \varphi_z, z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1), \varepsilon_1 \) sufficiently small, we get
\[
r \circ \varphi_z(\lambda) = |\lambda|^2 \left( Lr(z, a(z)) + \varepsilon(z, \lambda) \right),
\]
where \( |\varepsilon(z, \lambda)| \leq \varepsilon(\lambda) \to 0 \) if \( \lambda \to 0 \).

In particular, \( \varphi_z(\lambda) \in \mathbb{B}(z, \delta) \cap D \subset V \cap D \) when \( 0 < |\lambda| \leq \delta_0 \) for a certain positive \( \delta_0 \) and \( r \circ \varphi_z(\lambda) \leq -\delta_0^2 c/2 \) when \( |\lambda| = \delta_0 \).

Hence, \( K := \bigcup_{z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1)} \varphi_z(\lambda) \in D \cup V \). Choose an open set \( W = W(K) \subset D \cap V \). Then \( u \leq M \) on \( W \) for a positive \( M \).

Finally, choose a \( j_0 \) such that \( b^j = z^j - t_j \nu(z^j), j \geq j_0 \), where \( z^j \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1) \), \( 0 < t_j \leq t_0 \), and \( \varphi_{z^j}(\lambda) \in W \) when \( |\lambda| = \delta_0 \). Therefore, by construction, \( u(b^j) \leq M \), which contradicts the assumption. \( \square \)

**Proof of Proposition 4.** Assume that \( D \) is not pseudoconvex. Then, by Corollary 4.1.26 in [2], there is \( \varphi \in \mathcal{O}(\mathbb{D}, D) \) such \( \text{dist}(\varphi(0), \partial D) < \text{dist}(\varphi(\zeta), \partial D) \) for any \( \zeta \in \mathbb{D} \). To get a contradiction, it remains to use similar arguments as in the previous proof and we skip the details.

**Proof of Proposition 5.** It is enough to show that if \( \mathcal{O}(\mathbb{D}, D) \ni \psi_j \to \psi \) and \( \psi(\zeta) \in \partial D \) for some \( \zeta \in \mathbb{D} \), then \( \psi(\mathbb{D}) \subset \partial D \). Suppose the contrary. Then it is easy to find points \( \eta_k \to \eta \in \mathbb{D} \) such that \( \psi(\eta_k) \in \partial D \) but \( a = \psi(\eta) \in \partial D \). We may assume that \( \eta = 0 \) and \( g_a = \frac{1}{f_a} \) is bounded on \( D \cap U_a \). Let \( r \in (0, 1) \) be such that \( \psi(r\mathbb{D}) \in U_a \). Then \( \psi_j(r\mathbb{D}) \subset U_a \) for any \( j \geq j_0 \). Hence \( |g_a \circ \psi_j| < 1 \) and we may assume that \( g_a \circ \psi_j \to h_a \in \mathcal{O}(r\mathbb{D}, \mathbb{C}) \). Since \( h_a(\eta) = 0 \), it follows by the Hurwitz theorem that \( h_a = 0 \). This contradicts the fact that \( h_a(\eta_k) = g_a \circ \psi(\eta_k) \neq 0 \) for \( |\eta_k| < r \). \( \square \)
4. Proofs of Propositions 5, 6 and 7

Proof of Proposition 5. Our aim is to construct a non-pseudoconvex bounded domain $D \subset \mathbb{C}^2$ such that $\limsup_{z \to a} K_D(z) = \infty$ for any $a \in \partial D$.

Let us start with the domain $P \times \mathbb{D}$, where $P = \{\lambda \in \mathbb{C} : \frac{1}{2} < |\lambda| < \frac{3}{2}\}$. Let

$$S := \{(z_1, z_2) = (x_1 + iy_1, z_2) \in P \times \mathbb{D} : (x_1 - 1)^2 + \frac{1 + |z_2|^2}{1 - |z_2|^2} y_1^2 = \frac{1}{4}, y_1 > 0\}.$$  

Define $D := (P \times \mathbb{D}) \setminus S$. Note that $D$ is a domain. Its envelope of holomorphy is non-schlicht and consists of the union of $D$ and one additional 'copy' of the set

$$D_1 := \{(z_1, z_2) \in P \times \mathbb{D} : (x_1 - 1)^2 + \frac{1 + |z_2|^2}{1 - |z_2|^2} y_1^2 \leq \frac{1}{4}, y_1 > 0\}.$$  

In particular, $D$ is not pseudoconvex. Note that convexity of the interior $D^0$ of $D_1$ implies that $\lim_{z \to \partial D_1} K_{D^0}(z) = \infty$. Therefore, it follows from the localization result for the Bergman kernel due to Diederich-Fornaess-Herbort formulated for Riemann domains in the paper [4] that for all $a \in S \subset \partial D_1$ the following property holds: $\lim_{D \cap D_1 \ni z \to a} K_D(z) = \infty$ (on the other hand while tending to the points from $S$ from the 'other side' of the domain $D$ the Bergman kernel is bounded from above). Obviously $P \times \mathbb{D}$ is Bergman exhaustive, so for any $a \in \partial (P \times \mathbb{D})$ the following equality holds $\lim_{z \to a} K_D(z) = \infty$. □

Proof of Proposition 6. Recall the following facts that follow from [1].

