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Magnetic bottles on geometrically finite hyperbolic surfaces

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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^*(id + A)$ on a hyperbolic surface $M$, when the magnetic field $dA$ is infinite at the boundary at infinity. We prove that the counting function of the eigenvalues has a particular asymptotic behavior when $M$ has an infinite area.

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface $(M, g)$ and a smooth, real one-form $A$ on $M$. We define the magnetic Laplacian

$$-\Delta_A = (i d + A)^*(i d + A),$$

where $(i d + A)u = i du + uA$, $\forall u \in C^\infty_0(M; C)$. \hfill (1.1)

The magnetic field is the exact two-form $\rho_B = dA$.

If $dm$ is the Riemannian measure on $M$, then

$$\rho_B = \tilde{b} \, dm, \quad \text{with} \quad \tilde{b} \in C^\infty(M; R).$$

The magnetic intensity is $b = |\tilde{b}|$. \hfill (1.2)

\textsuperscript{1} Keywords: spectral asymptotics, magnetic bottles, hyperbolic surface.
It is well known, (see [Shu]), that \(-\Delta_A\) has a unique self-adjoint extension on \(L^2(M)\), containing in its domain \(C_0^\infty(M; \mathbb{C})\), the space of smooth and compactly supported functions.

When \(b\) is infinite at the infinity, (with some additional assumption), the spectrum of \(-\Delta_A\) is discrete, and we denote by \((\lambda_j)\) the increasing sequence of eigenvalues of \(-\Delta_A\), (each eigenvalue is repeated according to its multiplicity). Let

\[
N(\lambda) = \sum_{\lambda_j < \lambda} 1.
\]

We are interested by the hyperbolic surfaces \(M\), when the curvature of \(M\) is constant and negative.

In this case, when \(M\) has finite area, the asymptotic behavior of \(N(\lambda)\) seems to be the Weyl formula: \(N(\lambda) \sim_{+\infty} \frac{\lambda}{4\pi |M|}\).

S. Golénia and S. Moroianu in [Go-Mo] have such examples. In the case of the Poincaré half-plane, \(M = \mathbb{H}\), we prove in [Mo-Tr] that the Weyl formula is not valid: \(\lim_{\lambda \to +\infty} \lambda^{-1} N(\lambda) = +\infty\).

For example when \(b(z) = a_0^2 (x/y)^{2m_0} + a_1^2 y^{m_1} + a_2^2 y^{m_2}\), \(a_j > 0\) and \(m_j \in \mathbb{N}^*\), then

\[
N(\lambda) \sim_{+\infty} \lambda^{1+1/(2m_0)} \ln(\lambda) \alpha(m_0, m_1, m_2).
\]

In this paper, we are interested by the hyperbolic surfaces with infinite area. When \(M\) is a geometrically finite hyperbolic surface of infinite area and when the above example is arranged for this new situation, \((m_0)\) is absent, \(m_1\) appears in the cusps and \(m_2\) in the funnels), we get

\[
N(\lambda) \sim_{+\infty} \lambda^{1+1/m_2} \alpha(m_2) : \text{the cusps do not contribute to the leading part of } N(\lambda).
\]

## 2 Main result

We assume that \((M, g)\) is a smooth connected Riemannian manifold of dimension two, which is a geometrically finite hyperbolic surface of infinite area; (see [Per] or [Bor] for the definition and the related references). More precisely

\[
M = \left( \bigcup_{j=0}^{J_1} M_j \right) \cup \left( \bigcup_{k=1}^{J_2} F_k \right); \quad (2.1)
\]
where the $M_j$ and the $F_k$ are open sets of $M$, such that the closure of $M_0$ is compact, and if $J_1 > 0$, the other $M_j$ are cuspidal ends of $M$, and the $F_k$ are funnel ends of $M$.

This means that, for any $j$, $1 \leq j \leq J_1$, there exist strictly positive constants $a_j$ and $L_j$ such that $M_j$ is isometric to $S \times [a_j^2, +\infty)$, equipped with the metric
\[ ds_j^2 = y^{-2} (L_j^2 \, d\theta^2 + dy^2) \]
\[ (S = S^1 \text{ is the unit circle}) \]

In the same way, for any $k$, $1 \leq k \leq J_2$, there exist strictly positive constants $a_k$ and $L_k$ such that $F_k$ is isometric to $S \times [a_k^2, +\infty)$, equipped with the metric
\[ ds_k^2 = \tau_k^2 \cosh^2(t) d\theta^2 + dt^2 \]

moreover, for any two integers $j$, $k > 0$, we have $M_j \cap F_k = \emptyset$ and $M_j \cap M_k = F_j \cap F_k = \emptyset$ if $j \neq k$.

Let us choose some $z_0 \in M_0$ and let us define
\[ d : M \to \mathbb{R}_+ ; \quad d(z) = d_g(z, z_0) \]
\[ d_g(\cdot, \cdot) \text{ denotes the distance with respect to the metric } g. \]

We assume the smooth one-form $A$ to be given such that the magnetic field $\tilde{b}$ satisfies
\[ \lim_{d(z) \to \infty} b(z) = +\infty. \]

If $J_1 > 0$, there exists a constant $C_1 > 0$ such
\[ |X \tilde{b}(z)| \leq C_1 (b(z) + 1)e^{d(z)} |X|_g ; \]
\[ \forall z \in M_j, \forall X \in T_z M \text{ and } \forall j = 1, \ldots J_1. \]

There exists a constant $C_2 > 0$ such
\[ |X \tilde{b}(z)| \leq C_2 (b(z) + 1) |X|_g ; \]
\[ \forall z \in F_k, \forall X \in T_z M \text{ and } \forall k = 1, \ldots J_2. \]

For any self-adjoint operator $P$, and for any real $\lambda$, we will denote by $E_\lambda(P)$ its spectral projection, and when its trace is finite we will denote it by
\[ N(\lambda; P) = Tr(E_\lambda(P)) \]
\[ N(\lambda; P) \text{ is the number of eigenvalues of } P, \text{ (counted with their multiplicity), } \text{which are in } [-\infty, \lambda]. \]
Theorem 2.1 Under the above assumptions, $-\Delta_A$ has a compact resolvent and for any $\delta \in \left[\frac{1}{3}, \frac{2}{5}\right]$, there exists a constant $C > 0$ such that

$$\frac{1}{2\pi} \int_M \left(1 - \frac{C}{(b(m)+1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1-C\lambda^{-3\delta+1}) - \frac{1}{4}, b(m)) \, dm$$

$$\leq N(\lambda, -\Delta_A) \leq \frac{1}{2\pi} \int_M \left(1 + \frac{C}{(b(m)+1)^{(2-5\delta)/2}}\right) \mathcal{N}(\lambda(1+C\lambda^{-3\delta+1}) - \frac{1}{4}, b(m)) \, dm$$

where

$$\mathcal{N}(\mu, b(m)) = b(m) \sum_{k=0}^{+\infty} [\mu - (2k+1)b(m)]_+^0 \quad \text{if} \quad b(m) > 0 ,$$

and

$$\mathcal{N}(\mu, b(m)) = \mu/2 \quad \text{if} \quad b(m) = 0 .$$

$[\rho]_+^0$ is the Heaviside function:

$$[\rho]_+^0 = \begin{cases} 1, & \text{if} \quad \rho > 0 \\ 0, & \text{if} \quad \rho \leq 0 . \end{cases}$$

The Theorem remains true if we replace $\int_M$ by $\sum_{k=1}^{J_2} \int_{F_k}$, due to the fact that the other parts are bounded by $C\lambda$.