If the envelope of holomorphy $\bar{D}$ of the domain $D$ is a domain in $\mathbb{C}^n$ (is schlicht) then the Bergman kernel $K_D$ extends to a real analytic function $\bar{K}_D$ defined on $\bar{D}$.

Let $\emptyset \neq P_0 \subset D$, $P_0 \subset P$, $P \setminus D \neq \emptyset$ and $\bar{P}_0 \cap (\mathbb{C}^n \setminus D) \neq \emptyset$, where $P_0$, $P$ are polydiscs, and the following property is satisfied: for any $f \in \mathcal{O}(\bar{D})$ there is a function $\bar{f} \in \mathcal{O}(P)$ such that $f = \bar{f}$ on $P_0$. Then the Bergman kernel $K_D$ extends to a real analytic function on $P$. More precisely, there is a real analytic function $\bar{K}_D$ defined on $P$ such that $\bar{K}_D(z) = K_D(z)$, $z \in P_0$.

Both facts above complete the proof of Proposition 6. □

The proof of Proposition 7 is essentially contained in [3]. However, this PhD Thesis is not publically accessible. Therefore we repeat it here. The idea is the following: if $\limsup_{z \to a} K_D(z) = \infty$ for some $a \in \partial D$, then there is an $f \in L^2_h(D)$ such that $\limsup_{z \to a} |f(z)| = \infty$.

Proof of Proposition 7. In view of Proposition 5, $\limsup_{z \to a} K_D(z) = \infty$ for any $a \in \partial D$. □
Let \( a \in \partial D \). We claim that there is an \( L^2_h(D) \)-function \( h \) which is unbounded near \( a \).

Assume the contrary. Hence for any \( f \in L^2_h(D) \) there exists a neighborhood \( U_f \) of \( a \) and a number \( M_f \) such that \( |f| \leq M_f \) on \( D \cap U_f \).

Denote by \( L \) the unit ball in \( L^2_h(D) \) and by \( c = \pi^n \).

Let \( K_1 := \{ z \in D : \text{dist}(z, \partial D) \geq 1 \} \) (if this is empty take a smaller number than 1). By the meanvalue inequality we have for any \( f \in L \) that \( |f| \leq c \) on \( K_1 \). By assumption, there are \( z_1 \in D \) and \( f_1 \in L \) such that \( |z_1 - a| < 1 \) and \( |f_1(z_1)| > c \).

Set \( g_1 := f_1/c \). Then \( g \in L \) and therefore there are a neighborhood \( U_1 \) of \( a \) and number \( M_1 > 1 \) such that \( |g_1| \leq M_1 \) on \( D \cap U_1 \).

Set \( K_2 := \{ z \in D : \text{dist}(z, \partial D) \geq \text{dist}(z_1, \partial D) \} \) and \( d = c \text{dist}(z_1, \partial D) \).

Then \( K_1 \subset K_2 \). Choose \( z_2 \in U_1 \cap D \), \( z_2 \notin K_2 \), \( |z_2 - a| < 1/2 \), and \( f_2 \in L \) with \( |f_2(z_2)| \geq d(1^3 + 1^2 M_1) \). Moreover, \( |f_2| \leq d \) on \( K_2 \). Put \( g_2 := f_2/d \). Then \( g_2 \in L \). Choose now a neighborhood \( U_2 \) of \( a \) and a number \( M_2 \) such that \( |g_2| \leq M_2 \) on \( D \cap U_2 \).

Then we continue this process.

So we have points \( z_k \in K_{k-1} \), \( z_k \notin K_{k-1} \), \( |z_k - a| < 1/k \), and functions \( f_k \in L \) with

\[
|f_k(z_k)| \geq c \text{dist}(z_{k-1}, \partial D)^n(k^3 + k^2 \sum_{j=1}^{k-1} M_j).
\]

Setting \( g_k := f_k/d \) and \( h := \sum_{j=1}^{\infty} g_j/j^2 \), it is clear that \( h \in L^2_h(D) \).

Fix now \( k \geq 2 \). Then

\[
|h(z_k)| \geq \frac{|g_k(z_k)|}{k^2} - \sum_{j=1}^{k-1} \frac{|g_j(z)|}{j^2} - \sum_{j=k+1}^{\infty} \frac{|g_j(z)|}{j^2}
\]

\[
\geq k + \sum_{j=1}^{k-1} M_j - \sum_{j=1}^{k-1} \frac{M_j}{j^2} - \sum_{j=k+1}^{\infty} \frac{1}{j^2} > k - \frac{1}{6}.
\]

In particular, \( h \) is unbounded at \( a \) which is a contradiction.

It remains to choose a dense countable sequence \( (a_j) \subset \partial D \) such that any term repeats infinitely many times and to copy the proof of the Cartan-Thullen theorem. \( \square \)

References


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