Corollary 2.2 Under the assumptions of Theorem 2.1 and if the function

$$\omega(\mu) = \int_M [\mu - b(m)]_+^0 \, dm$$

satisfies, $\exists C_1 > 0$ s.t. $\forall \mu > C_1$, $\forall \tau \in ]0, 1[$,

$$\omega ((1+\tau) \mu) - \omega(\mu) \leq C_1 \tau \omega(\mu) ,$$

then

$$N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_M \mathcal{N}(\lambda - \frac{1}{4}, b(m)) \, dm .$$
For example this allows us to consider magnetic fields of the following type:

\[
\text{on } F_k \quad b(\theta, t) = p_k(\frac{1}{\cosh(t)}) ,
\]
and on \( M_j \), \( j > 0 \), \( b(\theta, y) = q_j(y) \),

where the \( p_k(s) \) and the \( q_j(s) \) are, for large \( s \), polynomial functions of order \( \geq 1 \). In this case, if \( d \) is the largest order of the \( p_k(s) \), then

\[
N(\lambda; -\Delta_A) \sim \alpha \lambda^{1+1/d},
\]

for some constant \( \alpha > 0 \), depending only on the funnels \( F_k \) where the order of \( p_k(s) \) is \( d \).

3 Estimate for Dirichlet operators

3.1 The main propositions

In this section, we consider some particular open set \( \Omega \) of \( M \) with smooth boundary. To \( \Omega \) and \( -\Delta_A \), we associate the Dirichlet operator \( -\Delta_A^\Omega \), and we estimate \( N(\lambda; -\Delta_A^\Omega) \).

**Proposition 3.1** Let \( \Omega \) an open set of \( M_0 \) with smooth boundary. Then there exists a constant \( C_\Omega > 0 \) s.t.

\[
\left| N(\lambda; -\Delta_A^\Omega) - \frac{|\Omega|}{4\pi} \lambda \right| \leq C_\Omega \sqrt{\lambda} ; \quad \forall \lambda > 1 .
\]

As \( \overline{\Omega} \) is compact, the above estimate is well known. See for example Theorem 29.3.3 in \[Hor\].

**Proposition 3.2** Let \( j > 0 \) and \( \Omega \) an open set of the cusp \( M_j \), isometric to \( S \times \mathbb{R} \), equipped with the metric

\[
ds^2 = y^{-2}( L^2 d\theta^2 + dy^2 ) ; \quad (a \text{ and } L \text{ are strictly positive constants}) .
\]

Then \( -\Delta_A^\Omega \) has a compact resolvent and

\[
N(\lambda; -\Delta_A^\Omega) \sim \frac{|\Omega|}{4\pi} \lambda ; \quad \text{as } \lambda \to +\infty .
\]
We will prove it in the next subsection.

**Proposition 3.3** Let $\Omega$ an open set of a funnel $F_k$, isometric to $S \times ]a^2, +\infty[,$ equipped with the metric
\[ ds^2 = L^2 \cosh^2(t) \, d\theta^2 + dt^2; \] (a and $L$ are strictly positive constants).

Then $-\Delta^\Omega_A$ has a compact resolvent and for any $\delta \in ]\frac{1}{5}, \frac{2}{5}[,$ there exists a constant $C > 0$ such that
\[ \frac{1}{2\pi} \int_{\Omega} \left( 1 - \frac{C}{(b(m) + 1)^{(2-5\delta)/2}} \right) N(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4}, b(m)) \, dm \leq N(\lambda, -\Delta^\Omega_A) \leq \frac{1}{2\pi} \int_{\Omega} \left( 1 + \frac{C}{(b(m) + 1)^{(2-5\delta)/2}} \right) N(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4}, b(m)) \, dm. \]

The proof comes easily following the ones in the Poincaré half-plane of $[\text{Mo-Tr}]$, using the method of $[\text{Col}]$, in the neighbourhood of the boundary at infinity. It corresponds to a context where the partitions of unity were fine, so they can be performed on $S \times ]a^2, +\infty[,$ (instead of $\mathbb{R} \times ]-\infty, 0[,$).

### 3.2 Proof of Proposition 3.2

For simplicity we change the unit circle $S = S_1$ into the circle $S_L$, of radius $L$, so
\[ \Omega = S_L \times ]a^2, +\infty[,$ \hspace{1em} $ds^2 = y^{-2}(dx^2 + dy^2), \quad \text{and} \]
\[ -\Delta^\Omega_A u(z) = y^2 [(D_x - A_1)^2 u(z) + (D_y - A_2)^2 u(z)]; \]
moreover $d(z, z') = \text{arg} \cosh \frac{y^2 + y'^2 + d^2_{S_L}(x, x')}{2yy'}$.

We begin by proving the compactness of the resolvent of $-\Delta^\Omega_A$.

**Lemma 3.4** There exists $C_0 > 1$ such that
\[ \int_{\Omega} (b(z) - C_0)|u(z)|^2 \, dm \leq \int_{\Omega} -\Delta^\Omega_A u(z)\overline{u(z)} \, dm; \quad \forall \, u \in C^\infty_0(\Omega). \]
Proof. Let us denote the quadratic form

\[ q_\Omega^\Omega(u) = \int_{\Omega} -\Delta_\Omega u(z) \overline{u(z)} dm \quad \forall \ u \in C_0^\infty(\Omega). \]  

(3.2)

Then \( q_\Omega^\Omega(u) = \int_{\Omega} \left[ |(D_x - A_1)u|^2 + (D_y - A_2)u|^2 \right] dxdy \),

and \( \left| \int_{\Omega} \tilde{b}(z)|u(z)|^2 dm \right| = \left| \int_{\Omega} [(D_x - A_1)u(z)(D_y - A_2)u(z) - (D_y - A_2)u(z)(D_x - A_1)u(z)]dxdy \right|. \)

Therefore we get that \( \left| \int_{\Omega} \tilde{b}(z)|u(z)|^2 dm \right| \leq q_\Omega^\Omega(u). \)

As \( b(z) = |\tilde{b}(z)| \to +\infty \) at the infinity, the Lemma comes easily.

The Lemma 3.4 and the assumption (2.5) prove that \( -\Delta_\Omega \) has compact resolvent.

Later on, we will need that the assumptions (2.5) and (2.6) ensure that there exists \( C > 1 \) such that \( \forall \ z = (x, y), \ z' = (x', y') \in \Omega \),

\( b(z)/C \leq b(z') \leq Cb(z) \), if \( |y - y'| \leq 1 \) and \( y > C \). \hspace{1cm} (3.3)

This comes from the fact that \( d(z) \) is equivalent to \( \ln(y) \) for \( y(> 1) \) large enough, so the assumption (2.6) ensures that \( |\partial_x b(z)| + |\partial_y b(z)| \leq C(|b(z)| + 1). \)

Lemma 3.5 There exists a constant \( C_0 > 1 \) such that, for any \( \lambda > 1 \) and for any \( K \subset \Omega \) isometric to \( I_1 \times I_2 \), endowed with the metric in (3.7), with

\[ I_1 = ]x_0 - \epsilon_1, x_0 + \epsilon_1[, \quad I_2 = ]y_0 - \epsilon_2, y_0 + \epsilon_2[, \]

\( \epsilon_1 \in ]C_0^{-1}, 1[, \quad \epsilon_2 = \sqrt{y_0}/\sqrt{b(z_0)}, \quad (y_0 > C_0) \);

the following estimates hold:

\[ \left[ \lambda(1 - \frac{1}{\sqrt{y_0}}) - C_0 \right]\frac{|K||y|}{4\pi} \leq N(\lambda; -\Delta_\Omega^K) \leq \left[ \lambda(1 + \frac{1}{\sqrt{y_0}}) + C_0 \right]\frac{|K||y|}{4\pi} \]. \hspace{1cm} (3.4)

Proof. If \( b(z_0) > C\lambda \), then, according to the estimate of Lemma 3.4 with \( K \) instead of \( \Omega \), \( N(\lambda; -\Delta_\Omega^K) = 0 \).

So we can assume that \( b(z_0) \leq C\lambda \).
We use that the spectrum of $-\Delta^K_A$ is gauge-invariant, so we can suppose that in $K$

$$A_2 = 0 \quad \text{and} \quad A_1(x, y) = -\int_{y_0}^y \frac{\tilde{b}(x, \rho)}{\rho^2} d\rho .$$

Then $|A_1(x, y)| \leq C\epsilon_2 \frac{b(z_0)}{y_0^2}$.

From this estimate, we get that for any $\epsilon \in [0, 1[$,

$$-(1 - \epsilon)\Delta^K_0 - C\epsilon_2 \frac{b^2(z_0)}{\epsilon y_0^2} \leq -\Delta^K_A \leq -(1 + \epsilon)\Delta^K_0 + C\epsilon_2 \frac{b^2(z_0)}{\epsilon y_0^2} .$$

We take $\epsilon = 1/\sqrt{y_0}$, to get

$$-(1 - \frac{1}{\sqrt{y_0}})\Delta^K_0 - C \frac{b(z_0)}{\sqrt{y_0}} \leq -\Delta^K_A \leq -(1 + \frac{1}{\sqrt{y_0}})\Delta^K_0 + C \frac{b(z_0)}{\sqrt{y_0}} .$$

As $b(z_0) \leq C\lambda$, the Lemma follows easily from the min-max principle and the well-known estimate for $N(\lambda; -\Delta^K_0)$.

**Proof of Proposition 3.2.**

It follows easily from Lemma 3.5 (for large $y$), using the same tricks as in [Mo-Tr].

### 4 Proof of the main Theorem 2.1

The proof comes easily from the three propositions 3.1 - 3.3, following the method developped in [Mo-Tr].

### 5 Remark on the case of constant magnetic field

It is not always possible to have a constant magnetic field on $M$, (for topological reason), but for any $(b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$, there exists a one-form $A$, such that the corresponding magnetic field $dA$ satisfies

$$dA = \tilde{b}(z) dm \quad \begin{cases} \tilde{b}(z) = b_j \forall z \in M_j \\ \tilde{b}(z) = \beta_k \forall z \in F_k \end{cases}$$

(5.1)
Theorem 5.1 Assume (2.1) and (5.1).

If \( J_1 = 0 \) and \( J_2 > 0 \), then the essential spectrum of \(-\Delta A\) is

\[
\text{sp}_{\text{ess}}(-\Delta A) = \left\{ \frac{1}{4} + \inf_k \beta_k^2, +\infty \right\} \bigcup \bigcup_{k=1}^{J_2} S(\beta_k) \]  

(5.2)

with \( S(\beta_k) = \emptyset \) when \(|\beta_k| \leq 1/2\) and when \(|\beta_k| > 1/2\)

\[
S(\beta_k) = \{(2j + 1)|\beta_k| - j(j + 1); j \in \mathbb{N}, j < |\beta_k| - 1/2\} .
\]

If \( J_1 \) and \( J_2 \) are \( > 0 \), then for any \( j, 1 \leq j \leq J_1 \) and for any \( z \in M_j \) there exists a unique closed curve through \( z \), \( C_{j,z} \) in \( (M_j, g) \), not contractible and with zero \( g \)-curvature. The following limit exists and is finite:

\[
[A]_{M_j} = \lim_{d(z) \to +\infty} \int_{C_{j,z}} A.
\]  

(5.3)

If \( J_1^A = \{ j \in \mathbb{N}, 1 \leq j \leq J_1 \ s.t. \ [A]_{M_j} \in 2\pi \mathbb{Z} \} \), then

\[
\text{sp}_{\text{ess}}(-\Delta A) = \left\{ \frac{1}{4} + \min \left\{ \inf_{j \in J_1^A} b_j^2, \inf_{1 \leq k \leq J_2} \beta_k^2 \right\}, +\infty \right\} \bigcup \bigcup_{k=1}^{J_2} S(\beta_k) .
\]  

(5.4)

If \( J_2 = 0 \) and \( J_1^A = \emptyset \), then \( \text{sp}_{\text{ess}}(-\Delta A) = \emptyset \) ;

\(-\Delta A\) has purely discrete spectrum, (its resolvent is compact).

Remark 5.2 In Theorem 5.1, one can change \( C_{j,z} \) into \( S_{j,z} \), the unique closed curve through \( z \), not contractible and with minimal \( g \)-length.

\( S_{j,z} \) is not smooth at \( z \), \( S_{j,z} \) is part of two geodesics through \( z \), so there is an out-going tangent and an incoming tangent at \( z \). It is easy to see that \( C_{j,z} \cap S_{j,z} = \{ z \} \), so by Stokes formula

\[
\int_{S_{j,z}} (A - A^0) = \int_{C_{j,z}} (A - A^0) ,
\]

where \( A^0 \) is a one-form on \( M_j \), such that

\[
dA = dA^0 \text{ on } M_j \text{ and } [A^0]_{M_j} = 0 ; \forall j .
\]

The orientation in both cases \( C_{j,z} \) and \( S_{j,z} \), is chosen such that, if \( u_z, v_z \in T_z M_j \), \( g_z(u_z, v_z) = 0 \), \( d\text{m}(u_z, v_z) > 0 \), and \( u_z \) is tangent to the curve (in the positive direction), then \( v_z \) points to boundary at infinity; (for \( S_{j,z} \), one can take as \( u_z \) the out-going tangent, or the incoming tangent).
Proof of Theorem 5.1. It is clear that

$$\text{sp}_{\text{ess}}(-\Delta_A) = \left( \bigcup_{j=1}^{J_1} \text{sp}_{\text{ess}}(-\Delta_{A_j}^A) \right) \bigcup \left( \bigcup_{k=1}^{J_2} \text{sp}_{\text{ess}}(-\Delta_{F_k}^A) \right); \quad (5.5)$$

so the proof will result on the two lemmas below.

Lemma 5.3

$$\text{sp}_{\text{ess}}(-\Delta_{F_k}^A) = \left[ \frac{1}{4} + \beta_k^2, +\infty \right] \cup S(\beta_k).$$

Proof. We have

$$-\Delta_{F_k}^A = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + \cosh^{-1}(t)(D_t - A_2)[\cosh(t)(D_t - A_2)].$$

Since \( \tilde{b} = \beta_k = \tau_k^{-1} \cosh^{-1}(t)(\partial_\theta A_2 - \partial_t A_1) \), there exists a function \( \varphi \) such that \( A - \tilde{A} = d\varphi \) if \( \tilde{A} = (\xi - \beta_k \tau_k \sinh(t))d\theta \), (for some constant \( \xi \)).

So we can assume that \( A = \tilde{A} \).

We change the density \( dm = \tau_k \cosh(t)d\theta dt \) for \( d\theta dt \), using the unitary operator \( Uf = (\tau_k \cosh(t))^{1/2} f \), so

$$P = -U\Delta_{F_k}^A U^* = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}(1 + \cosh^{-2}(t)).$$

We remind that \( \lambda \in \text{sp}_{\text{ess}}(-\Delta_{F_k}^A) \) iff there exists a sequence \( (u_j)_j \in \text{Dom}(-\Delta_{F_k}^A) \) converging weekly in \( L^2(F_k) \) to zero, \( \|u_j\|_{L^2(F_k)} = 1 \) and such that the sequence \( (-\Delta_{F_k}^A u_k - \lambda u_k)_k \) converges strongly to zero.

It is clear that \( \text{sp}(-\Delta_{F_k}^A) = \text{sp}\left( \bigoplus_{\ell \in \mathbb{Z}} P_\ell \right) \),

$$P_\ell = D_t^2 + \tau_k^{-2} \cosh^{-2}(\ell \tau_k)(\ell \beta_k + \beta_k \tau_k \sinh(t) - \xi)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)),$$

for the Dirichlet condition on \( L^2(I; dt); I = [\alpha_k^2, +\infty[. \)

So \( \text{sp}(-\Delta_{F_k}^A) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell) \).

Writing that \( P_\ell = D_t^2 + \left( \frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t) \right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) \),

we get easily that \( \text{sp}_{\text{ess}}(P_\ell) = \left[ \frac{1}{4} + \beta_k^2, +\infty \right] \), and that the number of eigenvalues \( < \frac{1}{4} + \beta_k^2 \) is finite for all \( \ell < \xi \) and equal to zero for all \( \ell \geq \xi \). Here
we assume $\beta_k > 0$. So $\left[\frac{1}{4} + \beta_k^2, +\infty\right] \subset \text{sp}_{\text{ess}}(-\Delta^F_k)$ and the other part of $\text{sp}_{\text{ess}}(-\Delta^F_k)$ is $S_{\infty} = \{ \lambda; \lambda = \lim_{\rho \to +\infty} \lambda_{\ell_j}, \lambda_{\ell_j} \in \text{sp}_d(P_{\ell_j}) \}$, where $(\ell_j(j))$ denotes any decreasing sequence of negative integers. Now we use again the formula

$$
P_\ell = D_t^2 + \left( \frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t) \right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)) \ .$$

Assuming $\ell - \xi < 0$, we set $\rho = |\ell - \xi|/\tau_k$ and we introduce the new variable $y = 2\rho e^{-t}$. We get that $P_\ell$ is unitarily equivalent to $\tilde{P}_\rho$ defined as a Dirichlet type operator in $L^2([0, 2\rho e^{-\alpha\xi}; dy])$, (zero boundary condition is only required on the right boundary):

$$
\tilde{P}_\rho = D_y(y^2 D_y) + W_\rho(y) , \quad \text{with}
$$

$$
W_\rho(y) = \left( \beta_k \frac{(1 - y^2/(4\rho^2))}{1 + y^2/(4\rho^2)} - \frac{y}{1 + y^2/(4\rho^2)} \right)^2 + \left( \frac{y/(2\rho)}{1 + y^2/(4\rho^2)} \right)^2 .
$$

So we have $\lim_{\rho \to +\infty} W_\rho(y) = W_\infty(y) = (\beta_k - y)^2$, and the operator

$$
\tilde{P}_\infty = D_y(y^2 D_y) + W_\infty(y) \text{ on } L^2([0, +\infty[; dy) \text{ satisfies, (see [Mo-Tr]),
$$

$$
\text{sp}(\tilde{P}_\infty) = \text{sp}_{\text{ess}}(\tilde{P}_\infty) \cup \text{sp}_d(\tilde{P}_\infty) \text{ with }
$$

$$
\text{sp}_{\text{ess}}(\tilde{P}_\infty) = \left[\frac{1}{4} + \beta_k^2, +\infty\right] ; \quad \text{sp}_d(\tilde{P}_\infty) = S(\beta_k) .
$$

We remind that the eigenfunctions associated to the eigenvalues in $S(\beta_k)$ of $\tilde{P}_\infty$ are exponentially decreasing, so if $\lambda_0(\rho) \leq \ldots \leq \lambda_j(\rho) \leq \lambda_{j+1}(\rho) \ldots$ are the eigenvalues of $\tilde{P}_\rho$ then for any $j$,

$$
\lim_{\rho \to +\infty} \lambda_j(\rho) = \lambda_j(\infty) = (2j + 1)\beta_k - j(j + 1) , \text{ if } \beta_k > 1/2 \text{ and } j < \beta_k - 1/2 ,
$$

otherwise

$$
\lim_{\rho \to +\infty} \lambda_j(\rho) = \frac{1}{4} + \beta_k^2 .
$$

Therefore we get that $S_{\infty} = S(\beta_k) \text{, or } S_{\infty} = S(\beta_k) \cup \left\{ \frac{1}{4} + \beta_k^2 \right\}$ : the formula of Lemma 5.3 follows.

**Lemma 5.4** If $1 \leq j \leq J_1$ and $j \notin J_1^A$, then

$$
\text{sp}_{\text{ess}}(-\Delta_A^{M_j}) = \emptyset .
$$

If $j \in J_1^A$, then

$$
\text{sp}_{\text{ess}}(-\Delta_A^{M_j}) = \left[\frac{1}{4} + b_j^2, +\infty\right] .
$$
Proof. Use the coordinate $t = \ln y$ instead of $y$, so

$$M_j = S \times |\alpha_j^2, +\infty| \quad \text{and} \quad ds_j^2 = L_j^2 e^{-2t} d\theta^2 + dt^2; \quad (\alpha_j = e^{a_j}).$$

Then $-\Delta_A^{M_j} = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + e^t (D_t - A_2)(e^{-t} (D_t - A_2)),$

$$\tilde{b} = L_j^{-1} e^t (\partial_\theta A_2 - \partial_t A_1) \quad \text{and} \quad dm = L_j e^{-t} d\theta dt.$$

As in Lemma 5.3, we have

$$A - \tilde{A} = d\varphi \quad \text{if} \quad \tilde{A} = (\xi + L_j b_j e^{-t}) d\theta,$$

(for some constant $\xi$).

So we can also assume that $A = \tilde{A}$.

We replace the density $dm$ by $d\theta dt$, using the unitary operator $Uf = \sqrt{L_j} e^{-t/2} f,$ so

$$P = -U \Delta_A^{M_j} U^* = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}.$$ 

Then we get also that

$$\text{sp}(\Delta_A^{M_j}) = \text{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell) ; \quad P_\ell = D_t^2 + \frac{1}{4} + \left( e^\ell (\ell + \xi) \right)^2 L_j + b_j,$$

for the Dirichlet condition on $L^2(I; dt) ; \quad I = [\alpha_j^2, +\infty[.$

When $\ell + \xi \neq 0$, the spectrum of $P_\ell$ is discrete. More precisely

$$\text{sp}(P_\ell) = \text{sp}(P^\pm), \quad \text{where} \quad P^\pm = D_t^2 + \frac{1}{4} + (\pm \ell + b_j)^2$$

for the Dirichlet condition on $L^2(I_{j,\ell}; dt) ; \quad I_{j,\ell} = [\alpha_j^2 + \ln(|\ell + \xi|/L_j), +\infty[, \quad \text{and} \quad \pm = \frac{\ell + \xi}{|\ell + \xi|}.$

So $\liminf_{|\ell| \to \infty} \text{sp}(P_\ell) = +\infty$, and then we get easily that the spectrum of $-\Delta_A^{M_j}$ is discrete, when $\xi = [A]_{M_j}/(2\pi) \notin \mathbb{Z}.$

If $\ell + \xi = 0$, the spectrum of $P_\ell$ is absolutely continuous:

$$\text{sp}(P_{-\ell}) = \text{sp} \text{ess}(P_{-\ell}) = \text{sp} \text{acc}(P_{-\ell}) = \left[ \frac{1}{4} + b_j^2, +\infty[ ;$$

and then, when $[A]_{M_j} \in 2\pi \mathbb{Z},$ \quad $\text{sp} \text{ess}(\Delta_A^{M_j}) = \left[ \frac{1}{4} + b_j^2, +\infty[.$

This achieves the proof of Lemma 5.4.
References


